TERNARY RELATIONS IN GEOMETRY AND ALGEBRA

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One of the simplest sets of axioms used in Mathematics is the set of propositions defining an equivalence relation. We will express the assertion that a is equivalent to b by writing (ab)'. For our present purposes it is convenient to state that this binary relation possesses the properties (the symbol > means "implies" in what follows):

(Se) Symmetry, that is, (ab)' > (ba)'
(Te) Transitivity, that is, (ab)', (bc)' > (ac)'.

The starting point of the following discussion was the observation that in many places in Mathematics similar although more complicated propositions occur, or that it is often possible to reformulate the discussion so that it can be stated in similar terms. An example of a ternary relation is that of collinearity. If we denote the fact that the points A, B, C lie on a straight line by writting (ABC)* we can say that this relation possesses the property

(Sg) Symmetry, that is (ABC)* > (BCA)*, (BAC)* etc.
and the property

(Tg) If A and B are distinct points

$$(ABC)^*, (ABD)^* > (ACD)^*.$$

Clearly, the last property is of the same general type as the transitivity property of the equivalence relation; the difference is essentially that there we deal with a binary relation and three elements and here, in the case of collinearity, with a ternary relation and four elements. There is also the difference—we will discuss it later—that here we must assume that the elements are distinct.

Geometry. It is possible to present the whole projective geometry in the plane as the study of properties of this relation of collinearity. To begin with we shall consider the propositions of Pappus and Desargues. The first states that if a hexagon is inscribed into a degenerate conic (that means, a conic consisting of two straight lines) the meets of its opposite sides are collinear. If the vertices of the hexagon are denoted by A, B, C, D, E, F the fact that it is inscribed into a degenerate conic is expressed by the collinearities (ACE)* and (BDF)*; if we denote the meets of the opposite sides by X, Y and Z we have six further collinearities, namely (AXB)*, (DXE)*, (BYC)*, (EYF)*, (CZD)*, (FZA)*. The conclusion is that (XYZ)*. Again, a proposition of type Tg.

The Desargues proposition states that if the joins of corresponding vertices of two triangles are concurrent then the meets of the corresponding sides are collinear. Denoting the vertices of the two triangles by A, B, C, and P, Q, R respectively the concurrence of the joins of corresponding vertices is expressed by (OAP)*, (OBQ)*, (OCR)* and if the meets of corresponding sides are denoted by X, Y, Z we have the further collinearities

(AXB)*, (PXQ)*, (BYC)*, (QYR)*, (CZA)*, (RZP)*,

and the conclusion is (XYZ)*. Again a proposition of type Tg (in both Pappus and Desargues propositions it is assumed that all points are distinct.

A very important concept in projective geometry is that of harmonic pairs. Consider a quadrifigure consisting of the four points A, B, C, D and the four lines AB, BC, CD, DA. The lines AC and BD are usually called diagonal lines; we shall call the meet E of AB and CD, and the meet F of BC and DA diagonal points of the quadrifigure. The pairs of points E, F is said to be harmonic to the pair of intersections of the diagonals with the line EF. In order for the concept of harmonicity to make sense we must be sure that these four points are distinct, that is that the meet of the diagonal lines is not collinear with the diagonal points. This then is an important proposition of projective geometry (that might be taken as an axiom), and we want to compare its structure as expressing a property of the relation of collinearity with the structure of other propositions giving properties of ternary relations. At first glance this seems to be a proposition of a different type in that its conclusion appears as a negation. However, it turns out that the situation is this: denote the meet of the diagonals AC and BD by G; then what we are given are the collinearities

(1) (EAB)*, (ECD)*, (FBC)*, (FDA)*, (ACG)* and (BDG)*.

It seemed that we want these relations to imply that E, F, G are <u>not</u> collinear; but there <u>is</u> a case when E, F, G, <u>are</u> collinear inspite of the above collinearities holding; this happens when the quadrifigure collapses into one line, that is, when the points A, B, C, D, are collinear. The proposition we want is then that if the above collinearities (1) hold and also (EFG)* then all 7 points are collinear. We have thus again a proposition of the same type Tg as those we had before. Here again we must assume that there are no identical points among the seven. Before we discuss this proviso we turn to algebra.

Algebra. We will speak here of addition; we may think of addition of numbers or, more generally, of the group operation of a commutative group. The statement

$$a + b = s \tag{2}$$

expresses the fact that a, b, s satisfy a certain ternary relation. This relation differs from the relations considered so far in that it is not symmetric, s plays here a different role from those played by a and b. However, it is easy to replace it by a symmetric relation, namely

$$a + b + c = 0 \tag{3}$$

where c satisfies another relation of the same type, namely

$$c + s + 0 = 0 . \tag{4}$$

We will now try to express the associative property of the operation of addition in terms of the relation (3). We write this associativity, as

$$(a + b) + c = a + (b + c) \tag{5}$$

and introduces the notation

$$a + b = p, \quad b + c = q, \quad p + c = r \tag{6}$$

Then the equality (5) takes the form

$$a + q = r . \tag{7}$$

In order to rewrite this in terms of the symmetric relation (3) we introduce e, f, g which satisfy

(8) $p + e + 0 = 0, \; q + f + 0 = 0, \; r + g + 0 = 0;$

then we can write (6) as

(9) $a + b + e = 0, \; b + c + f = 0, \; p + c + g = 0$

and the associativity (5) may be expressed by saying that

(10) $$a + q + g = 0$$

is a consequence of (8) and (9).

We have thus replaced the relation of addition (2) by a symmetric relation (3) and we found that the associativity of addition can be expressed in terms of that relation. We shall denote the fact that a, b, c satisfy the relation (3) by writing

$$(abc)'';$$

then the property of associativity may be expressed as follows:

Propositiona A.

$$(abe)'', (bcf)'', (pcg)'', (pez)'', (qfz)'' > (aqg)''.$$

We left out r because leaving it out does not affect the proposition. Also we denoted 0 by z in order to bring out the fact that it does not play any preferred role in the proposition. As a matter of fact, any element plays here the same role as any other element. This can be seen by writing the nine elements as a table

a	b	e
g	c	p
q	f	z

and noticing that interchanging the rows or the columns does not affect the situation.

We note that the ternary relation ( )'' which we have derived from the operation of addition possesses a property of the type T whose prototype was the transitivity property of the equivalence relation, the difference being that the hypothesis of this proposition involves 5 assertions instead of two, and that the total number of elements is 9 instead of 3, and, of course, that we have a ternary relation instead of a binary relation. Unlike the case of geometrical relations there is no additional condition that the elements must be distinct.

We have now translated the properties of commutativity and associativity of additon into properties of the ternary relation ( )''. Corresponding to closure of the group under addition we will want an existence property of our relation, namely

Proposition (E). Given u, v, there exists x such that

(uvx)''

Corresponding to the fact that the sum of two elements is uniquely determined we now want a proposition saying that x in the above relation is uniquely determined by u and v. In other words, if there is also y such that (uvy)'' we want x and y to be identical. How should we treat the question of identity? I believe Leibnitz would have said that x and y are identical if anything we may assert concerning x may also be asserted concerning y. Such a statement sounds too absolutistic, we should be satisfied with much less. We'll say: x and y are identical with repsect to a relation R, if whenever R(abx) we also have R(aby), and conversely. This is the relative identity we need. If we adopt this meaning of identity the uniqueness properity of our relation " will be

Proposition (U). (uvx)", (uvy)"; (abx)" > (aby)"

We see that we have again a proposition of type T. Is it independent of proposition (A)? There we have nine elements, in proposition (U) we have only six, so if we want to derive (U) from (A) we must make some identifications in (A). If we identify g with b and f with p, and write x for e, y for q, v for z and u for p (and f) it becomes

(abx)", (bcu)", (ucb)", (uvx)", (yuv)" > (ayb)" .

This differs from the proposition (U) we wanted to derive only in that the hypothesis contains (twice) the extra relation (ucb)". Since this is the only place where c appears all we need in order to derive (U) from (A) is the assurance that, given two elements b and u, there exists an element c which satisfies the relation (bcu)". We have the assurance if we accept proposition (E). This shows that the uniqueness (U) follows from associativity (A) and existence (E).

In what precedes we have been discussing some properties of addition (or the group operation of a commutative group) in terms of ternary relation. We want to see now whether these properties are sufficient to define a (commutative) group. We assume then a ternary relation which possesses the properties (S), (A) and (E), and we know that, as a consequence it also has the property (U). We want to see whether we may introduce the group operation and whether we'll have all the properties of a group.

The first thing we'll have to do is to <u>select</u> our element which is to play the part of identity, that is zero. If we select such an element—call it z_0 — we can, reversing our steps, define s as the sum of the elements a and b by the relations, analogous to (2) and (3), namely

$$(abc)'', \ (csz_0)'' .$$

If these relations hold we shall write

$$a + b = s.$$

The next question is whether the addition so defined possesses the necessary properties. Commutativity is an immediate consequence of A and existence and uniqueness of the sum also are implied by what precedes. But we must find out whether z_0 actually plays the part of zero, that is whether $a + z_0 = a$. With the above definition of addition this means that we must check the relations

$$(az_0c)'' \text{ and } (caz_0)'' .$$

Because of symmetry these relations are actually one relation and the question reduces to the question whether there exists c which satisfies it. The answer is given by (E) and is affirmative.

<u>Existence and Uniqueness in Geometry</u>. So far we have considered only points in geometry but no lines. It is possible to express all geometrical axioms in terms of points alone but in order to do that we must express in terms of collinearity of points the statement that two lines intersect or have a point in common, that is:

given four points A, B, C, D there exists a point X such that $(ABX)^*$ and $(CDX)^*$. We see that this is an existence type proposition, a proposition of the same type as the proposition (E) above.

It is interesting to remark that if we introduce in the course of a proof auxiliary points we have to use the above existence proposition. As a result, two propositions of type T may be independent in a geometry without existence propositions but if we assume the above existence proposition one may become derivable from the other, as is the case with the Desargues and Pappus propositions in plane projective geometry.

We come now to the role that the concept of identity or relative identity plays in geometry—it is quite different from the role it plays in algebra. There it appeared in connection with uniqueness. In geometry it is not needed for this purpose; if we want to discuss uniqueness of intersection of two straight lines we would have to say that if $(ABX)^*$, $(CDX)^*$ and $(ABY)^*$ and $(CDY)^*$ that X is identical with Y; but this is true only if A, B and C, D are not on the same line. In other words, if X and Y are not identical and we have the above collinearities then we have also $(ABC)^*$ etc. But if X and Y are not identical then from $(ABX)^*$ and $(ABY)^*$ follows $(AXY)^*$ and from $(CDX)^*$ and $(CDY)^*$ follows $(CXY)^*$; now $(AXY)^*$ and $(CXY)^*$ implies $(ACX)^*$ and $(ACX)^*$ and $(ABX)^*$ implies $(ABC)^*$ so that we have what we want from our proposition of type Tg. However, we have been using here heavily the concept of non-identity, the same proviso that appears in all our geometrical propositions. It is time now to consider it. The simplest proposition in which we used non-identity was: if A and B are distinct then $(ABC)^*$, $(ABD)^*$ implies $(ACD)^*$.

The statement that A and B are distinct means that they are not identical or the negation of the statement that they are identical. This last statement, as we understand it now, would mean that for every P we have $(ABP)^*$. To deny this we would have to say that there exists a P such that $(ABP)^*$ is false; we shall write this as $(ABP)^{-*}$. A complete statement of our proposition is now

$$(ABC)^*, (ABD)^*, \text{ there exists P such that } (ABP)^{-*} > (ACD)^*.$$

It is clear that the words "there exists a P such that" are superfluous: they merely express the fact that P appears in the hypothesis of the proposition but not in the conclusion; the same thing is true of B; omitting the above words we arrive at the final form of our proposition

$$(ABC)^*, (ABD)^*, (ABP)^{-*} > (ACD)^* .$$

All the other geometrical propositions of type T must be supplemented in a similar way by introducing additional points P, Q etc. and additional assertions, assertions of <u>negation</u>, assertions stating that certain collinearities do not hold.

We should note that in order to pass from our propositions (A) and (E) to group axioms we had to <u>select</u> an element to play the part of zero. We have analogous situations in geometry. For instance, we can pass from projective geometry to affine geometry by selecting a line which would play the role of the line at infinity. There are several other situations in which selection of an element, or of a set of elements, plays an important part in geometry; it, so to say, marks the boundaries between different geometries.

General Considerations. In what precedes we have been interested in the type of propositions expressing properties of ternary relations. We found that the simplest propositions are of what may be called generalized transitivity type, the hypothesis in these propositions consists of an assertion that relations on certain elements hold and the conclusion is that a relation on three of these elements holds.

In addition, some propositions are of the existence type, the conclusion asserts the existence of an element that together with elements appearing in the hypothesis satisfies a relation.

Finally, some properties are expressed by propositions whose hypothesis contains a negation of the truth of a relation. It is interesting to note that such propositions occur in geometry but not in the situation dealing with algebra we have been discussing.

It seems to be interesting to see how far we can go in a certain branch of mathematics if we limit ourselves to propositions of a certain type. Such an attitude is similar to what one does in geometry when one tries to find out how far we can go if we limit ourselves to using only some of the axioms, the classical example being that of Euclid who tried to push his theory so far as he could before assuming what we now call Euclid's axiom. Of course, there is also a difference because in geometry we may not only reject an axiom but also replace it by another, possibly by its negation, and this does not seem to have a counterpart in our case. We may mention also the analogy between our attitude and the attitude of mathematicians who reject the Law of the Excluded Middle or the Zermelo Axioms.

The question arises: what effect the attitude described above might have on the concepts of completeness and categoricity of a system of axioms. What we mean is the question of whether it is possible, in some sense, to add to a system of axioms giving properties of a certain relation additional axioms dealing with the same relation. It is clear that the question acquires a new meaning if we limit the type of these additional axioms. It seems plausible, for instance, that in order to distinguish real projective geometry from complex projective geometry the additional axiom will have to contain a negation in its conclusion. Is the set of axioms of real projective geometry complete in the sense that it is impossible in some sense to add to it an axiom that does not contain a negation in its conclusion?

It seems convenient to speak of a maximal property rather than of completeness. We shall say that a system of propositions S of a certain type is maximal within this type when it is "not possible" to add to it a proposition of the same type which is not a consequence of S; and it remains to say what "not possible" means in a given situation. As an example we'll consider the set of axioms defining the equivalence relation and we will show that this system is maximal in the sense that the addition of an independent axiom results in the system becoming trivial, that means that for the new system the relation is satisfied identically.

We consider then the binary relation (ab)' having the properties

1. (aa)',
2. (ab)' > (ba)',
3. (ac)', (bc)' > (ab)'.

and we consider a new proposition of the same type independent of these three. The conclusion of this proposition must be of the form (ab)'. The hypothesis must consists of assertions that the relation holds for certain pairs involving a, b and other letters. We do not change this new proposition if we add to the hypothesis all the consequences that follow from it on the basis of 1, 2, and 3. In the hypothesis so enlarged we replace by a all the letters which appear under the sign ' together with a, and by b all the other letters. The relation (ab)' cannot appear in the enlarged hypothesis, otherwise the new proposition would be a consequence of 1, 2, and 3. On the other hand, the hypothesis consists now of assertion (aa)' and (bb)' that are consequences of 1 and therefore the conclusion (ab)' applies to any pair; that is, the relation is satisfied identically, as asserted.

Conclusion. It seems appropriate to mention here a few situations in which instead of ternary relations we have relations involving more variables.

One quaternary relation is that of cocircularity, the relation expressing the fact that four points (in a Euclidean space) belong to the same circle; cosphericity is a relation expressing the fact that five points are on a surface of a sphere. Analagmatic or inversive geometry may be considered as the study of these relations.

Another relation involving four elements may be made the basis of the study of rings with commutative multiplication; it is the symmetric relation

$$xyz + xyu + yzu + xzu = 1.$$

The ring operations can be obtained from this by selecting two elements. If we set $u_0 = 0$ we obtain

$$xyz = 1$$

which after selecting a unit of multiplication will furnish an abelian group as in the earlier part of this paper, the difference being only in notation. Of course, in order to make a quaternary relation the basis for the study of a ring it is necessary to impose on the relation properties that correspond to the associative laws and to the distributive law.

Finally, we want to mention a relation on six variables the consideration of which was the starting point of this investigation. We call six points A, B, C, D, E, F Pascalian if the meets of the three pairs of the hexagon ABCDEF are collinear. It is a proposition of projective geometry which we may (with H. Liebmann) take as an axiom, that this Pascality relation is symmetric. It is then a theorem (the seven point theorem) that if, given seven points, two sets of six points taken from among them are Pascalian the remaining sets of six points are also Pascalian (certain degenerate cases form exceptions). One notices that this proposition is of the same type as transitivity of the equivalence relation or as propositions expressing the fundamental property of collinearity.