D. K. KAZARINOFF'S INEQUALITY FOR TETRAHEDRA

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1. INTRODUCTION

Let S be a tetrahedron, and P a point not exterior to S. Let the distances from P to the vertices and to the faces of S be denoted R_i and r_i, respectively. In this paper we establish an analogue of the Erdös-Mordell inequality for triangles [3, p. 12].

THEOREM 1. *For any tetrahedron whose circumcenter is not an exterior point,*

$$\sum R_i / \sum r_i > 2\sqrt{2}, \tag{1}$$

and $2\sqrt{2}$ *is the greatest lower bound.*

D. K. Kazarinoff stated that this inequality holds for all tetrahedra [3, p. 120]; but he refused to divulge his proof, probably because it was not simple enough, in his opinion, to be made public. Before his death, however, he did provide a simple proof of the Erdös-Mordell inequality [2], and he gave a generalization of this proof to three dimensions. This generalization and his proof of (1) for trirectangular tetrahedra are given in Sections 2 and 3. We use this work as a basis for the proof of Theorem 1.

2. THE FUNDAMENTAL INEQUALITY

Let the vertices of S be i, j, k, and l; let (i) and (jkl) denote the area of the face opposite i, and (ij) the length of the edge joining i and j; let H_i be the length of the altitude through i, R the radius of the circumsphere with center O, R_i the distance from P to i, and r_i the distance from P to the face opposite i.

A theorem of Pappus plays a leading role in the proof of the Erdös-Mordell inequality given in [2]. The following generalization of this theorem to three dimensions is of importance in the proof of Theorem 1. *Construct three triangular prisms which have for their bases three faces of* S, *which have a lateral edge in common, and of which all or none lie entirely outside of* S. *On the remaining face, construct a fourth prism whose lateral edges are translates of the common lateral edge of the first three prisms. Then the sum of the volumes of the first three prisms is equal to the volume of the fourth prism.*

LEMMA 1. *For any tetrahedron* S,

$$\sum R_i \geq \sum \frac{(ij)^2 + (ik)^2 + (il)^2}{2RH_i} r_i. \tag{2}$$

Equality holds if and only if P *and* O *coincide.*

Proof. The ingredients of the proof are the generalized Pappus theorem, an inversion (a reflection was used in [2]), and a theorem of von Staudt [1, p. 117]. We

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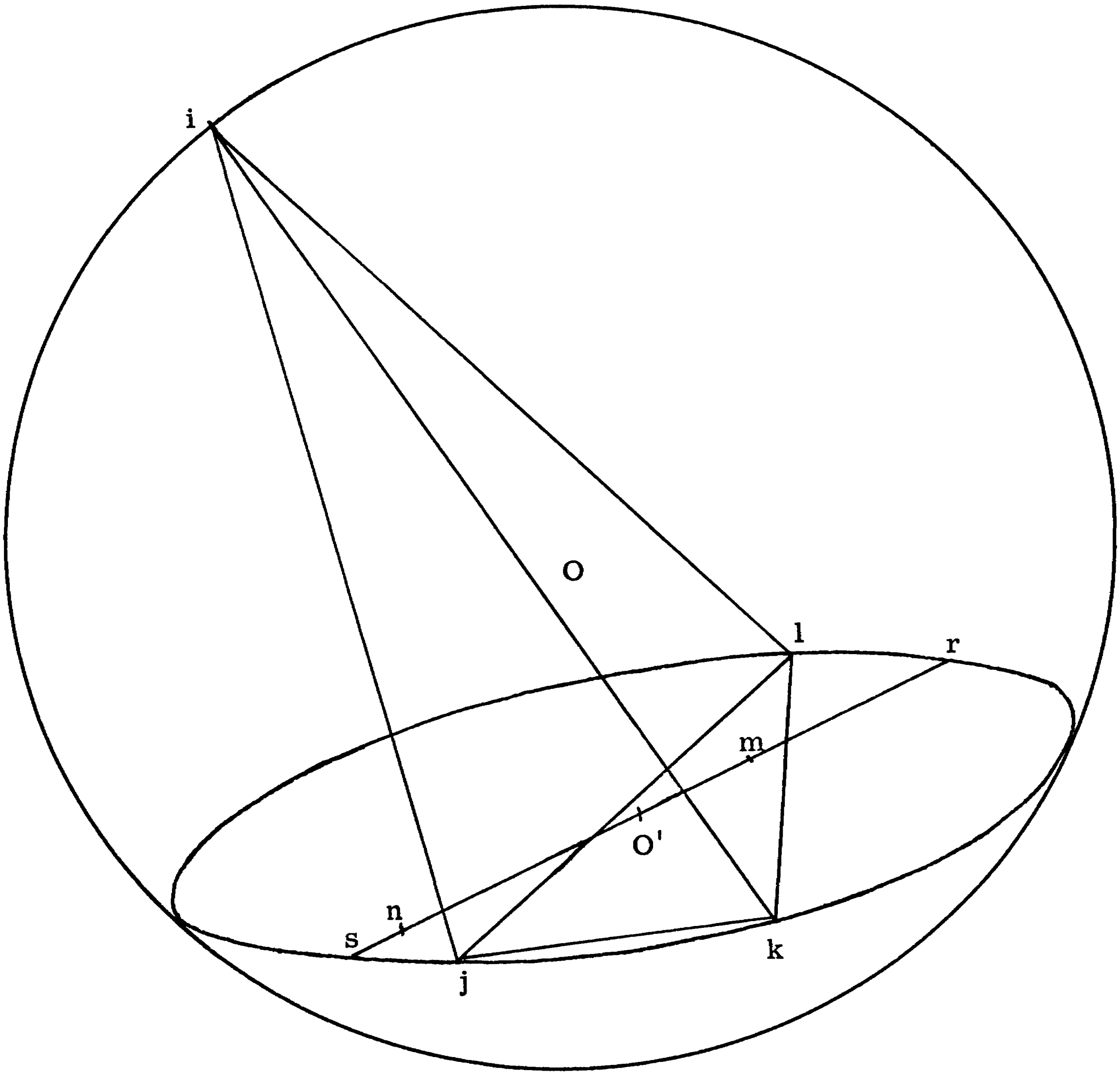

Figure 1.

transform S into a new tetrahedron ij'k'l' by a transformation of inversion $I(i, p_i)$, where $p_i^2 = (ij)(ik)(il)$. We consider the inversion as affecting only the vertices j, k, and l. Applying Pappus' theorem to ij'k'l' with iP as the common lateral edge, we find

$$R_i \cos(iP, iO)\,(j'k'l') = (j'k'i)\,r_l + (k'l'i)\,r_j + (l'j'i)\,r_k, \tag{3}$$

where cos (iP, iO) is the sine of the angle between iP and the plane of j'k'l'. Since $(ij')(ij) = p_i^2$, $(ij') = (il)(ik)$. Similarly, $(ik') = (ij)(il)$. Thus

$$(ij'k')/(ijk) = (il)^2, \tag{4}$$

and so forth. Since ijk and ik'j' are similar triangles, $(j'k')/(jk) = (ij')/(ik)$. But $(ij')/(ik) = (il)$; hence,

$$(j'k') = (jk)(il), \tag{5}$$

and so forth.

Let $T = (j'k'l')$, which by (5) is the area of the triangle whose sides have lengths (ij)(kl), (jk)(li), and (ki)(jl). Together, (3), (4), and (5) yield the conclusion that

$$TR_i \geq (il)^2(l)\, r_l + (ij)^2(j)\, r_j + (ik)^2(k)\, r_k, \tag{6}$$

and so forth. Now, a theorem of von Staudt states that $T = 6RV$, where V is the volume of S. Thus, since $3V = (i)H_i$,

$$T = 2RH_i(i), \tag{7}$$

and so forth. We now sum both sides of (6) over all vertices of S and divide by T, using (7). Equality will hold in the resulting relation (2) if and only if

$$\cos(iP, iO) = \cos(jP, jO) = \cdots = 1;$$

that is, if and only if O is not exterior to S and P coincides with O. This completes the proof of the lemma.

Remark. If the faces of S all have equal area, Pappus' theorem may be applied directly, without an inversion, to show that

$$\sum R_i \geq 3 \sum r_i.$$

3. TRIRECTANGULAR TETRAHEDRA

In this section we suppose that S is trirectangular, in other words, that at one vertex the edges are mutually perpendicular.

THEOREM 2. *For any trirectangular tetrahedron,*

$$\sum R_i > 2\sqrt{2} \sum r_i.$$

Proof. Let the vertex common to the orthogonal edges be i. It is known [1, p. 177] that

$$\frac{1}{H_i^2} = \frac{1}{(ij)^2} + \frac{1}{(ik)^2} + \frac{1}{(il)^2} \quad \text{and} \quad 2R = [(ij)^2 + (ik)^2 + (il)^2]^{1/2}.$$

Thus the coefficient of r_i in (2) is

$$\left\{ [(ij)^2 + (ik)^2 + (il)^2] \left[\frac{1}{(ij)^2} + \frac{1}{(ik)^2} + \frac{1}{(il)^2} \right] \right\}^{1/2},$$

which is greater than or equal to 3. The coefficient of any other r in (2), say r_j, is at least $2\sqrt{2}$, since

$$\frac{(jk)^2 + (jl)^2 + (ji)^2}{2RH_j} = \frac{3(ij)^2 + (ik)^2 + (il)^2}{2R(ij)} = \frac{(ij)}{R} + \frac{2R}{(ij)} \geq 2\sqrt{2}.$$

This completes the proof of Theorem 2. The circumcenter of a trirectangular tetrahedron is, of course, always outside the tetrahedron.

4. THE PROOF OF THEOREM 1

We prove the theorem by showing that the coefficients of the r's in (2) all exceed $2\sqrt{2}$. Moreover, we show that $\Sigma R_i = 2\sqrt{2}\Sigma r_i$, whenever S degenerates to an isosceles right triangle with two vertices (of S) at the vertex opposite the hypotenuse and with P at the circumcenter. The coefficients in (2) actually need not exceed $2\sqrt{2}$ if the circumcenter of S is allowed to be exterior to S, as the following example shows: Let jkl be an isosceles triangle with $(jk) = (jl) = \sqrt{10}$ and $(kl) = 6$, and let ij be perpendicular to the plane of jkl with $(ij) = 3$; then the coefficient of r_i in (2) is less than $2\sqrt{2}$. The reason that (2) ceases to be a "good approximation" in this instance is that in deriving (2) we have replaced the cosines in (3) by 1.

LEMMA 2. *If* O *is not exterior to* S, *then*

$$\frac{(ij)^2 + (ik)^2 + (il)^2}{2RH_i} > 2\sqrt{2}.$$

Proof. Clearly, we may assume that $R = 1$ and that O is fixed. The proof has two main steps. We first minimize $(ij)^2 + (ik)^2 + (il)^2$, keeping fixed the vertex i, the length H_i, and the plane of jkl, and keeping O in or on S. Let this minimum be $f(i, H_i)$. Second, we minimize $f(i, H_i)/2H_i$, subject to the same restriction relative to O and S; and we show that $f(i, H_i)/2H_i \geq 2\sqrt{2}$.

Since S is convex, iO extended must pierce the face jkl, say at m. Let n be the foot of the altitude from i, and note that n does not lie outside the circumcircle of the triangle jkl. In fact, the points m and n lie on a diameter rmns of this circle; and the center O' lies between them or coincides with them. Now,

$$(ij)^2 + (ik)^2 + (il)^2$$

$$= 3H_i^2 + (nj)^2 + (nk)^2 + (nl)^2.$$

Let us fix i, H_i, and the plane of jkl. The vertices j, k, and l may move only along their circumcircle, and they are subject to the restriction that m not lie outside jkl.

We proceed to minimize

$$\sigma = (nj)^2 + (nk)^2 + (nl)^2.$$

Suppose that k and l lie on opposite sides of rs. For fixed k and l, the minimum value of σ occurs when j is at s. We further decrease σ by moving l toward s until k, m, and l are collinear. Suppose that $(nk) \geq (nl)$, and rotate kl about m so that (nl) decreases. An application of the law of cosines to the triangles nO'k and nO'l shows that $(nk)^2 + (nl)^2$ also decreases, provided $\cos\alpha + \cos\beta$ increases, where α and β denote the angles at O' in nO'k and nO'l, respectively. Let the rotation of kl increase α by ω; then β decreases by at least ω. Now, $\sin\beta \geq \sin\alpha$; and

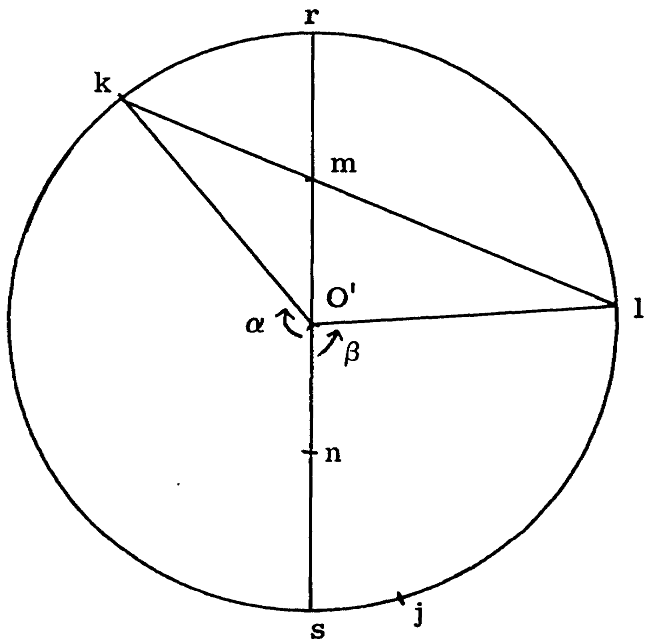

Figure 2.

$$\frac{d}{d\omega}[\cos(\alpha + \omega) + \cos(\beta - \omega)]_{\omega=0}$$

$$= \sin\beta - \sin\alpha.$$

The right member is positive, except when (nl) = (nk) or $\alpha = \pi$ and $\beta = 0$. From this it is easily seen that σ is a minimum when k = r and l = j = s. The minimum value $f(i, H_i)$ of

$$(ij)^2 + (ik)^2 + (il)^2,$$

for fixed H_i, thus occurs when i, j, k, l, and O are coplanar. This completes the first stage in the proof.

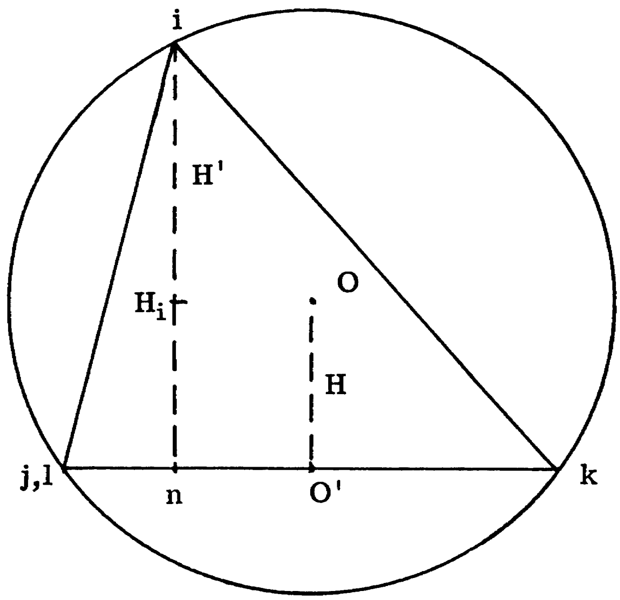

Figure 3.

We next minimize $f(i, H_i)/2H_i$. Let $(OO') = H$ and $H_i = H + H'$, where $1 \geq H' \geq H$. Also, let

$$F(H, H') = f(i, H_i)/2H_i.$$

Clearly,

$$F(H, H') = \frac{3(H + H')^2 + 2(\sqrt{1 - H^2} - \sqrt{1 - H'^2})^2 + (\sqrt{1 - H^2} + \sqrt{1 - H'^2})^2}{2(H + H')}$$

$$= \frac{3(HH' + 1) - \sqrt{1 - H^2}\sqrt{1 - H'^2}}{H + H'}.$$

We wish to find the minimum value of F in the isosceles right triangle containing all points (H, H') for which $H \leq H' \leq 1$ and $0 \leq H \leq 1$. On the perimeter of this triangle we find that

$$F(H, 1) = 3;$$

$$F(0, H') \geq 2\sqrt{2}, \text{ and } F(0, H') = 2\sqrt{2} \text{ if and only if } H' = 2\sqrt{2}/3;$$

$$F(H, H) \geq 2\sqrt{2}, \text{ and } F(H, H) = 2\sqrt{2} \text{ if and only if } H = 1/\sqrt{2}.$$

The minimum value of F, if any, over the interior of the triangle is taken on at a point where $\partial F/\partial H = \partial F/\partial H' = 0$. A computation reveals that this condition implies

$$H' = (3H - 2\sqrt{2})/(2\sqrt{2}H - 3).$$

But in this instance $F(H, H') = 2\sqrt{2}$. This completes the proof of the lemma.

Theorem 1 is an immediate consequence of the lemmas. Lemma 2 also provides a simple upper bound for R when O is not exterior to S.

Remark 1. The proof of Theorem 1 shows that the relation $\Sigma R_i = 2\sqrt{2}\Sigma r_i$ can hold only if P and O coincide and S is a degenerate tetrahedron in which two vertices coincide. If P = O, then $\Sigma R_i = 4R$; and Σr_i is a maximum when two vertices of S are at opposite ends of the hypotenuse of an isosceles right triangle and the two coincident vertices of S are at the third vertex of the triangle. In this case, and in this case alone, $\Sigma R_i = 2\sqrt{2}\,\Sigma r_i$.

Remark 2. One might suspect that in n-dimensional Euclidean space the relation $\Sigma R_i \geq 2^{n/2} \Sigma r_i$ holds. But A. L. Shields has pointed out a counter-example: if two vertices of a simplex in E_n $(n \geq 3)$ are opposite ends of a diagonal of a square, P is the midpoint, and the remaining n - 1 vertices approach one of the other corners, then

$$\sum R_i \to \frac{n+1}{2}\sqrt{2} \sum r_i .$$

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