On the Definition of Clifford Algebras

by

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Clifford algebras are usually defined in one of two ways. Let \( K \) be a field of characteristic not two. One method is to give a basis of the algebra [1]. The basis consists of the elements \( e_A \) where \( A \) ranges through the subsets of the set \( N = \{1, 2, \ldots, n\} \), including the null set \( \emptyset \). We write \( e_i \) for \( e_{\{i\}} \) and define

\[
(1) \quad e_i^2 = a_i e_\emptyset \quad (i = 1, \ldots, n)
\]

where the \( a_i \) are elements of \( K \); also

\[
(2) \quad e_i e_j = -e_j e_i \quad (i \neq j).
\]

Then if \( A = \{i_1, \ldots, i_r\} \) with \( i_1 < \cdots < i_r \), we require that \( e_A = e_{i_1} \cdots e_{i_r} \) and \( e_\emptyset = 1 \). From (1) and (2) products of the \( e_A \) can be defined. That multiplication is associative needs to be verified by computation.

A second method of definition is more intrinsic [2]. Let \( V \) be an \( n \)-dimensional vector space over \( K \). Let \( T(V) \) be the tensor algebra of \( V \), i.e., the free associative algebra over \( K \) consisting of sums of products of vectors in \( V \), where it is assumed that the product with a scalar is commutative. Let \( f \) be a symmetric bilinear scalar function on \( V \). Let \( J \) be the ideal of \( T(V) \) generated by \( \forall v w + w v - 2 f(v, w) \), where \( v \) and \( w \) range through \( V \). The difference algebra \( T(V)/J \) is defined to be a Clifford algebra.
The two definitions are connected by choosing an orthogonal basis in the space $V$ with the metric defined by $f$, i.e., a basis $u_1, \ldots, u_n$ of $V$ such that

$$f(u_i, u_j) = \delta_{ij} a_j.$$ 

Let $\overline{u_i}$ be the residue class of $u_i$ modulo $J$. The mapping

$$\theta: e_i \rightarrow \overline{u_i}$$

is clearly a homomorphism onto. In order to show that it is an isomorphism it is necessary to prove that the $\overline{u_{i_1}} \cdots \overline{u_{i_r}} (i_1 < \cdots < i_r)$ are linearly independent. This can be done by considering the inverse mapping $\theta^{-1}$ but then one must already have the algebra as given by the first definition. We shall prove directly that the $\overline{u_{i_1}} \cdots \overline{u_{i_r}}$, which we shall denote by $\overline{u}_A (A = \{i_1, \ldots, i_r\})$, are linearly independent.

The proof is by contradiction. Suppose $\sum c_A \overline{u}_A = 0$ ($c_A$ in $K$). Then $\sum c_A u_A$ is in $J$ and so

$$\sum c_A u_{i_1} \cdots u_{i_r} = \sum a_{ij} (u^2 - a_i) b_{ij}$$

$$+ \sum c_{ijk} (u_{i_1} u_{j_1} + u_{j_1} u_{i_1}) d_{ijk},$$

where the $a_{ij}$, $b_{ij}$, $c_{ijk}$, $d_{ijk}$ are non-commutative polynomials in the $u_i$. Suppose for some $B = \{j_1, \ldots, j_s\}$ we have $c_B \neq 0$; we may assume $c_B = 1$. Since (3) is an identity in the indeterminates $u_i$, we can equate those terms in which $u_{j_1}, \ldots, u_{j_s}$ appear to odd powers and the other $u_i$ to even powers. Hence

$$u_{j_1} \cdots u_{j_s} = F,$$

where
\[ F = \sum a_{ij}^l (u_i^2 - a_i)^b_{ij} \]
\[ + \sum c_{ijk}^l (u_i u_j + u_j u_i)^d_{ijk} ; \]

consequently

\[ (4) \quad u_{j_1}, \ldots, u_{j_s}, u_{j_1}, \ldots, u_{j_s} = (u_{j_1}, \ldots, u_{j_s})^F \]

and every term in this expression contains each \( u_i \) to an even power and every \( u_{j_1}, \ldots, u_{j_s} \) to a power at least 2.

Let \( x_1, \ldots, x_n \) be \( n \) commutative independent indeterminates over \( K \). To each expression \( \sum c u_{i_1} \cdots u_{i_p} \) where each subscript appears an even number of times in each term we make correspond \( \sum (-1)^v cx_{i_1} \cdots x_{i_p} \) where \( v \) is the number of inversions from the natural order in \( i_1, \ldots, i_p \). This is a homomorphism onto \( K[x_1, \ldots, x_n] \). Clearly addition is preserved. In order to show that multiplication is preserved it is sufficient to prove that if \( i = (i_1, \ldots, i_p) \) and \( k = (k_1, \ldots, k_q) \) are both in natural order, then \( (i_1, \ldots, i_p, k_1, \ldots, k_q) \) has an even number of inversions. But this is true because the numbers in \( i \) appear in pairs of adjacent equal numbers.

Under this mapping \( c' (u_i u_j + u_j u_i) d' \) has image 0. Hence from (4) we find that

\[ (x_{j_1} \cdots x_{j_s})^2 = (x_{j_1} \cdots x_{j_s})^2 \sum g_i(x_i^2 - a_i), \]

where \( g_i \) is a polynomial in \( x_1^2, \ldots, x_n^2 \). Division by \( (x_{j_1} \cdots x_{j_s})^2 \) gives

\[ 1 = \sum g_i(x_i^2 - a_i). \]

But this is impossible because under the mapping
\[ x_i \rightarrow a_i^{1/2} \quad (i = 1, \ldots, n) \]

into the field \( K(a_1^{1/2}, \ldots, a_n^{1/2}) \), the right side has image 0.

References


2. M. Eichler, Quadratische Formen und orthogonale Gruppen, p. 22 (1952).

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