A Footnote to "Statistical Decision Functions"

by

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The following theorem is one of the important results in the theory of two-person games given by Wald in his book [1].

Theorem 2.22 (p. 54). If A and B are the spaces of mixed strategies for a two-person zero-sum game, A being the strategies for player 1, and A is separable and B is weakly compact, then the class of all admissible strategies of player 2 is a complete class [2].

Wald says of this result, in a footnote, "This theorem is related to a theorem of Zorn [3] on partially ordered sets ... but cannot be derived from it, since Zorn assumes that each simply ordered subset has an upper bound in the system, whereas in our case merely each denumerable and simply ordered subset can be shown to have an upper bound." Wald's

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1. Statistical Decision Functions, Wiley Publications in Statistics, 1950.

2. Throughout, I use Wald's notation and definitions. I will remind the reader, however, that a strategy is admissible if there is no uniformly better strategy; and that a family of strategies is complete if for any strategy not in the family, one in the family is uniformly better.

3. See Lefschetz, Algebraic Topology, Amer. Math. Soc. Colloquium Publ., 27 (1942).

remark does not seem to give a clear picture of the situation. Actually, the hypotheses of Zorn's lemma are satisfied in this case. Since clarification of this point casts some light on the role of the respective hypotheses on A and on B, and yields some information on the structure of A, it seems worth while to discuss it.

The strategies $\{\eta\}$ of the second player are partially ordered by the condition that $\eta' < \eta''$ means that η' is uniformly better than η''; i.e., for every ξ in A, $K(\xi, \eta') \leq K(\xi, \eta'')$, and for some ξ_0, $K(\xi_0, \eta') < K(\xi_0, \eta'')$, where $K(\xi, \eta)$ is the "payoff" function for the game. To satisfy the hypotheses of Zorn's lemma, we have to show that if $\{\eta_\alpha\}$ is a simply ordered subset of B, then there is an element η' of B such that $\eta' \leq \eta_\alpha$ for every α. If $\{\eta_\alpha\}$ were countable, as Wald remarks, the existence of η' would be immediate from the definition of weak compactness of B. We have, however, the following theorem.

Theorem. If A is separable, and $\{\eta_\alpha\}$ is a simply ordered set of strategies of player 2, then there is a countable subset $\{\eta_i\}$ of $\{\eta_\alpha\}$ such that for each η_α there is an i for which $\eta_i \leq \eta_\alpha$.

Proof: Let $\{a_i\}$ be a countable sequence of elements of A dense in A. It is immediate from the definition of the metric in A that if $\eta < \eta'$, then there is an i such that $K(a_i, \eta) < K(a_i, \eta')$ [4]. For each i, let K_i be the set of all numbers $K(a_i, \eta)$, η in $\{\eta_\alpha\}$. Then there is a countable sequence $k_{i,n}$ of points of K_i, with

$$\lim_{n\to\infty} k_{i,n} = \underset{\alpha}{\text{g.l.b.}}\ K(a_i, \eta_\alpha).$$

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4. Used by Wald in his proof.

Each $k_{i,n}$ is the value of $K(a_i, \ )$ for some choice $\eta_{i,n}$ of η in $\{\eta_\alpha\}$. For each integer j, of the strategies $\eta_{i,j}$, $i = 1, 2, \ldots, j$, one, η_j, is such that for each i, $\eta_j \leq \eta_{i,j}$. (Of course, we have no assurance, for any given j, that this set $\eta_{i,j}$, $i \leq j$, has j distinct elements.) The set $\{\eta_j'\}$ is the desired countable subset. For, given any η_α in $\{\eta_\alpha\}$, if it is not a lower bound, then there exist an a_i and a $j > i$ such that $k_{i,j} < K(a_i, \eta_\alpha)$. But then $K(a_i, \eta_j) < K(a_i, \eta_\alpha)$, and since η_j and η_α are comparable, we must have $\eta_j < \eta_\alpha$.

If we now require that B be weakly compact, it follows that every simply ordered system $\{\eta_\alpha\}$ of elements of B has a lower bound, namely, any lower bound of the corresponding $\{\eta_i\}$. By Zorn's lemma, then, given any simply ordered subset of B, there is an element which is a lower bound for the subset, and which is not greater than any other element of B — that is, is admissible. This proves 2.22.

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