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Determinant Functors on Exact Categories and Their Extensions to Categories of Bounded Complexes

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Introduction

In this paper I revisit a theme unsatisfactorily treated in [KM]. The methods used here are more natural and more general. The theorem we prove was suggested to me by Grothendieck in a letter dated May 19, 1973 (see Appendix B), and it states that the category of determinants on the derived category of an exact category is equivalent via restriction to the category of determinants on the exact category itself.

Here is how the problem comes about [KM]. Consider the following category. The objects are bounded complexes of locally free finite quasi-coherent sheaves of $\mathcal{O}_X$-modules on a fixed scheme (site) X. The morphism $\operatorname{Mor}(A, B)$ of two such complexes is the group of global sections of the sheaf of germs of homotopy classes of homomorphisms from A to B. If we assign to every complex the invertible sheaf

$$f(A) = \left(\bigotimes_{i\in\mathbb{Z}} \bigwedge^{\max} A^{2i}\right) \otimes \left(\bigotimes_{i\in\mathbb{Z}} \bigwedge^{\max} A^{2i+1}\right)^{-1},$$

then the problem is to assign to every quasi-isomorphism $\alpha \in \operatorname{Mor}(A, B)$ an isomorphism $f(\alpha)\colon f(A) \to f(B)$ in such a way that f becomes a functor and such that $f = \bigwedge^{\max}$ in case of a complex consisting of a single locally free sheaf supported in degree 0. The existence of such an f follows immediately from the theorem. The theorem is quite general and depends (a) on certain properties of projective modules over a *commutative* ring and short exact sequences of such, and (b) on certain properties of tensor products of modules of rank 1.

The appropriate notions are that of an exact category (see [Q, Sec. 2]) and that of a commutative Picard category. The reader not familiar with the notion of an exact category is advised to have in mind the category of finitely generated *projective* modules over a commutative ring, where exact sequences are what they are. An admissible monomorphism is an injection whose cokernel is projective, and similarly an admissible epimorphism is a surjection with projective kernel. Of course, in this particular case all surjections are admissible.

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The axioms and some important results about commutative Picard categories are given in Appendix A. In particular we find the notion of an *inverse structure* (see Definition A.16) quite useful. Such a structure always exists and is unique up to unique isomorphism.

In Section 1 we define the notion of a determinant and state some fundamental properties (cf. [D]).

In Section 2, we state and prove the main theorem. Even though we give an explicit construction of the determinant of a quasi-isomorphism, the verification of its properties is usually done by induction with respect to length of complexes. The good complexes for induction are the *admissible* complexes (see Definition 2.13). Unfortunately, in some silly exact categories there are acyclic complexes that are not admissible. Fortunately, by [TT, A.7.16b], for every acyclic complex A there exists a *split* exact admissible complex E supported in the same degrees as A and such that $A \oplus E$ is admissible and acyclic; this is sufficient for the proof to go through. In the case of projective modules, every acyclic complex is admissible (in fact, split-exact), so most readers should disregard this technicality.

In Section 3 we establish, under certain conditions, natural isomorphisms between (a) the determinant of a complex and that of its cohomology and (b) the determinant of a filtered complex and that of the rth term of its associated spectral sequence.

In Section 4 we generalize the main theorem to *multideterminants* and prove a result suggested to me by Pierre Deligne. In Section 5, we give a formula for the determinant of a homotopy equivalence in terms of a *good pair* of homotopies (see Definition 5.4). It is then possible to compare our construction with that of Ranicki [R].

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1. Definitions and First Properties

In order to fix the definition of a determinant functor on an exact category, and on the exact category of bounded complexes of an exact category, we will consider certain special subcategories of exact categories (see also [Q; TT, Apx. A]).

DEFINITION 1.1. Let $\mathcal{E}$ be an exact category. We call a class w of morphisms an *SQ-class* if it satisfies the following axioms.

SQ 1 Every isomorphism is in w.

SQ 2 If any two of α, β, and $\beta\alpha$ are in w, then so is the third.

SQ 3 If α', α, and α'' are morphisms of short exact sequences and if any two of them are in w, then so is the third.

Let $\mathcal{E}$ be an exact category, w a SQ-class of morphisms, and P a Picard category. We will use the following notation: $\mathcal{E}_{\mathrm{w}}$ is the subcategory determined by w, and $\{\mathcal{E}\}_{\mathrm{w}}$ is the category of short exact sequences and morphisms in w^3. We have three functors $p', p, p''\colon \{\mathcal{E}\}_{\mathrm{w}} \to \mathcal{E}_{\mathrm{w}}$ defined by $p^i(A' \rightarrowtail A \twoheadrightarrow A'') = A^i$ for $i \in \{\cdot', \cdot, \cdot''\}$, and likewise for morphisms.

Definition 1.2. A *predeterminant* f on $\mathcal{E}_{\mathrm{w}}$ with values in P consists of a functor $f_1\colon \mathcal{E}_{\mathrm{w}} \to P$ together with a natural isomorphism $f_2\colon f_1 \circ p \to f_1 \circ p' \otimes f_1 \circ p''$.

Remark 1.3. For any 0-object Z of $\mathcal{E}$, the sequence $Z \rightarrowtail Z \twoheadrightarrow Z$ is a short exact one. Applying f_2 to this sequence gives $f_1(Z)$ the structure of a reduced unit and so, by Remark A.8, $f_1(Z)$ is a unit.

Definition 1.4. A predeterminant f on $\mathcal{E}_{\mathrm{w}}$ with values in P is a *determinant* if the following three conditions are fulfilled.

(i) *Compatibility.* For any object A, if $\Sigma = (A = A \twoheadrightarrow 0)$ then the morphisms $f_2(\Sigma)$ and $\delta^R_{f_1(0)}(f_1(A))$ are inverse to each other:

$$\begin{tikzcd} f_1(A) \arrow[r, bend left, "f_2(\Sigma)"] & f_1(A) \otimes f_1(0). \arrow[l, bend left, "\delta^R_{f_1(0)}(f_1(A))"] \end{tikzcd}$$

(ii) *Associativity.* For any short exact sequence of short exact sequences and for any exact square (as in the left-hand diagram), the right-hand diagram is commutative:

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail] \\ A \arrow[r, tail] \arrow[d, two heads] & C \arrow[r, two heads] \arrow[d, two heads] & B' \arrow[d, two heads] \\ 0 \arrow[r, tail] & A' \arrow[r, equal] & A'; \end{tikzcd} \qquad \begin{tikzcd} f_1(C) \arrow[r, "f_2"] \arrow[d, "f_2"'] & f_1(A) \otimes f_1(B') \arrow[d, "1\otimes f_2"] \\ f_1(B) \otimes f_1(A') \arrow[r, "f_2\otimes 1"] & f_1(A) \otimes f_1(C') \otimes f_1(A'). \end{tikzcd}$$

(iii) *Commutativity.* The two short exact sequences on the left give rise to the commutative diagram on the right:

$$\Sigma_1 = A \xrightarrow{\binom{1}{0}} A \oplus B \xrightarrow{(0\ 1)} B, \qquad \Sigma_2 = B \xrightarrow{\binom{0}{1}} A \oplus B \xrightarrow{(1\ 0)} A;$$

$$\begin{tikzcd} f_1(A \oplus B) \arrow[r, equal] \arrow[d, "f_2(\Sigma_1)"'] & f_1(A \oplus B) \arrow[d, "f_2(\Sigma_1)"] \\ f_1(A) \otimes f_1(B) \arrow[r, "\psi"] & f_1(B) \otimes f_1(A). \end{tikzcd}$$

PROPOSITION 1.5.

(a) *If* $\alpha\colon A \to B$ *is an isomorphism, then*

$$\delta^R \circ (f_2(A \xrightarrow{\alpha} B = 0)) = [f_1(\alpha)]^{-1}$$

and

$$\delta^L \circ (f_2(0 = A \xrightarrow{\alpha} B)) = f_1(\alpha).$$

(b) *If we consider* $\mathcal{E}_{\mathrm{w}}$ *as an AC (associative and commutative) tensor category with* $\oplus$ *as its tensor functor and consider the isomorphism* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\colon A \oplus B \to B \oplus A$ *for its commutation, then the functor* f_1 *together with the natural isomorphism* $f_2\colon f_1(A \oplus B) \to f_1(A) \otimes f_1(B)$ *makes the pair* f_1, f_2 *an AC tensor functor of AC tensor categories.*

(c) *For any* A, *we have* $f_1(-1_A) = \varepsilon(f_1(A))$ *considered as an automorphism of* 1.

Proof. The proofs of (a) and (b) follow directly from functoriality, compatibility, and commutativity. Now (c) follows from (b) and the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{1}"] \arrow[d, equal] & A \oplus A \arrow[r, "(1\ -1)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)"] & A \arrow[d, "-1"] \\
A \arrow[r, "\binom{1}{1}"] & A \oplus A \arrow[r, "(1\ -1)"] & A\,.
\end{tikzcd}$$

□

DEFINITION 1.6. By an *admissible* filtration we shall mean a finite sequence of admissible monomorphisms $0 = A^0 \rightarrowtail A^1 \rightarrowtail \cdots \rightarrowtail A^n = C$.

If $0 = A^0 \rightarrowtail A^1 \rightarrowtail \cdots \rightarrowtail A^n = C$ and $0 = A'^0 \rightarrowtail A'^1 \rightarrowtail \cdots \rightarrowtail A'^n = C'$ are admissible filtrations and $\alpha\colon C \to C'$ is a morphism, then we will say that α *respects* the filtrations if the induced maps $A^i \to C'$ factor through A'^i.

The proofs of the next two propositions are outlined in [D]. Actually, he first proves Corollary 1.10 and then Proposition 1.9 by induction. The next proposition follows from associativity by induction.

PROPOSITION 1.7. *Let* $0 = A^0 \rightarrowtail A^1 \rightarrowtail \cdots \rightarrowtail A^n = C$ *be an admissible filtration, and let* $A^{i-1} \rightarrowtail A^i \twoheadrightarrow C^i$ *be short exact sequences in* $\mathcal{E}$. *Then, by repeated use of* f_2, *we can construct an isomorphism* $f_1(C) \to \bigotimes_{i=1}^n f_1(C^i)$.

Moreover, if $0 = A'^0 \rightarrowtail A'^1 \rightarrowtail \cdots \rightarrowtail A'^n = C'$ *is an admissible filtration,* $A'^{i-1} \rightarrowtail A'^i \twoheadrightarrow C'^i$ *are short exact sequences, and* α *is a morphism* $C \to C'$ *that respects the filtrations and induces* w-*morphisms* $\alpha^i\colon C^i \to C'^i$ *for each* i $(i \leq 1 \leq n)$, *then* α *is a* w-*morphism and the following diagram is commutative*:

$$\begin{tikzcd}
f_1(C) \arrow[r, "f_1(\alpha)"] \arrow[d] & f_1(C') \arrow[d] \\
\bigotimes_{i=1}^{n} f_1(C^i) \arrow[r, "\bigotimes_{i=1}^{n} f_1(\alpha^i)"] & \bigotimes_{i=1}^{n} f_1(C'^i).
\end{tikzcd}$$

DEFINITION 1.8. We call two filtrations $0 = A^0 \rightarrowtail A^1 \rightarrowtail \cdots \rightarrowtail A^n = F$ and $0 = B^0 \rightarrowtail B^1 \rightarrowtail \cdots \rightarrowtail B^n = F$ *compatible* if the lattice generated by the $i(A)$s and the $i(B)$s in the Gabriel–Quillen embedding $i : \mathcal{E} \to \mathcal{A}$ is admissible (see [G]).

PROPOSITION 1.9. *Let $0 = A^0 \rightarrowtail A^1 \rightarrowtail \cdots \rightarrowtail A^m = F$ and $0 = B^0 \rightarrowtail B^1 \rightarrowtail \cdots \rightarrowtail B^n = F$ be compatible filtrations. Let $A^{i-1} \rightarrowtail A^i \twoheadrightarrow C^i$ and $B^{j-1} \rightarrowtail B^j \twoheadrightarrow D^j$ be short exact sequences and, for each i and j, let*

$$\frac{B^{j-1} + (A^i \cap B^j)}{B^{j-1} + (A^{i-1} \cap B^j)} \approx E^{i,j} \approx \frac{A^{i-1} + (B^j \cap A^i)}{A^{i-1} + (B^{j-1} \cap A^i)}$$

be the butterfly isomorphisms. Then the $E^{i,j}$ are the successive quotients of the two extreme admissible filtrations

$$0 \rightarrowtail \cdots \rightarrowtail B^{j-1} + (A^{i-1} \cap B^j) \rightarrowtail B^{j-1} + (A^i \cap B^j) \rightarrowtail \cdots \rightarrowtail C,$$

$$0 \rightarrowtail \cdots \rightarrowtail A^{i-1} + (B^{j-1} \cap A^i) \rightarrowtail A^{i-1} + (B^j \cap A^i) \rightarrowtail \cdots \rightarrowtail C,$$

and the following diagram is commutative:

$$\begin{tikzcd}
\bigotimes_{i=1}^{m} f_1(C^i) \arrow[d] & f_1(F) \arrow[l] \arrow[r] & \bigotimes_{j=1}^{n} f_1(D^j) \arrow[d] \\
\bigotimes_{i=1}^{m} \bigotimes_{j=1}^{n} f_1(E^{i,j}) \arrow[rr] & & \bigotimes_{j=1}^{n} \bigotimes_{i=1}^{m} f_1(E^{i,j}).
\end{tikzcd}$$

COROLLARY 1.10. *For any exact square as shown in the first diagram, the second diagram is commutative*:

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail] \\
A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\
A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C'';
\end{tikzcd}$$

$$\begin{tikzcd}
f_1(A) \otimes f_1(C) \arrow[r] & f_1(A') \otimes f_1(A'') \otimes f_1(C') \otimes f_1(C'') \arrow[dd, "1\otimes\psi\otimes 1"] \\
f_1(B) \arrow[u] \arrow[d] & \\
f_1(B') \otimes f_1(B'') \arrow[r] & f_1(A') \otimes f_1(C') \otimes f_1(A'') \otimes f_1(C'').
\end{tikzcd}$$

Proof. Since exact categories are closed under extensions, the two filtrations $A' \rightarrowtail B' \rightarrowtail B$ and $A' \rightarrowtail A \rightarrowtail B$ are compatible. The extremal filtrations are $A' \rightarrowtail B' \rightarrowtail A + B' \rightarrowtail B$ and $A' \rightarrowtail A \rightarrowtail A + B' \rightarrowtail B$, with successive quotients A', C', A'', C'' and A', A'', C', C'', respectively. □

DEFINITION 1.11. A *morphism* of determinants $q\colon f \to g$ is a natural isomorphism $q\colon f_1 \to g_1$ such that, for every short exact sequence $\Sigma = A' \rightarrowtail A \twoheadrightarrow A''$, the following diagram is commutative:

$$\begin{tikzcd}
f_1(A) \arrow[r, "f_2(\Sigma)"] \arrow[d, "q(A)"'] & f_1(A') \otimes f_1(A'') \arrow[d, "q(A')\otimes q(A'')"] \\
g_1(A) \arrow[r, "g_2(\Sigma)"] & g_1(A') \otimes g_1(A'').
\end{tikzcd}$$

DEFINITION 1.12. For any determinants f, g, h, any morphism $\alpha\colon A \to B$ in $\mathcal{E}_{\mathrm{w}}$, and any short exact sequence $\Sigma = A' \rightarrowtail A \twoheadrightarrow A''$, we define:

$$\begin{aligned}
(f \otimes g)_1(A) &= f_1(A) \otimes g_1(A),\\
(f \otimes g)_1(\alpha) &= f_1(\alpha) \otimes g_1(\alpha),\\
(f \otimes g)_2(\Sigma) &= (1 \otimes \psi \otimes 1) \circ (f_1(\Sigma) \otimes g_1(\Sigma)),\\
\phi(f, g, h)(A) &= \phi(f_1(A), g_1(A), h_1(A)),\\
\psi(f, g)(A) &= \psi(f_1(A), g_1(A)).
\end{aligned}$$

PROPOSITION 1.13. *The determinants on a category $\mathcal{E}_{\mathrm{w}}$ with values in a Picard category P, together with morphisms of determinants, form a category that we denote by* $\det(\mathcal{E}_{\mathrm{w}}, P)$. *The tensor product together with ϕ and ψ* (*as defined previously*) *induce on* $\det(\mathcal{E}_{\mathrm{w}}, P)$ *the structure of a Picard category.*

Proof. It follows from the general coherence theorem that (a) $\phi(f, g, h)$ and $\psi(f, g)$ are morphisms of determinants and (b) ϕ and ψ are natural and satisfy both the pentagonal and the hexagonal axiom. □

In the rest of this section, $\mathcal{E}$ and $\mathcal{E}'$ will denote exact categories, w and w$'$ will denote SQ-classes of morphisms in $\mathcal{E}$ and $\mathcal{E}'$ respectively, and P and P' will denote Picard categories.

DEFINITION 1.14. We denote by $\mathrm{Ex}(\mathcal{E}_{\mathrm{w}}, \mathcal{E}'_{\mathrm{w}'})$ the category of covariant exact functors $F : \mathcal{E} \to \mathcal{E}'$ with the property that $F(\alpha) \in \mathrm{w}'$ for all $\alpha \in \mathrm{w}$. Morphisms are natural transformations. We will denote by ew' the class of natural transformations $\eta : F \to G$ with the property that $\eta(A) \in \mathrm{w}'$ for all objects A of $\mathcal{E}$.

PROPOSITION 1.15. *The category* $\mathrm{Ex}(\mathcal{E}_{\mathrm{w}}, \mathcal{E}'_{\mathrm{w}'})$ *is an exact category, and* ew' *is an SQ-class of morphisms.*

Proof. We leave the proof to the reader. □

The next two propositions follow from the general coherence Theorem A.2.

PROPOSITION 1.16. *Composition induces a determinant, the* tautological *determinant*

$$*: \mathrm{Ex}(\mathcal{E}_{\mathrm{w}}, \mathcal{E}'_{\mathrm{w}'})_{\mathrm{ew}'} \to \mathrm{Hom}^{\otimes}(\det(\mathcal{E}'_{\mathrm{w}'}, P), \det(\mathcal{E}_{\mathrm{w}}, P)).$$

PROPOSITION 1.17. *Composition induces an AC tensor functor*

$$*: \mathrm{Hom}^{\otimes}(P', P) \to \mathrm{Hom}^{\otimes}(\det(\mathcal{E}_{\mathrm{w}}, P'), \det(\mathcal{E}_{\mathrm{w}}, P)).$$

COROLLARY 1.18. *Any inverse structure σ on P pulls back via the tautological functor* $*: \mathrm{Hom}^{\otimes}(P, P) \to \mathrm{Hom}^{\otimes}(\det(\mathcal{E}_{\mathrm{w}}, P), \det(\mathcal{E}_{\mathrm{w}}, P))$ *to an inverse structure σ_*. Because there can be no confusion, we will drop the asterisks in the induced inverse structure. We then have*

$$(f^{\sigma})_1(A) = (f^{\sigma_*})_1(A) = (f_1(A))^{\sigma},$$

$$(f^{\sigma})_2(\Sigma) = (f^{\sigma_*})_2(\Sigma) = \sigma_2(f_1(A'), f_1(A'')) \circ (f_2(A))^{\sigma},$$

$$\sigma_3(f)(A) = \sigma_{3*}(f)(A) = \sigma_3(f_1(A)).$$

REMARK 1.19. Let $i : \mathcal{E} \to \mathcal{A}$ denote the Gabriel–Quillen embedding of $\mathcal{E}$ into the abelian category $\mathcal{A}$. The functor i is fully faithful, exact, and reflects exactness (see also [TT, A.7]).

We consider the full subcategory $\mathcal{E}'$ of $\mathcal{A}$ of objects A with the property that there exists an object $A' \in \mathcal{E}$ such that $A \oplus A' \in \mathcal{E}$. The category $\mathcal{E}'$ might be called the *stabilization* of $\mathcal{E}$, and we leave it to the reader to check that $\mathcal{E}'$ is an exact category. Moreover, $\mathcal{E}'$ satisfies [TT, Axiom A.1.5], which states that any morphism t for which there exists a morphism s such that $ts = 1$ is an admissible epimorphism. It follows from [TT, A.7.16b] that every morphism in $\mathcal{E}'$ that is also an epimorphism in $\mathcal{A}$ is admissible.

If w is an SQ-class of morphisms, then we say that a morphism $\alpha : A \to B$ belongs to the class w' if there exists an object E in $\mathcal{E}$ such that (a) both $A \oplus E$ and $A' \oplus E$ belong to $\mathcal{E}$ and (b) the morphism $\alpha \oplus 1_E$ belongs to w. We leave it to the reader to check that if α belongs to w' then $\alpha \oplus 1_E$ belongs to w for all such E and that the class w' is an SQ-class of morphisms. Moreover, the restriction functor $\det(\mathcal{E}'_{\mathrm{w}'}, P) \to \det(\mathcal{E}_{\mathrm{w}}, P)$ is an equivalence of categories. For this reason we will assume from now on that every morphism in $\mathcal{E}$ that is an epimorphism in $\mathcal{A}$ is admissible.

2. The Main Theorem

In this section, $\mathcal{E}$ is an exact category and $\mathrm{C}(\mathcal{E})$ denotes the exact category of bounded complexes of objects in $\mathcal{E}$. We consider $\mathcal{E}$ as the full subcategory of $\mathrm{C}(\mathcal{E})$ consisting of complexes supported only in degree 0; P is a Picard category with a fixed inverse structure σ. All determinants considered will have values in P, so for short we will write $\det(\mathcal{E}_{\mathrm{w}})$ instead of $\det(\mathcal{E}_{\mathrm{w}}, P)$.

DEFINITION 2.1. A *quasi-isomorphism* in $\mathrm{C}(\mathcal{E})$ is a morphism whose image in $\mathrm{C}(\mathcal{A})$ induces an isomorphism in cohomology. The morphism class of quasi-isomorphisms (resp., isomorphisms) will be denoted by qis (resp., iso).

REMARK 2.2. By the long exact sequence in cohomology associated to a short exact sequence, it follows that qis is an SQ-class of morphisms.

We now state the main theorem. It is a consequence of Lemma 2.22 and Proposition 2.25.

THEOREM 2.3 (Main Theorem). *The restriction functor* $\det(\mathrm{C}(\mathcal{E})_{\mathrm{qis}}) \to \det(\mathcal{E}_{\mathrm{iso}})$ *is an equivalence and an AC tensor functor.*

DEFINITION 2.4. A complex A is *acyclic* if $i(A)$ has vanishing cohomology in $\mathcal{A}$.

DEFINITION 2.5. For any complex A, we denote by $A[1] = TA$ the complex defined by $TA^i = A^{i+1}$ and $d_{TA} = -d_A$. Note that T is an exact functor.

DEFINITION 2.6. Let $\alpha\colon A \to B$ be a morphism of complexes. The *mapping cone* of α is the complex $\mathrm{C}(\alpha)$, given by

$$\mathrm{C}(\alpha)^i = B^i \oplus A^{i+1},$$

$$d_{\mathrm{C}}(\alpha)^i = \begin{pmatrix} d^i & \alpha^{i+1} \\ 0 & -d^{i+1} \end{pmatrix}.$$

PROPOSITION 2.7. *We have the short exact sequences*

$$B \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \mathrm{C}(\alpha) \xrightarrow{(0\ 1)} A[1] \tag{1}$$

and

$$A \xrightarrow{\begin{pmatrix}1\\0\\\alpha\end{pmatrix}} \mathrm{C}(-1_A \oplus B) \xrightarrow{\left(\begin{smallmatrix}-\alpha & 0 & 1\\ 0 & 1 & 0\end{smallmatrix}\right)} \mathrm{C}(\alpha), \tag{2}$$

and a commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "\begin{pmatrix}1\\0\\\alpha\end{pmatrix}"] \arrow[dr, "\alpha"'] & \mathrm{C}(-1_A) \oplus B \arrow[d, two heads, "(0\ 0\ 1)"] \\ & B. \end{tikzcd} \tag{3}$$

COROLLARY 2.8. *A morphism α is a quasi-isomorphism if and only if its mapping cone $\mathrm{C}(\alpha)$ is acyclic, and in this case both the horizontal and the vertical morphisms in diagram* (3) *are quasi-isomorphisms.*

DEFINITION 2.9. For any complex A, we denote by $A \otimes I$ the mapping cone of the antidiagonal

$$-\Delta = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) \colon A \to A \oplus A,$$

and by ∂_0 and ∂_1 we denote the maps

$$\partial_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \colon A \to A \otimes I \quad \text{and} \quad \partial_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \colon A \to A \otimes I.$$

DEFINITION 2.10. Two morphisms $\alpha_0, \alpha_1 \colon A \to B$ will be called *homotopic*, and a map $h \colon TA \to B$ will be called a *homotopy* from α_0 to α_1, if $\alpha_0 - \alpha_1 = dh + hd$.

PROPOSITION 2.11.

(a) *The map* $\mathrm{sum} = (1\,1\,0) \colon A \otimes I \to A$ *is a quasi-isomorphism, and it is an equalizer of the homotopic quasi-isomorphisms ∂_0 and ∂_1.*
(b) *If $h \colon TA \to B$ is a homotopy from α_0 to α_1 and if $\tilde h = (\alpha_0, \alpha_1, h)$ then, for all $i \in \{0, 1\}$, the following diagram is a commutative diagram of morphisms of complexes*:

$$\begin{tikzcd}
 & A \otimes I \arrow[dd, two heads, "\tilde h"] \\
A \arrow[ur, tail, "\partial_i"] \arrow[dr, tail, "\alpha_i"'] & \\
 & B.
\end{tikzcd}$$

COROLLARY 2.12. *Given a functor f from $\mathrm{C}(\mathcal{E})_{\mathrm{qis}}$ to a category Q all of whose morphisms are invertible, it follows that f factors through $\mathrm{D}(\mathcal{E})_{\mathrm{qis}}$. This means that $f(\alpha_0) = f(\alpha_1)$ for any two homotopic quasi-isomorphisms α_0 and α_1.*

Proof. Since $\mathrm{sum}\,\partial_0 = \mathrm{sum}\,\partial_1$, it follows by cancellation that $f(\partial_0) = f(\partial_1)$. Hence $f(\alpha_0) = f(\tilde h \partial_0) = f(\tilde h) f(\partial_0) = f(\tilde h) f(\partial_1) = f(\tilde h \partial_1) = f(\alpha_1)$. ☐

DEFINITION 2.13. We will say that a complex A is *admissible* if the Z^i and B^i are isomorphic to objects of $\mathcal{E}$. By Remark 1.19, every acyclic complex is admissible.

DEFINITION 2.14. For any admissible complex A, the complex $Z = Z(A)$ is the complex given by $Z^i = \ker(d_A^i)$ and $d_Z^i = 0$ for all i. We define similarly the complex $B = B(A)$, and we have the short exact sequence $Z \rightarrowtail A \overset{d}{\twoheadrightarrow} B[1]$.

DEFINITION 2.15. We will say that a morphism in $\mathrm{C}(\mathcal{E})$ is *admissible* if its mapping cone is admissible. By Remark 1.19, every quasi-isomorphism is admissible.

DEFINITION 2.16. A complex A is called *split exact* if there exists an isomorphism $A \to \mathrm{C}(1_Z)$ that makes the following diagram commutative:

$$\begin{tikzcd} Z \arrow[r, tail] \arrow[d, equal] & A \arrow[r, two heads] \arrow[d] & Z[1] \arrow[d, equal] \\ Z \arrow[r, tail] & \mathrm{C}(1_Z) \arrow[r, two heads] & Z[1]. \end{tikzcd}$$

DEFINITION 2.17 (The Brutal Truncation). For every integer k and every complex A, we denote by $\sigma^{\geq k}A$ the kth *upper brutally truncated* subcomplex of A. It is the complex that remains when the objects in degrees $j < k$ are killed. Similarly, we denote by $\sigma^{<k}A$ the kth *lower brutally truncated* quotient complex of A. It is the complex that remains when the objects in degrees $j \geq k$ are killed. We denote by $\Sigma_k(A)$ the kth *brutal truncation sequence* of A, the short exact sequence $\sigma^{\geq k}A \rightarrowtail A \twoheadrightarrow \sigma^{<k}A$.

DEFINITION 2.18 (The Good Truncation). For every integer k and every admissible complex A, we denote by $\gamma^{<k}A$ the kth *lower well truncated* subcomplex of A. It is the complex that remains when the objects in degrees $j \geq k$ are killed and A^{k-1} is replaced by $\ker(d^{k-1})$. Similarly, we denote by $\gamma^{\geq k}A$ the kth *upper well truncated* quotient complex of A. It is the complex obtained by augmenting $\sigma^{\geq k}A$ with the map $\mathrm{im}(d^{k-1}) \to A^k$. We denote by $\Gamma_k(A)$the kth *good truncation sequence* of A, the short exact sequence $\gamma^{<k}A \rightarrowtail A \twoheadrightarrow \gamma^{\geq k}A$.

LEMMA 2.19. *The brutal truncation is a functor* $\Sigma_k \colon \mathrm{C}(\mathcal{E}) \to \{\mathrm{C}(\mathcal{E})\}$, *and it maps isomorphisms to isomorphisms. The good truncation is a functor* $\Gamma_k \colon \mathrm{C}(\mathcal{E})^{\mathrm{adm}} \to \{\mathrm{C}(\mathcal{E})^{\mathrm{adm}}\}$, *and it maps quasi-isomorphisms to quasi-isomorphisms.* (Note: "adm" denotes admissible.)

DEFINITION 2.20. An *S-determinant* on $\mathcal{E}_{\mathrm{w}}$ is a sequence $(f_n, \mu_n)_{n\in\mathbb{Z}}$, where each f_n is a determinant on $\mathcal{E}_{\mathrm{w}}$ and each μ_n is an isomorphism of determinants $f_n \otimes f_{n-1} \to 1$.

DEFINITION 2.21. A morphism of S-determinants $q \colon (f_n, \mu_n) \to (f'_n, \mu'_n)$ is a sequence of morphisms of determinants $q_n \colon f_n \to f'_n$ such that $\mu_n = \mu'_n \circ (q_n \otimes q_{n-1})$ for all $n \in \mathbb{Z}$. We denote the category of S-determinants by $\mathrm{Sdet}(\mathcal{E}_{\mathrm{w}})$.

LEMMA 2.22. *The forgetful functor* $\mathrm{Sdet}(\mathcal{E}_{\mathrm{w}}) \to \det(\mathcal{E}_{\mathrm{w}})$ *is an equivalence.*

Proof. For any determinant f on $\mathcal{E}_{\mathrm{w}}$, we define $S^\sigma(f) = (f_n, \mu_n)$ by

$$f_n = \begin{cases} f^\sigma & \text{for } n \text{ odd,} \\ f & \text{for } n \text{ even;} \end{cases}$$

$$\mu_n = \begin{cases} \sigma \circ \psi & \text{for } n \text{ odd,} \\ \sigma & \text{for } n \text{ even.} \end{cases}$$

It follows from Proposition 1.17 that $S^\sigma(f)$ is an S-determinant, and the categories are equivalent by Remark A.17. □

DEFINITION 2.23. For any determinant f on $C\mathcal{E}_{\mathrm{qis}}$, we define the S-determinant $T^{\bullet}(f) = (f_n, \mu_n)$ on $\mathcal{E}_{\mathrm{iso}}$ by $f_n(A) = f(A[-n])$, $f_n(\Sigma) = f(\Sigma[-n])$, and $\mu_n(A)$ via

$$f(A[-n]) \otimes f(A[-n+1]) \xrightarrow{f_2^{-1}} f(C(1_{A[-n]})) \xrightarrow{f(0)} f(0) \longrightarrow 1.$$

Note that f_n corresponds to restricting f to complexes supported only in degree n.

DEFINITION 2.24. For any S-determinant (f_n, μ_n) on $\mathcal{E}_{\mathrm{iso}}$, we define the two maps $g(f_n, \mu_n) = (g_1, g_2)$ on $C\mathcal{E}_{\mathrm{qis}}$ as follows.

(a) For a complex A, we define $g_1(A) = \bigotimes (f_n)_1(A^n)$ and $g_1(0) = 1$.

(b) For a short exact sequence Σ, we define $g_2(\Sigma)$ via

$$\bigotimes f_n(A^n) \xrightarrow{\bigotimes (f_n)_2(\Sigma^n)} \bigotimes (f_n(A'^n) \otimes f_n(A''^n)) \longrightarrow \bigotimes f_n(A'^n) \otimes \bigotimes f_n(A''^n).$$

(c) For an acyclic complex Q, we define $g_1(0)\colon g_1(Q) \to 1$ via g_2 of the short exact sequence $Z(Q) \rightarrowtail Q \overset{d}{\twoheadrightarrow} TZ(Q)$ and the isomorphism

$$\bigotimes f_n(Z^n) \otimes \bigotimes f_n(Z^{n+1}) \longrightarrow \bigotimes (f_n(Z^n) \otimes f_{n-1}(Z^n)) \xrightarrow{\bigotimes \mu_n(Z^n)} 1.$$

(d) For a quasi-isomorphism that is an admissible epimorphism $Q \rightarrowtail A \overset{\alpha}{\twoheadrightarrow} B$, we define $g_1(\alpha)$ as the composition

$$g_1(A) \xrightarrow{g_2} g_1(Q) \otimes g_1(B) \xrightarrow{g_1(0) \otimes 1} 1 \otimes g_1(B) \longrightarrow g_1(B).$$

(d*) For a quasi-isomorphism that is an admissible monomorphism $A \overset{\alpha}{\rightarrowtail} B \twoheadrightarrow Q$, we define g_1 as the inverse of the composition

$$g_1(B) \xrightarrow{g_2} g_1(A) \otimes g_1(Q) \xrightarrow{1 \otimes g_1(0)} g_1(A) \otimes 1 \longrightarrow g_1(A).$$

(e) For an arbitrary quasi-isomorphism $A \xrightarrow{\alpha} B$, we use the factorization of Proposition 2.7, $A \overset{\alpha_2}{\rightarrowtail} C(1_A) \oplus B \overset{\alpha_1}{\twoheadrightarrow} B$, and define

$$g_1(\alpha) = g_1(\alpha_1) g_1(\alpha_2).$$

(f) For a morphism $q_n\colon (f_n, \mu_n) \to (f'_n, \mu'_n)$, we define $g(q)\colon g_1 \to g'_1$ by

$$g(q)(A) = \bigotimes q_n(A^n).$$

PROPOSITION 2.25. *The maps $T^{\bullet}$ and g are functors, and they establish an equivalence of categories* $\det(C\mathcal{E}_{\mathrm{qis}})$ *and* $\mathrm{Sdet}(\mathcal{E}_{\mathrm{iso}})$.

We will prove the proposition through a series of lemmas.

LEMMA 2.26. *On the full exact subcategory of acyclic complexes, we have that g is well-defined, is a determinant, and factors through the* rigid *subcategory* $\mathrm{Unit}(P)$ (see [S, 2.2.5.1]).

Proof. We apply Proposition 1.10 to the exact square

$$\begin{tikzcd}
Z' \arrow[r, tail] \arrow[d, tail] & Z \arrow[r, two heads] \arrow[d, tail] & Z'' \arrow[d, tail] \\
Q' \arrow[r, tail] \arrow[d, two heads] & Q \arrow[r, two heads] \arrow[d, two heads] & Q'' \arrow[d, two heads] \\
TZ' \arrow[r, tail] & TZ \arrow[r, two heads] & TZ''.
\end{tikzcd}$$

The lemma then follows because the μ_n are morphisms of determinants. □

LEMMA 2.27. *For a composition of admissible epimorphisms $A \overset{\alpha}{\twoheadrightarrow} B \overset{\beta}{\twoheadrightarrow} C$, we have $g_1(\beta\alpha) = g_1(\beta)g_1(\alpha)$.*

Proof. We apply Proposition 1.10, Lemma 2.26, and Remark A.8 to the exact square

$$\begin{tikzcd}
Q' \arrow[r, equal] \arrow[d, tail] & Q' \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
Q \arrow[r, tail] \arrow[d, two heads] & A \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, equal] \\
Q'' \arrow[r, tail] & B \arrow[r, two heads] & C.
\end{tikzcd}$$

□

LEMMA 2.28. *For a composition of admissible monomorphisms $A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\rightarrowtail} C$, we have $g_1(\beta\alpha) = g_1(\beta)g_1(\alpha)$.*

Proof. The dual construction of the previous proof. □

LEMMA 2.29. *The two possible definitions for g_1 on isomorphisms agree and are given by $g_1(\alpha) = \bigotimes(f_n)_1(\alpha^n)$.*

Proof. This is Proposition 1.5(a) applied to the f_n. □

LEMMA 2.30. *For two factorizations $A \overset{\alpha}{\rightarrowtail} C \overset{\beta}{\twoheadrightarrow} B$ and $A \overset{\alpha''}{\rightarrowtail} C'' \overset{\beta''}{\twoheadrightarrow} B$ with $\beta\alpha = \beta''\alpha''$, we have $g_1(\beta)g_1(\alpha) = g_1(\beta'')g_1(\alpha'')$.*

Proof. Since the two factorizations can be covered by the fiber product of C and C'' over B and since fiber products with at least one epimorphism exists in exact categories, we can reduce the lemma to the case of

$$\begin{tikzcd}
& C \arrow[dr, two heads, "\beta"] \arrow[dd, two heads, "\gamma"] & \\
A \arrow[ur, tail, "\alpha"] \arrow[dr, tail, "\alpha''"'] & & B \\
& C''. \arrow[ur, two heads, "\beta''"'] &
\end{tikzcd}$$

After applying g, the right triangle commutes by Lemma 2.27. To see that the left triangle is commutative, we apply Proposition 1.10, Lemma 2.26, and Remark A.8 to the exact square

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] & Q' \arrow[r, equal] \arrow[d, tail] & Q' \arrow[d, tail] \\
A \arrow[r, tail] \arrow[d, equal] & C \arrow[r, two heads] \arrow[d, two heads] & Q \arrow[d, two heads] \\
A \arrow[r, tail] & C'' \arrow[r, two heads] & Q''.
\end{tikzcd}$$
□

LEMMA 2.31. *For a composition $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$, $g_1(\beta\alpha) = g_1(\beta)g_1(\alpha)$.*

Proof. The lemma follows by Lemmas 2.29 and 2.27 applied to the commutative diagram

$$\begin{tikzcd}
A \arrow[r, tail, "\begin{pmatrix}1\\0\\\alpha\end{pmatrix}"] \arrow[dr, "\alpha"'] & \mathrm{C}(-1_A) \oplus B \arrow[r, tail, "\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&\beta\end{pmatrix}"] \arrow[d, two heads, "(0\ 0\ 1)"] & \mathrm{C}(-1_A) \oplus \mathrm{C}(-1_B) \oplus C \arrow[d, two heads, "\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}"] \\
& B \arrow[r, tail, "\begin{pmatrix}1\\0\\\beta\end{pmatrix}"] \arrow[dr, "\beta"'] & \mathrm{C}(-1_B) \oplus C \arrow[d, two heads, "(0\ 0\ 1)"] \\
& & C
\end{tikzcd}$$

and Corollary 1.10 applied to the exact square

$$\begin{tikzcd}
\mathrm{C}(-1_A) \arrow[r, equal] \arrow[d, tail] & \mathrm{C}(-1_A) \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
\mathrm{C}(-1_A) \oplus B \arrow[r, tail] \arrow[d, two heads] & \mathrm{C}(-1_A) \oplus \mathrm{C}(-1_B) \oplus C \arrow[r, two heads] \arrow[d, two heads] & \mathrm{C}(\beta) \arrow[d, equal] \\
B \arrow[r, tail] & \mathrm{C}(-1_B) \oplus C \arrow[r, two heads] & \mathrm{C}(\beta).
\end{tikzcd}$$
□

LEMMA 2.32. *For any morphism $q_n\colon (f_n, \mu_n) \to (f'_n, \mu'_n)$, $g(q)$ is a morphism of determinants $(g_1, g_2) \to (g'_1, g'_2)$. In fact, g is an AC tensor functor.*

Proof. For any short exact sequence $\Sigma = A' \rightarrowtail A \twoheadrightarrow A''$ we have, by general coherence and since each q_n is natural, a commutative diagram

$$\begin{tikzcd} g_1(A) \arrow[r, "g_2(\Sigma)"] \arrow[d, "g(q)(A)"'] & g_1(A') \otimes g_1(A'') \arrow[d, "g(q)(A')\otimes g(q)(A'')"] \\ g'_1(A) \arrow[r, "g'_2(\Sigma)"] & g'_1(A') \otimes g'_1(A'') . \end{tikzcd} \tag{$*$}$$

Consider a quasi-isomorphism $\alpha\colon A \to B$. We need to prove that the following diagram is commutative:

$$\begin{tikzcd} g_1(A) \arrow[r, "g_1(\alpha)"] \arrow[d, "g(q)(A)"'] & g_1(B) \arrow[d, "g(q)(B)"] \\ g'_1(A) \arrow[r, "g'_1(\alpha)"] & g'_1(B) . \end{tikzcd} \tag{$**$}$$

When B is acyclic and $A = 0$, commutativity follows because each q_n is a morphism of determinants—and dually for A acyclic and $B = 0$. This, together with Lemma 2.26, shows commutativity for all quasi-isomorphisms of acyclic complexes. The diagram $(*)$ then shows that $(**)$ is commutative for quasi-isomorphisms that are admissible epimorphsms or monomorphisms; by Lemma 2.31, $g(q)$ is a morphism of determinants. That g is an AC tensor functor follows from general coherence. $\square$

LEMMA 2.33. *The composition $T^{\bullet} \circ g$ is the identity, and g is faithful.*

Proof. Let $(f'_n, \mu'_n) = (T^{\bullet} \circ g)(f_n, \mu_n)$. For any object A of $\mathcal{E}$,

$$(f_n)'_1(A) = T^{*-n}(g_1(A)) = g_1(A[-n]) = (f_n)_1(A).$$

Similarly, we see that $(f_n)'_2 = (f_n)_2$. Hence by Proposition 1.5(a), $(f_n)'_1 = (f_n)_1$ for all isomorphisms. Finally, $\mu'_n = \mu_n$ because both $T^{\bullet}$ and g are AC tensor functors.

Let g and g' be determinants on $\mathrm{C}(\mathcal{E})_{\mathrm{qis}}$, and let q and q' be two morphisms $g \to g'$ such that $T^{\bullet}(q) = T^{\bullet}(q')$. This means that q and q' agree on all complexes of length 1. By the brutal truncation and the condition of Definition 1.11 for morphisms of determinants, it follows by induction with respect to length of complexes that $q = q'$ on all complexes. $\square$

COROLLARY 2.34. *Both $T^{\bullet}$ and g are fully faithful.*

LEMMA 2.35. *There is an isomorphism of functors* $\mathrm{id} \to g \circ T^{\bullet}$.

Proof. Let h be a determinant on $\mathrm{C}(\mathcal{E})_{\mathrm{qis}}$, let $T^{\bullet}(h) = (f_n, \mu_n)$, and let $(g_1, g_2) = g(f_n, \mu_n)$. Again we have $h_1(A) = g_1(A)$ for all complexes of length 1, and $h_1(\alpha) = g_1(\alpha)$ for all isomorphisms of such complexes. We use the brutal filtration $\cdots \rightarrowtail \sigma^{\geq k}A \rightarrowtail \sigma^{\geq k-1}A \rightarrowtail \cdots \rightarrowtail A$ to construct $q(A) = q(h)(A)\colon h_1(A) \to g_1(A)$. It follows from Proposition 1.7 and general coherence that we have commutative diagrams

$$\begin{array}{ccc} h_1(A) & \xrightarrow{h_1(\alpha)} & h_1(B) \\ {\scriptstyle q(A)}\downarrow & & \downarrow{\scriptstyle q(B)} \\ g_1(A) & \xrightarrow{g_1(\alpha)} & g_1(B) \end{array} \quad \text{and} \quad \begin{array}{ccc} h_1(A) & \xrightarrow{h_2(\Sigma)} & h_1(A') \otimes h_1(A') \\ {\scriptstyle q(A)}\downarrow & & \downarrow{\scriptstyle q(A')\otimes q(A'')} \\ g_1(A) & \xrightarrow{g_2(\Sigma)} & g_1(A') \otimes g_1(A') \end{array}$$

for every isomorphism $\alpha\colon A \to B$ and every short exact sequence

$$\Sigma = A' \rightarrowtail A \twoheadrightarrow A''.$$

By definition of the μ_n we have a commutative diagram

$$\begin{array}{ccc} h_1(Q) & \xrightarrow{q(Q)} & g_1(Q) \\ {\scriptstyle h_1(0)}\downarrow & & \downarrow{\scriptstyle g_1(0)} \\ 1 & \longrightarrow & 1 \end{array}$$

for every complex Q isomorphic to a complex of the form $\mathrm{C}(1_A)$, where A is a complex of length 1. This in particular includes all acyclic complexes of length 2. Using good truncations, it follows by induction that the diagram just displayed commutes for all acyclic complexes Q, and this proves that $q = q(h)$ is a morphism of determinants. That q is natural follows from Corollary 2.34. This proves the last lemma. Hence Proposition 2.25 and thus the main theorem are proved. □

DEFINITION 2.36. In the rest of the paper we will denote the composition of the functors S^σ and g by $\mathrm{C}^\sigma = g \circ S^\sigma : \det(\mathcal{E}_{\mathrm{iso}}) \to \det(\mathrm{C}\mathcal{E}_{\mathrm{qis}})$.

3. Determinants, Homology, and Spectral Sequences

In this section, f is a determinant on $\mathrm{C}(\mathcal{E})_{\mathrm{qis}}$ with values in a Picard category P. We denote the restriction of f to $\mathcal{E}_{\mathrm{iso}}$ by f as well.

DEFINITION 3.1. For any admissible complex A, we denote by $C(i_A)$ the mapping cone of the monomorphism $i_A\colon B(A) \to Z(A)$. The morphism $c(A)$ is the unique isomorphism that makes the following diagram commutative:

$$\begin{array}{ccc} f(A) & \longrightarrow & f(Z(A)) \otimes f(TB(A)) \\ {\scriptstyle c(A)}\downarrow & & \| \\ f(C(i_A)) & \longrightarrow & f(Z(A)) \otimes f(TB(A)). \end{array}$$

For any quasi-isomorphism $\alpha\colon A \to B$ of admissible complexes, we denote by $c(\alpha)$ the induced morphism $C(i_A) \to C(i_B)$, and we define the assignment $g = (g_1, g_2)$ on the subcategory of admissible complexes as follows. For any short exact sequence $\Sigma = A' \rightarrowtail A \twoheadrightarrow A''$ of admissible complexes,

$$
\begin{aligned}
g_1(A) &= f_1(C(i_A)),\\
g_1(\alpha) &= f_1(c(\alpha)),\\
g_2(\Sigma) &= f_2(c(A') \otimes c(A'')) \circ f_2(\Sigma) \circ c(A)^{-1}.
\end{aligned}
$$

If $H(A)$ is in $\mathrm{C}(\mathcal{E})$, we have a quasi-isomorphism $C(i_A) \to H(A)$ and we define $h(A)$ to be the composition $f(A) \to g(A) \to f(H(A))$.

PROPOSITION 3.2. *Except for the possibility that the admissible complex is not an exact category, the pair (g_1, g_2) is a determinant and more importantly, c is a morphism of determinants.*

Proof. By definition, c satisfies the condition of Definition 1.11, so we need only prove that c is natural. We prove this by induction with respect to length. If a complex A is of length 1 or of length 2 and if the differential d is a monomorphism, then A and $C(i_A)$ are naturally isomorphic and so there is nothing to prove. Let A be an admissible complex. Then, by the good filtration, we have a short exact sequence. Let $\Sigma = A' \rightarrowtail A \twoheadrightarrow A''$ of admissible complexes such that either (a) both A' and A'' are strictly shorter than A or (b) A is of length 2, A' is of length 2 with the differential a monomorphism, and A'' is of length 1. For such a short exact sequence, the sequences $Z(A') \rightarrowtail Z(A) \twoheadrightarrow Z(A'')$ and $B(A') \rightarrowtail B(A) \twoheadrightarrow B(A'')$ are also short exact. Hence so is the sequence $C(\Sigma) = C(i_{A'}) \rightarrowtail C(i_A) \twoheadrightarrow C(i_{A''})$, and it follows from Corollary 1.10 that $g_2(\Sigma) = f_2(C(\Sigma))$. Let $\alpha\colon A \to B$ be a quasi-isomorphism of admissible complexes and consider the following diagram, where $A' = \gamma^{<k}A$, $B' = \gamma^{<k}B$, $A'' = \gamma^{\geq k}A$, and $B'' = \gamma^{\geq k}B$.

$$
\begin{tikzcd}
 & f(A') \otimes f(A'') \arrow[rr] \arrow[dd, "f(\alpha') \otimes f(\alpha'')"] & & g(A') \otimes g(A'') \arrow[dd, "g(\alpha') \otimes g(\alpha'')"] \\
f(A) \arrow[ur] \arrow[rr] \arrow[dd, "f(\alpha)"] & & g(A) \arrow[ur] \arrow[dd, "g(\alpha)"] & \\
 & f(B') \otimes f(B'') \arrow[rr] & & g(B') \otimes g(B'') \\
f(B) \arrow[ur] \arrow[rr] & & g(B) \arrow[ur] &
\end{tikzcd}
$$

We have just observed that the right square is commutative because $g_2(\Sigma) = f_2(C(\Sigma))$ in this case. The left square commutes by naturality of f_2, the back square commutes by induction, and the top and bottom squares commute by definition. Hence the front square is commutative. $\square$

PROPOSITION 3.3. *Let $\alpha, \alpha'\colon A \to B$ be two quasi-isomorphisms of admissible complexes. If the induced morphisms in cohomology $H(\alpha) = H(\alpha')$, then $f(\alpha) = f(\alpha')$. Moreover, if $H(A)$ and $H(B)$ are objects of $\mathrm{C}(\mathcal{E})$, then the following diagram is commutative*:

$$\begin{tikzcd} f(A) \arrow[r, "f(\alpha)"] \arrow[d, "h(A)"'] & f(B) \arrow[d, "h(B)"] \\ f(H(A)) \arrow[r, "f(H(\alpha))"] & f(H(B)). \end{tikzcd}$$

Proof. By the previous proposition we may assume that A and B are of the form $C(i_A)$ and $C(i_B)$. In this case, if $H(\alpha) = H(\alpha')$ then α and α' are homotopic, so $f(\alpha) = f(\alpha')$ by Proposition 2.12. If $H(A)$ and $H(B)$ are objects of $\mathrm{C}(\mathcal{E})$, the result follows because we have a commutative diagram of quasi-isomorphisms

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d] \\ H(A) \arrow[r, "H(\alpha)"] & H(B). \end{tikzcd}$$

□

In the following we consider the category $\mathrm{FC}(\mathcal{E})$ of finitely decreasingly filtered complexes and morphisms respecting the filtrations. We denote the pth filtered subcomplex of a complex A by $F^p(A)$. The following is a convenient way of viewing spectral sequences from the standpoint of determinants.

DEFINITION 3.4. For any filtered complex A, the rth *derived filtration* DF_r is

$$DF_r^n(A^m) = \mathrm{Ker}(F^{n+mr}(A^m) \to A^{m+1}/F^{n+(m+1)r}(A^{m+1})),$$

and its successive quotients are

$$DF_r^{n+1}(A) \rightarrowtail DF_r^n(A) \twoheadrightarrow DG_r^n(A).$$

PROPOSITION 3.5. *In the abelian category $\mathcal{A}$, we have a canonical quasi-isomorphism*

$$DG_r(A) = \bigoplus DG_r^n(A) \to E_r(A).$$

DEFINITION 3.6. For any filtered complex A, the rth *spectral filtration* SF_r is

$$SF_r^n(A^m) = \begin{cases} DF_r^{((n-m)/2)}(A^m) & \text{for } n-m \text{ even}, \\ DF_{r-1}^{((n-m+1)/2)}(A^m) & \text{for } m-n \text{ odd}, \end{cases}$$

and its successive quotients are

$$SF_r^{n+1}(A) \rightarrowtail SF_r^n(A) \twoheadrightarrow SG_r^n(A).$$

PROPOSITION 3.7. *The induced differentials $d_r^{n,m} : SG_r^n(A^m) \to SG_r^n(A^{m+1})$ satisfy $d_r^{n,m} = 0$ when $n-m$ is even, and $d_r^{n,m}$ is a monomorphism when $n-m$ is odd. In $\mathcal{A}$ we have*

$$E_r^{p,q} = H^{n+2m}(SG_r^n(A)),$$

where the integers p, q, m, and n are related by

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} r & 2r-1 \\ 1-r & 3-2r \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}.$$

By the property of the differentials, it follows that if the rth *spectral filtration is admissible then so are the complexes* $SG_r^n(A)$.

PROPOSITION 3.8. *Let* $\alpha, \alpha' \colon A \to B$ *be two morphisms of filtered complexes such that the induced morphisms* $E_r(\alpha)$ *and* $E_r(\alpha')$ *are quasi-isomorphisms. Then* $f(\alpha) = f(\alpha')$ *if either*:

(a) *the* rth *derived filtration is admissible,* $E_r(A)$ *and* $E_r(B)$ *are objects of* $\mathrm{C}(\mathcal{E})$, *and* $f(E_r(\alpha)) = f(E_r(\alpha'))$; *or*
(b) *the* $(r+1)$th *spectral filtration is admissible and the induced morphisms in cohomology* $E_{r+1}(\alpha) = E_{r+1}(\alpha')$.

If the rth *derived filtration is admissible and* $E_r(A)$ *and* $E_r(B)$ *are objects of* $\mathrm{C}(\mathcal{E})$, *let* $e_r(A)$ *denote the composition* $f(A) \to \bigotimes_m f(DG_r^m(A)) \to f(E_r(A))$. *Then, if* $E_r(\alpha)$ *is a quasi-isomorphism, the following diagram is commutative*:

$$\begin{tikzcd} f(A) \arrow[r, "f(\alpha)"] \arrow[d, "e_r(A)"'] & f(B) \arrow[d, "e_r(B)"] \\ f(E_r(A)) \arrow[r, "f(E_r(\alpha))"] & f(E_r(B)). \end{tikzcd}$$

Proof. This is just Propositions 1.7, 3.3, 3.5, and 3.7. □

4. Multifunctors and Multideterminants

Let I be a finite set, and let $\{\mathcal{E}_i\}_{i\in I}$ and $\mathcal{F}$ be categories. If all the $\mathcal{E}_i$ are equal, we consider an automorphism σ of I to be also an automorphism of $\prod_{i\in I} \mathcal{E}_i$ via $\sigma(A)_i = A_{\sigma^{-1}(i)}$.

DEFINITION 4.1. An *order invariant* functor S from $\prod_{i\in I} \mathcal{E}_i$ to $\mathcal{F}$ is a functor $S\colon O(I) \to \mathrm{Funct}\big(\prod_{i\in I} \mathcal{E}_i, \mathcal{F}\big)$, where $O(I)$ is the category with the total orderings of I as objects and with one and only one morphism between any two objects.

Note that any functor on $\prod_{i\in I} \mathcal{E}_i$ is order invariant by simply letting the functor on $O(I)$ be constant. If $\rho\colon S \to T$ is a morphism of order invariant functors, we have for any ordering $\prec$ a morphism $\rho(\prec)\colon S(\prec) \to T(\prec)$, and this induces an isomorphism $\mathrm{Mor}(S,T) \approx \mathrm{Mor}(S(\prec), T(\prec))$. By abuse of notation we will write $S\colon \prod_{i\in I} \mathcal{E}_i \to \mathcal{F}$ instead of $S\colon O(I) \to \mathrm{Funct}\big(\prod_{i\in I} \mathcal{E}_i, \mathcal{F}\big)$ when S is order invariant.

DEFINITION 4.2. Suppose all the $\mathcal{E}_i$ are equal. A *symmetric* functor S from $\prod_{i\in I} \mathcal{E}_i$ to $\mathcal{F}$ consists of a functor $S\colon \prod_{i\in I} \mathcal{E}_i \to \mathcal{F}$ together with natural isomorphisms $\psi_S(\sigma)\colon S \to S\circ\sigma$, for each automorphism σ of I, satisfying $\psi_S(\sigma\tau)(A) = \psi_S(\sigma)(\tau(A)) \circ \psi_S(\tau)(A)$ for any pair of automorphisms σ and τ.

An order invariant functor S is symmetric if each $S(\prec)$ is symmetric, and the following diagram is commutative for every σ and any pair of orderings $\prec_1$ and $\prec_2$:

$$\begin{tikzcd} S(\prec_1) \arrow[r, "S(\prec_2,\prec_1)"] \arrow[d, "\psi_{S(\prec_1)}(\sigma)"'] & S(\prec_2) \arrow[d, "\psi_{S(\prec_2)}(\sigma)"] \\ S(\prec_1)\circ\sigma \arrow[r, "S(\prec_2,\prec_1)\circ\sigma"'] & S(\prec_2)\circ\sigma. \end{tikzcd}$$

PROPOSITION 4.3. *If $\{\mathcal{E}_i\}_{i\in I}$ and $\mathcal{F}$ are additive categories, then any order invariant additive multifunctor $S\colon \prod_{i\in I}\mathcal{E}_i \to \mathcal{F}$ has an extension to an order invariant additive multifunctor* $\mathrm{C}(S)$ *on the category of bounded* (*or bounded below or above*) *complexes* $\mathrm{C}(S)\colon \prod_{i\in I}\mathrm{C}(\mathcal{E}_i)\to \mathrm{C}(\mathcal{F})$. *In fact,* C *is a functor and, for every $i\in I$, we have a natural isomorphism $\rho_i\colon \mathrm{C}\circ T_i \to T\circ \mathrm{C}$. Moreover,* $\mathrm{C}(S)$ *maps quasi-isomorphisms to quasi-isomorphisms, and if S is symmetric then so is* $\mathrm{C}(S)$.

Proof. We use the sign conventions of SGA 4 (XVII, Sec. 1; there are corrections in SGA $4\frac{1}{4}$, but we don't need them here). We denote by $\varepsilon_i\in\mathbb{Z}^I$ the function that takes the value 0 except at i, where it takes the value 1. If $A\in\mathrm{Ob}\big(\prod_{i\in I}\mathrm{C}(\mathcal{E}_i)\big)$ is a multicomplex and if $k\in\mathbb{Z}^I$, then $A^k\in\prod_{i\in I}\mathcal{E}_i$ is the object whose ith component is given by $(A^k)_i = A_i^{k_i}$ and $d_i^k(A)\colon A^k\to A^{k+\varepsilon_i}$ is the map that is the identity on $(A^k)_j$ for $i\neq j$ and $d^k_{A_i}$ on $(A^k)_i$. Similarly, we have $f^k\colon A^k\to B^k$ for any morphism $f\colon A\to B$. With the integral functions

$$\kappa(\prec,k,i)=\sum_{j\prec i} k_i \quad\text{and}\quad \lambda(\prec_1,\prec_2,k)=\sum_{\substack{i\prec_1 j\\ j\prec_2 i}} k_i k_j,$$

the functor C is defined by the equations

$$\mathrm{C}S_\prec(A)^m = \sum_{|k|=m} S_\prec(A^k),$$

$$\mathrm{C}S_\prec(f)^m = \sum_{|k|=m} S_\prec(f^k),$$

$$d^m_{\mathrm{C}S_\prec(A)} = \sum_{|k|=m} d^k(\mathrm{C}S_\prec(A)),$$

$$d^k(\mathrm{C}S_\prec(A)) = \sum_{i\in I}(-1)^{\kappa(\prec,k,i)} S_\prec(d_i^k(A)),$$

$$\rho_i(S)(A) = \sum_k (-1)^{\kappa(\prec,k,i)} 1_{S(A^k)},$$

$$\mathrm{C}S(\prec_2,\prec_1)(A^k) = \sum_k (-1)^{\lambda(\prec_1,\prec_2,k)} S(\prec_2,\prec_1)(A).$$

We leave the verification to the reader. □

The next definition is a formal definition of a multideterminant. Let the $\mathcal{E}_i$ be exact categories, and let w_i be SQ-classes of morphisms. Informally, a multideterminant on the product category $\prod_{i\in I}\mathcal{E}_{i\,\mathrm{w}_i}$ with values in a Picard category P is a multifunctor that is a determinant for every choice of $|I|-1$ frozen variables and such that we obtain a certain commutative diagram for every pair of indices $i \neq j$. To state the definition formally, we need some notation.

For any subset $K \subseteq I$, the isomorphism

$$\mathrm{Ev}^K : \mathrm{Funct}\Big(\prod_{i\in I}\mathcal{E}_i, P\Big) \to \mathrm{Funct}\Big(\prod_{i\in K}\mathcal{E}_i, \mathrm{Funct}\Big(\prod_{i\in I\setminus K}\mathcal{E}_i, P\Big)\Big)$$

is given by

$$\mathrm{Ev}^K(S)(A')(A'') = S(A), \quad \text{where} \quad A_i = \begin{cases} A'_i & \text{for } i \in K, \\ A''_i & \text{for } i \in I \setminus K. \end{cases}$$

Let p', p, and p'' be the projections $\{\mathcal{E}_i\} \to \mathcal{E}_i$ as in Definition 1.2, and let

$$\mathcal{E}_K = \prod_{i\in I}\mathcal{E}_{K,i}, \quad \text{where} \quad \mathcal{E}_{K,i} = \begin{cases} \mathcal{E}_{i\,\mathrm{w}_i} & \text{for } i \in K, \\ \{\mathcal{E}_i\}_{\{\mathrm{w}_i\}} & \text{for } i \in I \setminus K. \end{cases}$$

For any subsets $J \subset K$ and $L \subset I$ and for $s \in \{\cdot', \cdot''\}^{K\setminus J}$, we have the two projections $p^s_{K,J} : \mathcal{E}_J \to \mathcal{E}_K$ and $p_J : \mathcal{E}_L \to \mathcal{E}_{L\cup I\setminus J}$ given by

$$(p^s_{K,J}(A))_i = \begin{cases} A_i \in \mathrm{Ob}(\{\mathcal{E}_i\}) & \text{for } i \in I \setminus K, \\ p^{s(i)}(A_i) \in \mathrm{Ob}(\mathcal{E}_i) & \text{for } i \in K \setminus J, \\ A_i \in \mathrm{Ob}(\mathcal{E}_i) & \text{for } i \in J; \end{cases}$$

$$(p_J(A))_i = \begin{cases} A_i \in \mathrm{Ob}(\{\mathcal{E}_i\}) & \text{for } i \in J \setminus L, \\ p(A_i) \in \mathrm{Ob}(\mathcal{E}_i) & \text{for } i \in I \setminus (L \cup J), \\ A_i \in \mathrm{Ob}(\mathcal{E}_i) & \text{for } i \in L. \end{cases}$$

DEFINITION 4.4. A *multideterminant* f on the category $\mathcal{E}_I = \prod_{i\in I}\mathcal{E}_{i\,\mathrm{w}_i}$ with values in P consists of a multifunctor $f : \mathcal{E}_I \to P$, together with natural isomorphisms

$$f_{K,J} : f \circ p_J \to \bigotimes_{s\in\{\cdot',\cdot''\}^{K\setminus J}} f \circ p_K \circ p^s_{K,J}$$

on $\mathrm{Funct}(\mathcal{E}_J, P)$ for each pair of subsets $J \subset K$, satisfying the following conditions.

(a) For each $A \in \mathrm{Ob}\big(\prod_{i\in K}\mathcal{E}_i\big)$ with $|K| = |I| - 1$, we have that $(f_1, f_2) = (\mathrm{Ev}^K f(A), \mathrm{Ev}^K f_{I,K}(A))$ is a determinant.

(b) The isomorphism $f_{K,J}(A)$ depends only on $p_K(A)$, and for any subsets $J \subset K \subset L$ and $A \in \mathcal{E}_J$ we have a commutative diagram

$$\begin{tikzcd}
f \circ p_J(A) \arrow[r, "f_{K,J}(A)"] \arrow[d, "f_{L,J}(A)"'] & \bigotimes_{s\in\{\cdot',\cdot''\}^{K\setminus J}} f \circ p_K \circ p^s_{K,J}(A) \arrow[d, "\bigotimes_{s\in\{\cdot',\cdot''\}^{K\setminus J}} f_{L,K}(p^s_{K,J}(A))"] \\
\bigotimes_{u\in\{\cdot',\cdot''\}^{L\setminus J}} f \circ p_L \circ p^u_{L,J}(A) \arrow[r, "\sim"] & \bigotimes_{s\in\{\cdot',\cdot''\}^{K\setminus J}} \left(\bigotimes_{t\in\{\cdot',\cdot''\}^{L\setminus K}} f \circ p_L \circ p^t_{L,K}(p^s_{K,J}(A)) \right).
\end{tikzcd}$$

REMARK 4.5. Since $f_{K,J}(A)$ depends only on $p_K(A)$, it follows that $f_{K,J}$ is determined by $f_{I,I\setminus(K\setminus J)}$; hence it suffices to have (b) satisfied for all $J \subset K \subset L$ with $|J| = |I| - 2$.

DEFINITION 4.6. A *morphism* of multideterminants $\rho\colon f \to g$ is a natural isomorphism of multifunctors with the property that, for all subsets $J \subset K$ and for all $A \in \mathrm{Ob}(\mathcal{E}_J)$, the following diagram is commutative:

$$\begin{tikzcd}
f(p_J(A)) \arrow[r, "f_{K,J}(A)"] \arrow[d, "\rho(p_J(A))"'] & \bigotimes_{s\in\{\cdot',\cdot''\}^{K\setminus J}} f(p_K(p^s_{K,J}(A))) \arrow[d, "\bigotimes_{s\in\{\cdot',\cdot''\}^{K\setminus J}} \rho(p_K(p^s_{K,J}(A)))"] \\
g(p_J(A)) \arrow[r, "g_{K,J}(A)"] & \bigotimes_{s\in\{\cdot',\cdot''\}^{K\setminus J}} g(p_K(p^s_{K,J}(A)))
\end{tikzcd}$$

We denote by $\det\big(\prod_{i\in I} \mathcal{E}_{i\,\mathrm{w}_i}, P\big)$ the category of multideterminants.

PROPOSITION 4.7. *The category of multideterminants is a Picard category, and for any multideterminant f in $\det(\mathcal{E}_I, P)$ and any $K \subset I$, $\mathrm{Ev}^K(f)$ is a multideterminant on $\prod_{i\in K} \mathcal{E}_{i\,\mathrm{w}_i}$ with values in $\det\big(\prod_{i\in I\setminus K} \mathcal{E}_{i\,\mathrm{w}_i}, P\big)$. In fact, we have an AC tensor functor and an isomorphism of categories*

$$\mathrm{Ev}^K : \det\left(\prod_{i\in I} \mathcal{E}_{i\,\mathrm{w}_i}, P\right) \to \det\left(\prod_{i\in K} \mathcal{E}_{i\,\mathrm{w}_i}, \det\left(\prod_{i\in I\setminus K} \mathcal{E}_{i\,\mathrm{w}_i}, P\right)\right).$$

THEOREM 4.8. *The restriction functor*

$$\det\left(\prod_{i\in I} \mathrm{C}\mathcal{E}_{i\,\mathrm{qis}}, P\right) \to \det\left(\prod_{i\in I} \mathcal{E}_{i\,\mathrm{iso}}, P\right)$$

is an equivalence and also an AC tensor functor.

Proof. We construct an inverse functor $\mathrm{C}^{\sigma,\prec}$ depending upon an inverse structure σ on P and a total ordering $\prec$ on I. We proceed by induction with respect to $|I|$, and we denote the restriction of $\prec$ to any subset of I by $\prec$ as well. By the main Theorem 2.3, Theorem 4.8 holds for $|I| = 1$. Let j be the maximum member of I. By the induction hypothesis and Proposition 4.7, we have a commutative diagram

$$\begin{tikzcd}
\det\Big(\prod_i \mathrm{C}\mathcal{E}_{i\,\mathrm{qis}}, P\Big) \arrow[r, "\mathrm{Ev}^{\{j\}}"] \arrow[dd, "\mathrm{res}_I"'] & \det\Big(\mathrm{C}\mathcal{E}_{j\,\mathrm{qis}}, \det\Big(\prod_{i\neq j} \mathrm{C}\mathcal{E}_{i\,\mathrm{qis}}, P\Big)\Big) \arrow[d, "\mathrm{res}_j"] \\
 & \det\Big(\mathcal{E}_{j\,\mathrm{iso}}, \det\Big(\prod_{i\neq j} \mathrm{C}\mathcal{E}_{i\,\mathrm{qis}}, P\Big)\Big) \arrow[u, bend left, "\mathrm{C}^{\sigma}"] \arrow[d, "(\mathrm{res}_{I\setminus\{j\}})_*"] \\
\det\Big(\prod_i \mathcal{E}_{i\,\mathrm{iso}}, P\Big) \arrow[r, "\mathrm{Ev}^{\{j\}}"] & \det\Big(\mathcal{E}_{j\,\mathrm{iso}}, \det\Big(\prod_{i\neq j} \mathcal{E}_{i\,\mathrm{iso}}, P\Big)\Big), \arrow[u, bend left, "(\mathrm{C}^{\sigma,\prec})_*"]
\end{tikzcd}$$

where $\mathrm{C}^{\sigma,\prec}$ is an inverse to $\mathrm{res}_{I\setminus\{j\}}$. Again by Theorem 2.3, C^{σ} is an inverse to res_j, and since composition of AC tensor functors yields an AC tensor functor, the theorem follows. □

REMARK 4.9. For any pair $(\prec_1, \prec_2)$ of total orderings, both $\mathrm{C}^{\sigma,\prec_1}$ and $\mathrm{C}^{\sigma,\prec_2}$ are canonically isomorphic because they are inverses to the restriction. Hence we may view C^{σ} as a functor of order invariant multideterminants.

The following is a generalization of [D, 4.14] that can be thought of as a formula for the determinant of the Kronecker product of two matrices in terms of the determinants of those matrices.

Let $\{\mathcal{E}_i\}_{i\in I}$ and $\mathcal{F}$ be exact categories, and let $\mathrm{v} = \{\mathrm{v}_i\}_{i\in I}$ and w be SQ-classes of morphisms in $\{\mathcal{E}_i\}_{i\in I}$ and $\mathcal{F}$, respectively. Let $\{P_i\}_{i\in I}$ and Q be Picard categories, let $S\colon \prod_i \mathcal{E}_i \to \mathcal{F}$ be a multiexact functor sending v to w, and let $T\colon \prod_i P_i \to Q$ be a multi-AC tensor functor, by which we mean a multifunctor that is an AC tensor functor for any $|I|-1$ frozen variables and that satisfies the commutativity of the obvious diagrams for each pair of indices.

LEMMA 4.10. *Let $\{\mathcal{E}_i\}_{i\in I}$, $\mathcal{F}$, $\{\mathrm{v}_i\}_{i\in I}$, w, $\{P_i\}_{i\in I}$, Q, S, and T be as just defined. If $f = \{f_i\}_{i\in I}$ and g are determinants on $\mathcal{E}_{i\,\mathrm{v}_i}$ and $\mathcal{F}_{\mathrm{w}}$ with values in P_i and Q, respectively, then the compositions $g\circ S$ and $T\circ f$ are both multideterminants.*

DEFINITION 4.11. With notation as in Lemma 4.10, an $\langle S, T, \mathrm{v}, \mathrm{w}\rangle$-*determinant* is a triple (f, g, η), where $f = \{f_i\}_{i\in I}$ and g are determinants on $\mathcal{E}_{i\,\mathrm{v}_i}$ and $\mathcal{F}_{\mathrm{w}}$ with values in P_i and Q (respectively) and where $\eta\colon g\circ S \to T\circ f$ is an isomorphism of multideterminants. A morphism of $\langle S, T, \mathrm{v}, \mathrm{w}\rangle$-determinants from (f, g, η) to (f', g', η') is a pair of natural transformations $q\colon f \to f'$ and $r\colon g \to g'$ commuting with η and η'. If S and T are order invariant, we say that (f, g, η) is order invariant if η is an isomorphism of order invariant functors. If S and T are symmetric, we say that (f, g, η) is symmetric if η is an isomorphism of symmetric functors and if all of the f_i are the same determinant. We denote the category of $\langle S, T, \mathrm{v}, \mathrm{w}\rangle$-determinants by $\det\langle S, T, \mathrm{v}, \mathrm{w}\rangle$.

LEMMA 4.12. *With notation as in Lemma 4.10,* $\det\langle S, T, \mathrm{v}, \mathrm{w}\rangle$ *is a Picard category with tensor product defined componentwise.*

COROLLARY 4.13. *The restriction functor*

$$\det\langle \mathrm{C}(S), T, \mathrm{qis}, \mathrm{qis}\rangle \to \det\langle S, T, \mathrm{iso}, \mathrm{iso}\rangle$$

is an equivalence and also an AC tensor functor. Moreover, (f, g, η) *is symmetric if and only if* $\mathrm{res}(f, g, \eta)$ *is.*

EXAMPLE 4.14 [D]. In this example we let $I = \{1, 2\}$ be an index set, the standard ordering is $<$, the permutation σ is the transposition $(1, 2)$, and $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{F}$ is the exact category of locally free sheaves on a scheme V. For any such sheaf A, we let n_A be the rank function. The Picard category $P_1 = P_2 = Q$ is the category of $\mathbb{Z}^V$-graded invertible sheaves on V. An object in this category is a pair $\bar{X} = \langle X, n_X\rangle$, where X is an invertible sheaf on V and n_X is a *continuous* integral function on V.

The order invariant and symmetric biexact functor $S\colon \prod_{i\in I}\mathcal{E}_i \to \mathcal{F}$ is given by

$$S_<(A) = A_1 \otimes A_2 \quad \text{and} \quad S_>(A) = S_<(\sigma(A)) = A_2 \otimes A_1$$

for any object $A = (A_1, A_2) \in \mathrm{Ob}\big(\prod_{i\in I}\mathcal{E}_i\big)$, and the morphism

$$S_{>,<}(A)\colon S_<(A) \to S_>(A)$$

is given stalkwise by

$$S_{>,<}(A)(a_1 \otimes a_2) = S_>(\sigma)(A)(a_1 \otimes a_2) = a_2 \otimes a_1,$$

where a_1 and a_2 are germs of sections of A_1 and A_2, respectively.

The classical determinant $\det\colon \mathcal{F} \to P$ is defined by $\det(A) = \big\langle \bigwedge^{n_A} A, n_A\big\rangle$, and the composition $\det \circ S$ is an order invariant, symmetric bi-determinant.

The order invariant and symmetric functor $T\colon \prod_{i\in I} P_i \to Q$ is given by

$$T_<(\bar{X}) = \langle X_1^{\bigotimes n_{X_2}} \otimes X_2^{\bigotimes n_{X_1}}, n_{X_1} + n_{X_2}\rangle \quad \text{and} \quad T_>(\bar{X}) = T_<(\sigma(\bar{X}))$$

for any object $\bar{X} = (\bar{X}_1, \bar{X}_2) \in \mathrm{Ob}\big(\prod_{i\in I} P_i\big)$, and the morphism

$$T_{>,<}(\bar{X})\colon T_<(\bar{X}) \to T_>(\bar{X})$$

is given stalkwise by

$$T_{>,<}(\bar{X})\Bigg(\bigotimes_{j=1}^{n_{X_2}} x_{1,j} \otimes \bigotimes_{i=1}^{n_{X_1}} x_{2,i}\Bigg) = (-1)^{\binom{n_{X_1}}{2}\binom{n_{X_2}}{2}} \bigotimes_{i=1}^{n_{X_1}} x_{2,i} \otimes \bigotimes_{j=1}^{n_{X_2}} x_{1,j}.$$

The functor $T_<$ is a bi-AC tensor functor via the morphisms

$$T_{<,2}(\bar{X}_1', \bar{X}_1'', \bar{X}_2)\colon T_<(\bar{X}_1' \otimes \bar{X}_1'', \bar{X}_2) \to T_<(\bar{X}_1', \bar{X}_2) \otimes T_<(\bar{X}_1'', \bar{X}_2)$$

and

$$T_{<,1}(\bar{X}_1, \bar{X}_2', \bar{X}_2'')\colon T_<(\bar{X}_1, \bar{X}_2' \otimes \bar{X}_2'') \to T_>(\bar{X}_1, \bar{X}_2') \otimes T_>(\bar{X}_1, \bar{X}_2''),$$

given on stalks by

$$\bigotimes_{j=1}^{n_{X_2}}(x'_{1,j}\otimes x''_{2,j})\otimes\bigotimes_{i=1}^{n_{X'_1}+n_{X''_1}}x_{2,i}\mapsto\bigotimes_{j=1}^{n_{X_2}}x'_{1,j}\otimes\bigotimes_{i=1}^{n_{X'_1}}x_{2,i}\otimes\bigotimes_{j=1}^{n_{X_2}}x''_{1,j}\otimes\bigotimes_{i=n_{X'_1}+1}^{n_{X'_1}+n_{X''_1}}x_{2,i}$$

and

$$\bigotimes_{j=1}^{n_{X'_2}+n_{X''_2}}x_{1,j}\otimes\bigotimes_{i=1}^{n_{X_1}}(x'_{2,i}\otimes x''_{2,i})$$
$$\mapsto(-1)^{n_{X'_2}n_{X''_2}\binom{n_{X_1}}{2}}\bigotimes_{j=1}^{n_{X'_2}}x_{1,j}\otimes\bigotimes_{i=1}^{n_{X_1}}x'_{2,i}\otimes\bigotimes_{j=n_{X'_2}+1}^{n_{X'_2}+n_{X''_2}}x_{1,j}\otimes\bigotimes_{i=1}^{n_{X_1}}x''_{2,i}.$$

The reader can check that the diagram

$$\begin{tikzcd}[column sep=small]
 & T_<(\bar{X}'_1\otimes\bar{X}''_1,\bar{X}'_2)\otimes T_<(\bar{X}'_1\otimes\bar{X}''_1,\bar{X}''_2) \arrow[dr] & \\
 & & T_<(\bar{X}'_1,\bar{X}'_2)\otimes T_<(\bar{X}''_1,\bar{X}'_2)\otimes T_<(\bar{X}'_1,\bar{X}''_2)\otimes T_<(\bar{X}''_1,\bar{X}''_2) \arrow[d, "1\otimes\psi\otimes1"] \\
T_<(\bar{X}'_1\otimes\bar{X}''_1,\bar{X}'_2\otimes\bar{X}''_2) \arrow[uur] \arrow[ddr] & & T_<(\bar{X}'_1,\bar{X}'_2)\otimes T_<(\bar{X}'_1,\bar{X}''_2)\otimes T_<(\bar{X}''_1,\bar{X}'_2)\otimes T_<(\bar{X}''_1,\bar{X}''_2) \\
 & & \\
 & T_<(\bar{X}'_1,\bar{X}'_2\otimes\bar{X}''_2)\otimes T_<(\bar{X}''_1,\bar{X}'_2\otimes\bar{X}''_2) \arrow[uur] &
\end{tikzcd}$$

commutes, and if $T_{>,1}=T_{<,2}$ and $T_{>,2}=T_{<,1}$, then $T_{>,<}$ is a morphism of bi-AC tensor functors. If we denote the functor $\det\colon\prod_{i\in I}\mathcal{E}_i\to\prod_{i\in I}P_i$ given by $\det(A_1,A_2)=(\det(A_1),\det(A_2))$ by det as well, then the composition $T\circ\det$ is also an order invariant and symmetric bi-determinant. We define $\eta\colon\det\circ S\to T\circ\det$ stalkwise by

$$\eta_<(A)\Bigg(\bigwedge_{(i,j)\in J(A)}a^j_{1,i}\otimes a^i_{2,j}\Bigg)=\bigotimes_{j=1}^{n(A_2)}\bigwedge_{i=1}^{n(A_1)}a^j_{1,i}\otimes\bigotimes_{i=1}^{n(A_1)}\bigwedge_{j=1}^{n(A_2)}a^i_{2,j},$$

where $J(A)$ is the ordered set $\{1,\dots,n(A_1)\}\times\{1,\dots,n(A_2)\}$ (with lexicographical ordering) and $\eta_>(A)=\eta_<(\sigma(A))$. The diagram

$$\begin{tikzcd}
\det\circ S_< \arrow[r] \arrow[d, "\eta_<"'] & \det\circ S_> \arrow[d, "\eta_>"'] \\
T_<\circ\det \arrow[r] & T_>\circ\det
\end{tikzcd}$$

commutes because the pullback by the transposition $J(A)\to J(\sigma(A))$ of the lexicographical ordering on $J(\sigma(A))$ differs from the lexicographical ordering on $J(A)$

by a permutation of signature $(-1)^{\binom{n_{A_1}}{2}\binom{n_{A_2}}{2}}$. The reader may check that η is a morphism of order invariant and symmetric bi-determinants. Therefore $(\det, \det, \eta)$ is an order invariant and symmetric $\langle S, T, \mathrm{iso}, \mathrm{iso}\rangle$-determinant, and by Corollary 4.13 $(\det, \det, \eta)$ has an essentially unique extension to an order invariant and symmetric $\langle \mathrm{C}(S), T, \mathrm{qis}, \mathrm{qis}\rangle$-determinant.

Next we take a quick look at contravariant functors.

DEFINITION 4.15. Let $\mathcal{E}$ and $\mathcal{F}$ be exact categories. For any contravariant functor $S\colon \mathcal{E} \to \mathcal{E}$, we define the extended contravariant functor $\mathrm{C}S\colon \mathrm{C}\mathcal{E} \to \mathrm{C}\mathcal{E}$ by the formulas in SGA 4 (XVII, 1.1.5.1):

$$[\mathrm{C}S(A)]^k = A^{-k},$$

$$[\mathrm{C}S(\alpha)]^k = \alpha^{-k},$$

$$[d_{\mathrm{C}S(A)}]^k = (-1)^{k+1} S(d_A^{-(k+1)}).$$

LEMMA 4.16. *If S is exact, then so is* $\mathrm{C}S$. *If T denotes the translation functor and if $C(\alpha)$ denotes the mapping cone of the morphism α, then there are canonical isomorphisms of functors*

$$T^{-1} \circ \mathrm{C}S \approx \mathrm{C}S \circ T,$$

$$TC(\mathrm{C}S(\alpha)) \approx \mathrm{C}S(C(\alpha)).$$

COROLLARY 4.17. *The restriction functor on the Picard category of contravariant* $\langle \mathrm{C}S, T, \mathrm{iso}, \mathrm{iso}\rangle$*-determinants is an equivalence and also an AC tensor functor.*

EXAMPLE 4.18 [D]. Let $\mathcal{E} = \mathcal{F}$ and $P = Q$ be as in Example 4.14, and let the functors $S\colon \mathcal{E} \to \mathcal{F}$ and $T\colon P \to Q$ be given by

$$S(A) = A^{\vee}, \qquad S(\alpha) = \alpha^{\vee},$$

$$T\langle X, m\rangle = \langle X^{\vee}, m\rangle, \qquad T(\alpha) = \alpha^{\vee}.$$

Then S is a contravariant exact functor, T is a contravariant AC tensor functor, and we have a natural isomorphism of contravariant determinants

$$\eta\colon \det \circ S \to T \circ \det.$$

Hence $(\det, \det, \eta)$ is an $\langle S, T, \mathrm{iso}, \mathrm{iso}\rangle$-determinant, and by Corollary 4.17 the canonical isomorphism for finite locally free sheaves $\bigwedge^{n_A}(A^{\vee}) \to \left(\bigwedge^{n_A} A\right)^{\vee}$ extends uniquely to complexes.

5. The Homotopy Formula

In this section the terms $\mathcal{E}$, $\mathrm{C}(\mathcal{E})$, P, and σ are as in Section 2, and $f = (f_1, f_2)$ is a determinant on $\mathcal{E}_{\mathrm{iso}}$ with values in P.

DEFINITION 5.1. For complexes A, B and a map $\alpha\colon A \to B$, we have the following objects and maps in $\mathcal{E}$:

$$A^+ = \bigoplus_i A^{2i},$$
$$A^- = \bigoplus_i A^{2i+1};$$
$$\alpha^+ = \bigoplus_i \alpha^{2i}\colon A^+ \to B^+,$$
$$\alpha^- = \bigoplus_i \alpha^{2i+1}\colon A^- \to B^-.$$

LEMMA 5.2. *Any zero-homotopic complex is split exact.* (See Definition 2.16.)

Proof. Let h be a homotopy for the complex A, and let $h' = h - dh^3d$. We leave it to the reader to verify that $1 = dh' + h'd$ and $h'^2 = 0$. Hence we may assume that $h^2 = 0$. The maps $p^i = d^{i-1}h^i$ and $q^i = h^{i+1}d^i$ are projections, and the isomorphisms $A^i \to Z^i \oplus Z^{i+1}$ are given by the two maps p^i and d^iq^i. □

PROPOSITION 5.3. *Let A be a homotopically trivial complex, and let h be a homotopy for A. Then the map $d^- + h^-\colon A^- \to A^+$ is an isomorphism, the map $f(d^- + h^-)\colon f(A^-) \to f(A^+)$ does not depend on the choice of h, and $f(d^+ + h^+) = (f(d^- + h^-))^{-1}$.*

Proof. By Lemma 5.2, A is split exact. The composition $(d^+ + h^+)(d^- + h^-) = 1 + h^+h^-$ is an isomorphism because h^+h^- is nilpotent. Also $1 + h^+h^-$ respects the natural filtration and induces the identity on each quotient A^{2n+1}; hence, by Proposition 1.7, $f(d^+ + h^+)f(d^- + h^-) = 1_{f(A^-)}$. If h' is another homotopy for A, then $h' - h$ is a *morphism* $TA \to A$. We leave it to the reader to check that any morphism of split exact complexes is homotopic to zero, so there is a map $s\colon T^2A \to A$ such that $h' - h = ds - sd$. The two maps $d^- + h'^-$ and $(1 - s^+)(d^- + h^-)(1 + s^-)$ induce the same map $\operatorname{gr} A^- \to \operatorname{gr} A^+$, and since $f(1 - s^+) = 1_{f(A^+)}$ and $f(1 + s^-) = 1_{f(A^-)}$, the proposition follows from Proposition 1.7. □

DEFINITION 5.4. Consider a pair (α, β) of morphisms of complexes $\alpha\colon A \to B$ and $\beta\colon B \to A$. A pair (h, k) of maps $h\colon TA \to A$ and $k\colon TB \to B$ will be called an (α, β)*-good pair of homotopies* if (i) they are homotopies (i.e., $dh + hd = 1 - \beta\alpha$ and $dk + kd = 1 - \alpha\beta$) and (ii) there exists a map $l\colon T^2A \to B$ such that $k\alpha - \alpha h + dl - ld = 0$. Symmetrically, we say that (k, h) is (β, α)-good if $h\beta - \beta k + dm - md = 0$ for some $m\colon T^2B \to A$. We say that (h, k) is a *good pair* if (h, k) is (α, β)-good and (k, h) is (β, α)-good.

REMARK 5.5. Note that the relations just described simply say that the maps

$$\begin{pmatrix} k & l \\ \beta & -h \end{pmatrix}\colon TC(\alpha) \to C(\alpha) \quad \text{and} \quad \begin{pmatrix} h & m \\ \alpha & -k \end{pmatrix}\colon TC(\beta) \to C(\beta)$$

are homotopies for $C(\alpha)$ and $C(\beta)$, respectively.

Proposition 5.6. *Let $\alpha\colon A \to B$ and $\beta\colon B \to A$ be a pair of morphisms of complexes, and let $h\colon TA \to A$ and $k\colon TB \to B$ be a pair of homotopies for the pair α, β. If we define $h_1 = h + \beta(k\alpha - \alpha h)$ and $k_1 = \alpha(h\beta - \beta k)$, then both pairs h_1, k and h, k_1 are good.*

Proof. The goodness of h_1, k is readily checked by setting $l = k\alpha h - k^2\alpha$ and $m = h\beta k - h^2\beta - \beta k^2$. □

Proposition 5.7. *Let $\alpha\colon A \to B$ be a morphism, let $\beta\colon B \to A$ be a homotopy inverse, and let $h\colon TA \to A$ and $k\colon TB \to B$ be an (α, β)-good pair of homotopies. Then the map*

$$\begin{pmatrix} \alpha^+ & d^- + k^- \\ d^+ + h^+ & -\beta^- \end{pmatrix}\colon A^+ \oplus B^- \to B^+ \oplus A^-$$

is an isomorphism, the map

$$f\begin{pmatrix} \alpha^+ & d^- + k^- \\ d^+ + h^+ & -\beta^- \end{pmatrix}\colon f(A^+ \oplus B^-) \to f(B^+ \oplus A^-)$$

does not depend upon the choice of β, h, and k, and

$$f\begin{pmatrix} \beta^+ & d^- + h^- \\ d^+ + k^+ & -\alpha^- \end{pmatrix} = \left(f\begin{pmatrix} \alpha^+ & d^- + k^- \\ d^+ + h^+ & -\beta^- \end{pmatrix}\right)^{-1}.$$

Proof. Let $l\colon T^2A \to B$ be as in Definition 5.4. In order to simplify the computations we use the following commutative diagram, where the vertical arrows are shuffling morphisms:

$$\begin{tikzcd}
A^+ \oplus B^- \arrow[r, "\begin{pmatrix} \alpha^+ + l^+ & d^- + k^- \\ -(d^+ + h^+) & \beta^- \end{pmatrix}"] \arrow[d] & B^+ \oplus A^- \arrow[r, "\begin{pmatrix} \beta^+ & -(d^- + h^-) \\ d^+ + k^+ & \alpha^- + l^- \end{pmatrix}"] \arrow[d] & A^+ \oplus B^- \arrow[d] \\
C(\alpha)^- \arrow[r, "\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix}^- + \begin{pmatrix} k & l \\ \beta & -h \end{pmatrix}^-"] & C(\alpha)^+ \arrow[r, "\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix}^+ + \begin{pmatrix} k & l \\ \beta & -h \end{pmatrix}^+"] & C(\alpha)^- .
\end{tikzcd}$$

Consider the three compositions

$$\left(\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix} + \begin{pmatrix} k & l \\ \beta & -h \end{pmatrix}\right)^+ \left(\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix} + \begin{pmatrix} k & l \\ \beta & -h \end{pmatrix}\right)^-$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^- + \begin{pmatrix} k^2 + l\beta & kl - lh \\ \beta k - h\beta & \beta l + h^2 \end{pmatrix}^-,$$

$$\left(\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix} + \begin{pmatrix} k & l \\ \beta & -h \end{pmatrix}\right)^+ \left(\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix} + \begin{pmatrix} k & 0 \\ \beta & -h \end{pmatrix}\right)^-$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^- + \begin{pmatrix} k^2 + l\beta & -dl - lh \\ \beta k - h\beta & h^2 \end{pmatrix}^-,$$

$$\left(\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix} + \begin{pmatrix} k & 0 \\ \beta & -h \end{pmatrix}\right)^{+} \left(\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix} + \begin{pmatrix} k & l \\ \beta & -h \end{pmatrix}\right)^{-}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-} + \begin{pmatrix} k^2 & kl - ld \\ \beta k - h\beta & \beta l + h^2 \end{pmatrix}^{-}.$$

All these matrices respect the fine admissible filtration on $C(\alpha)^-$ given by $C(\alpha)^- = \cdots \oplus B^{2i-1} \oplus A^{2i} \oplus B^{2i+1} \oplus \cdots$ and induce the identity on each successive quotient. It follows from this and from Proposition 1.5(c) that

$$\begin{aligned}
\left(f\begin{pmatrix} \alpha^+ & d^- + k^- \\ d^+ + h^+ & -\beta^- \end{pmatrix}\right)^{-1} &= \varepsilon(f(A^-))\left(f\begin{pmatrix} \alpha^+ & d^- + k^- \\ -(d^+ + h^+) & \beta^- \end{pmatrix}\right)^{-1} \\
&= \varepsilon(f(A^-))\left(f\begin{pmatrix} \alpha^+ + l^+ & d^- + k^- \\ -(d^+ + h^+) & \beta^- \end{pmatrix}\right)^{-1} \\
&= \varepsilon(f(A^-))f\begin{pmatrix} \beta^+ & -(d^- + h^-) \\ d^+ + k^+ & \alpha^- + l^- \end{pmatrix} \\
&= f\begin{pmatrix} \beta^+ & d^- + h^- \\ d^+ + k^+ & -(\alpha^- + l^-) \end{pmatrix} \\
&= f\begin{pmatrix} \beta^+ & d^- + h^- \\ d^+ + k^+ & -\alpha^- \end{pmatrix}.
\end{aligned}$$

The proposition now follows from Propositions 5.3 and 5.6. □

DEFINITION 5.8. Let $\alpha : A \to B$ be a homotopy equivalence. We denote by $\tilde{f}(\alpha)$ the morphism that makes the following diagram commutative (here β is any homotopy inverse, and h and k is any (α, β)-good pair of homotopies):

$$\begin{tikzcd}
f(A^+ \oplus B^-) \arrow[r] \arrow[d, "f\begin{pmatrix} \alpha^+ & d^- + k^- \\ d^+ + h^+ & -\beta^- \end{pmatrix}"'] & f(A^+) \otimes f(B^-) \arrow[d, "\tilde{f}(\alpha)"] \\
f(B^+ \oplus A^-) \arrow[r] & f(B^+) \otimes f(A^-).
\end{tikzcd}$$

THEOREM 5.9 (The Homotopy Formula). *Let $\mathrm{C}^\sigma(f)$ be the σ-extension of the determinant f to $\mathrm{C}(\mathcal{E})_{\mathrm{qis}}$. Then, for any homotopy equivalence $\alpha : A \to B$, the following diagram is commutative*:

$$\begin{tikzcd}
f(A^+) \otimes f(B^-) \otimes f^\sigma(A^-) \otimes f^\sigma(B^-) \arrow[r, "\sigma(\{2,4\})"] \arrow[d, "\tilde{f}(\alpha)\otimes 1 \otimes 1"'] & f(A^+) \otimes f^\sigma(A^-) \arrow[r] & \mathrm{C}^\sigma(f)(A) \arrow[d, "\mathrm{C}^\sigma(f)(\alpha)"] \\
f(B^+) \otimes f(A^-) \otimes f^\sigma(A^-) \otimes f^\sigma(B^-) \arrow[r, "\sigma(\{2,3\})"] & f(B^+) \otimes f^\sigma(B^-) \arrow[r] & \mathrm{C}^\sigma(f)(B).
\end{tikzcd}$$

Proof. We start with the case $B = 0$. In this case A is split exact and, assuming $h^2 = 0$, we have the commutative diagram

$$\begin{tikzcd}
A^+ \arrow[r, "d^+ + h^+"] \arrow[d, "\binom{d^- h^+}{d^+}"'] & A^- \arrow[d, "\binom{d^+ h^-}{d^-}"] \\
Z^+ \oplus Z^- \arrow[r, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"] & Z^- \oplus Z^+ .
\end{tikzcd}$$

This and the properties of the inverse structure σ (see Definition A.16) shows that we have a commutative diagram

$$\begin{tikzcd}
f(A^+) \otimes f^\sigma(A^-) \arrow[r, "f(d^+ + h^+) \otimes 1"] \arrow[d] & f(A^-) \otimes f^\sigma(A^-) \arrow[r] \arrow[d] & 1 \arrow[d] \\
f(Z^+) \otimes f(Z^-) \otimes f^\sigma(Z^-) \otimes f^\sigma(Z^+) \arrow[r, "\psi \otimes 1 \otimes 1"] \arrow[d] & f(Z^-) \otimes f(Z^+) \otimes f^\sigma(Z^-) \otimes f^\sigma(Z^+) \arrow[r] \arrow[d] & 1 \arrow[d] \\
\mathrm{C}^\sigma(f)(Z) \otimes \mathrm{C}^\sigma(f)(TZ) \arrow[r] & \mathrm{C}^\sigma(f)(Z) \otimes \mathrm{C}^\sigma(f)(TZ) \arrow[r] & 1,
\end{tikzcd}$$

and this proves the theorem in the case $B = 0$. The case $A = 0$ follows by taking inverses.

For the special complex $\mathrm{C}(1_A)$ we have that $0^R \colon \mathrm{C}(1_A) \to 0$ is a homotopy equivalence with homotopy $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and from the theorem in the case $B = 0$ we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{C}^\sigma(f)(A) \otimes \mathrm{C}^\sigma(f)(TA) \arrow[r] \arrow[d] & \mathrm{C}^\sigma(f)(\mathrm{C}(1_A)) \arrow[d, "\mathrm{C}^\sigma(f)(0^R)"] \\
f(A^+) \otimes f^\sigma(A^-) \otimes f(A^-) \otimes f^\sigma(A^+) \arrow[r, "\sigma(\{1,4\},\{2,3\})"] & 1.
\end{tikzcd}$$

We also have the commutative diagrams

$$\begin{tikzcd}
\mathrm{C}^\sigma(f)(A) \otimes \mathrm{C}^\sigma(f)(TA) \arrow[r] \arrow[d, "\mathrm{C}^\sigma(f)(\alpha) \otimes 1"'] & \mathrm{C}^\sigma(f)(\mathrm{C}(1_A)) \arrow[d] \\
\mathrm{C}^\sigma(f)(B) \otimes \mathrm{C}^\sigma(f)(TA) \arrow[r] & \mathrm{C}^\sigma(f)(C(\alpha))
\end{tikzcd}$$

and

$$\begin{tikzcd}
f(C(\alpha)^-) \arrow[r] \arrow[d, "\tilde{f}(0^L)"] & f(A^+) \otimes f(B^-) \arrow[d, "\varepsilon(f(A^-))\tilde{f}(\alpha)"] \\
f(C(\alpha)^+) \arrow[r] & f(B^+) \otimes f(A^-) .
\end{tikzcd}$$

These diagrams, together with the theorem in the case $A = 0$, show that the right vertical composition from top to bottom in the following diagram is the map $\mathrm{C}^{\sigma}(f)(0^L)$ and that the theorem follows if the whole diagram is commutative. The top and bottom squares commute by definition, and the middle square commutes by Theorem A.22.

$$\begin{tikzcd}
\mathrm{C}^{\sigma}(f)(A)\otimes \mathrm{C}^{\sigma}(f)(TA) \arrow[r] & 1 \\
f(A^+)\otimes f(B^-)\otimes f^{\sigma}(A^-)\otimes f^{\sigma}(B^-)\otimes f(A^-)\otimes f^{\sigma}(A^+) \arrow[u, "\sigma(\{2,4\})"] \arrow[r, "\sigma(\{3,5\})"] \arrow[d, "\tilde{f}(\alpha)\otimes 1\otimes 1\otimes 1\otimes 1"'] & f(A^+)\otimes f(B^-)\otimes f^{\sigma}(B^-)\otimes f^{\sigma}(A^+) \arrow[u] \arrow[d, "\varepsilon(f(A^-))\tilde{f}(\alpha)\otimes 1\otimes 1"] \\
f(B^+)\otimes f(A^-)\otimes f^{\sigma}(A^-)\otimes f^{\sigma}(B^-)\otimes f(A^-)\otimes f^{\sigma}(A^+) \arrow[r, "\sigma(\{2,3\})"] \arrow[d, "\sigma(\{2,3\})"'] & f(B^+)\otimes f(A^-)\otimes f^{\sigma}(B^-)\otimes f^{\sigma}(A^+) \arrow[d] \\
\mathrm{C}^{\sigma}(f)(B)\otimes \mathrm{C}^{\sigma}(f)(TA) \arrow[r] & \mathrm{C}^{\sigma}(f)(C(\alpha))
\end{tikzcd} \qquad \square$$

A. Picard Categories

We recall the definition of an associative and commutative (AC) tensor category P (see also [Ke; L; S]).

DEFINITION A.1. An *AC tensor category* $P = (P, \otimes, \phi, \psi)$ consists of a category P, a bifunctor $\otimes\colon P \times P \to P$, and two natural isomorphisms,

$$\phi(X, Y, Z)\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),$$

$$\psi(X, Y)\colon X \otimes Y \to Y \otimes X,$$

that satisfy the following two axioms.

Pentagonal Axiom. The following diagram is commutative:

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \otimes W \arrow[dl, "\phi\otimes 1"'] \arrow[dr, "\phi"] & \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[d, "\phi"'] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "\phi"] \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "1\otimes\phi"] & & X \otimes (Y \otimes (Z \otimes W)).
\end{tikzcd}$$

Hexagonal Axiom. The following diagram is commutative:

$$
\begin{tikzcd}[column sep=small]
 & (X \otimes Y) \otimes Z \arrow[dl, "\phi"'] \arrow[dr, "\psi \otimes 1"] & \\
X \otimes (Y \otimes Z) \arrow[d, "\psi"] & & (Y \otimes X) \otimes Z \arrow[d, "\phi"] \\
(Y \otimes Z) \otimes X \arrow[dr, "\phi"] & & Y \otimes (X \otimes Z) \arrow[dl, "1 \otimes \psi"] \\
 & Y \otimes (Z \otimes X). &
\end{tikzcd}
$$

REMARK A.2. The general coherence theorem is proved in [L]. It states that all diagrams involving just the ϕs and the ψs commute. This means that, if I and J are disjoint finite sets and if $\{X_i\}_{i \in I \cup J}$ is an indexed set of objects of P, then it makes sense to talk about the "object" $\bigotimes_{i \in I} X_i$ and the unique isomorphism induced by ϕ and ψ: $\bigotimes_{i \in I} X_i \otimes \bigotimes_{i \in J} X_i \to \bigotimes_{i \in I \cup J} X_i$. For this reason we will often drop parentheses and names of these canonical morphisms in diagrams.

DEFINITION A.3. An AC tensor functor $h = (h_1, h_2) \colon (P', \otimes', \phi', \psi') \to (P, \otimes, \phi, \psi)$ consists of a functor $h_1 \colon P \to P'$ and a natural isomorphism $h_2(X, Y) \colon h_1(X \otimes' Y) \to h_1(X) \otimes h_1(Y)$ that together make the following two diagrams commutative:

$$
\begin{tikzcd}
h_1((X \otimes' Y) \otimes' Z) \arrow[r, "h_2(X \otimes' Y, Z)"] \arrow[d, "h_1(\phi')"] & h_1(X \otimes' Y) \otimes h_1(Z) \arrow[r, "h_2(X,Y) \otimes 1"] & (h_1(X) \otimes h_1(Y)) \otimes h_1(Z) \arrow[d, "\phi"] \\
h_1(X \otimes' (Y \otimes' Z)) \arrow[r, "h_2(X, Y \otimes' Z)"] & h_1(X) \otimes h_1(Y \otimes' Z) \arrow[r, "1 \otimes h_2(Y,Z)"] & h_1(X) \otimes (h_1(Y) \otimes h_1(Z));
\end{tikzcd}
$$

$$
\begin{tikzcd}
h_1(X \otimes' Y) \arrow[r, "h_2(X,Y)"] \arrow[d, "h_1(\psi')"] & h_1(X \otimes h_1(Y) \arrow[d, "\psi"] \\
h_1(Y \otimes' X) \arrow[r, "h_2(Y,X)"] & h_1(X) \otimes (h_1(Y).
\end{tikzcd}
$$

DEFINITION A.4. If $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are AC tensor functors from an AC tensor category P' to an AC tensor category P, then an AC natural transformation $\eta \colon f_1 \to g_1$ is an AC natural transformation if the following diagram is commutative:

$$\begin{tikzcd} f_1(X \otimes' Y) \arrow[r, "f_2(X,Y)"] \arrow[d, "\eta(X \otimes' Y)"] & f_1(X) \otimes f_1(Y) \arrow[d, "\eta(X) \otimes \eta(Y)"] \\ g_1(Y \otimes' X) \arrow[r, "g_2(Y,X)"] & g_1(X) \otimes (g_1(Y). \end{tikzcd}$$

DEFINITION A.5. If $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are AC tensor functors from an AC tensor category P' to an AC tensor category P, then we define the tensor product $f \otimes g$ as follows:

$$(f \otimes g)_1(X) = f_1(X) \otimes g_1(X),$$

$$(f \otimes g)_2(X, Y) = (1 \otimes \psi' \otimes 1) \circ (f_2(X, Y) \otimes g_2(X, Y)).$$

PROPOSITION A.6. *The AC tensor functors* (*from an AC tensor category P' to an AC tensor category P*) *and the AC natural transformations form a category that we denote by* $\mathrm{Hom}^{\otimes}(P', P)$. *The tensor product together with ϕ and ψ induce on* $\mathrm{Hom}^{\otimes}(P', P)$ *the structure of an AC tensor category.*

Proof. It follows from the general coherence theorem that ϕ and ψ induce natural transformations that satisfy both the pentagonal and the hexagonal axiom. □

DEFINITION A.7. A unit (U, δ^L, δ^R) in a commutative tensor category $(P, \otimes, \phi, \psi)$ consists of an object U together with two natural isomorphisms,

$$\delta^L(X)\colon U \otimes X \to X,$$

$$\delta^R(X)\colon X \otimes U \to X,$$

that satisfy the following axioms.

Unit Axioms. The following three diagrams are commutative:

$$\begin{tikzcd} X \otimes U \arrow[dr, "\delta^R(X)"] \arrow[dd, "\psi(X,U)"'] & \\ & X; \\ U \otimes X \arrow[ur, "\delta^L(X)"'] & \end{tikzcd} \quad \begin{tikzcd} (X \otimes Y) \otimes U \arrow[dr, "\delta^R(X \otimes Y)"] \arrow[dd, "\phi(X,Y,U)"'] & \\ & X \otimes Y; \\ X \otimes (Y \otimes U) \arrow[ur, "1 \otimes \delta^R(Y)"'] & \end{tikzcd} \quad \begin{tikzcd} (X \otimes U) \otimes Y \arrow[dr, "\delta^R(X) \otimes 1"] \arrow[dd, "\phi(X,U,Y)"'] & \\ & X \otimes Y. \\ X \otimes (U \otimes Y) \arrow[ur, "1 \otimes \delta^L(Y)"'] & \end{tikzcd}$$

REMARK A.8. It is shown in [S, 2.4.1] that the left diagram is redundant and that $\psi(U, U) = 1_{U \otimes U}$. For any two units U and U', there is a unique isomorphism $\gamma(U', U)\colon U \to U'$ such that, for any X, we have $\delta'^R(X) \circ 1 \otimes \gamma(U', U) = \delta^R(X)$. An object U together with an isomorphism $\delta\colon U \otimes U \to U$ is called a *reduced unit*. In [S, 2.2.5.1] it is shown that, for any reduced unit (U, δ), there is a unique unit (U, δ^L, δ^R) such that $\delta(U) = \delta^L(U) = \delta^R(U)$. Furthermore, if $J \subseteq I$ are finite sets and if $\{X_i\}_{i \in I}$ is an indexed set of objects of P such that $(X_j, X_j \otimes X_j \to X_j)$ is a unit for each $j \in J$, then we have a unique "cancellation isomorphism" $\bigotimes_{i \in I} X_i \to \bigotimes_{i \in I \setminus J} X_i$.

For any unit U, $\mathrm{End}(U)$ acts via δ on any object of P. In particular, $\mathrm{End}(U)$ acts on U and endows $\mathrm{End}(U)$ with two operations. The naturality of δ and the functoriality of $\otimes$ show that the two operations are identical, that $\mathrm{End}(U)$ is a commutative monoid, and that $\mathrm{Aut}(U)$ is an abelian group.

COROLLARY A.9. *If P and P' are AC tensor categories and P has units, then any assignment $X \mapsto u_1(X)$, $X \mapsto \delta(X)\colon u_1(X) \otimes u_1(X) \to u_1(X)$ of a unit in P to every object X of P' defines a unique unit $(u, \delta\colon u \otimes u \to u)$ in* $\mathrm{Hom}^{\otimes}(P', P)$.

DEFINITION A.10. A *right inverse* to an object X in a tensor category P consists of an object Y and an isomorphism $\rho\colon X \otimes Y \to U$ with U a unit. We say that an object X of a tensor category P is *invertible* if a right inverse exists. For any right inverse $\rho\colon X \otimes Y \to U$, we have an associated left inverse $\rho \circ \psi(Y, X)\colon Y \otimes X \to U$.

REMARK A.11. For an invertible object X, we derive (via Y and ρ) isomorphisms of monoids $\mathrm{End}(X) \approx \mathrm{End}(X \otimes Y) \approx \mathrm{End}(U)$, and these isomorphisms do not depend on the choice of Y and ρ.

From now on we shall consider only tensor categories that have units, and we pick a particular unit $(1, \delta^L, \delta^R)$.

DEFINITION A.12. For any invertible object X, the automorphism $\psi\colon X \otimes X \to X \otimes X$ induces an automorphism of order 2 of $\mathrm{Aut}(1)$ that we call $\varepsilon(X)$.

PROPOSITION A.13. *The assignment $X \mapsto \varepsilon(X)$ is a function* $[\mathrm{Inv}\, P] \to \mathrm{Aut}(1)$ *from isomorphism classes of invertible objects of P to the automorphism group of the identity object. Furthermore, $\varepsilon(1) = 1$ and $\varepsilon(X \otimes Y) = \varepsilon(X)\varepsilon(Y)$.*

PROPOSITION A.14. *If $\rho\colon X \otimes Y \to 1$ is an isomorphism, then the composition*

$$Y \xrightarrow{(\delta^R(Y))^{-1}} Y \otimes 1 \xrightarrow{(1\otimes\rho)^{-1}} Y \otimes X \otimes Y \xrightarrow{\psi\otimes 1} X \otimes Y \otimes Y \xrightarrow{\rho\otimes 1} 1 \otimes Y \xrightarrow{\delta^L(Y)} Y$$

is $\varepsilon(Y)1_Y$.

Proof. The proof can be seen from the following diagram:

$$\begin{tikzcd}
Y \otimes X \otimes Y \arrow[rr, "\psi\otimes 1"] \arrow[d, equal] & & X \otimes Y \otimes Y \arrow[d, equal] \\
Y \otimes (X \otimes Y) \arrow[r, "\psi"] \arrow[d, "1\otimes\rho"] & (X \otimes Y) \otimes Y \arrow[r, "1\otimes\psi"] \arrow[d, "\rho\otimes 1"] & X \otimes Y \otimes Y \arrow[d, "\rho\otimes 1"] \\
Y \otimes 1 \arrow[r, "\psi"] \arrow[d] & 1 \otimes Y \arrow[r, "1\otimes\varepsilon(Y)1"] \arrow[d] & 1 \otimes Y \arrow[d] \\
Y \arrow[r, equal] & Y \arrow[r, "\varepsilon(Y)1"] & Y.
\end{tikzcd}$$

□

DEFINITION A.15. A *Picard category* is an AC tensor category with units and with the properties that every object is invertible and every morphism is an isomorphism.

DEFINITION A.16. An *inverse structure* $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ on a Picard category P consists of an AC tensor functor $(\sigma_1, \sigma_2)\colon P \to P$ and an AC natural isomorphism $\sigma_3\colon \mathrm{id} \otimes \sigma_1 \to 1$.

REMARK A.17. Note that an inverse structure is simply an inverse to the identity functor in $\mathrm{Hom}^{\otimes}(P, P)$. Any two inverse structures are canonically isomorphic, and an inverse structure is uniquely determined by a choice of an inverse $\sigma_3\colon X \otimes \sigma_1(X) \to 1$ for every object X of P.

The rest of this section will be devoted to a theorem that is an elaboration of Proposition A.14. We fix a Picard category P and an inverse structure σ on P. We will make use of the notation $X^\sigma = \sigma_1(X)$ and $\sigma(X)\colon X \otimes X^\sigma \to 1$.

DEFINITION A.18. We will call a pair of objects $\{X, Y\}$ of P an *inverse couple* if $X = Y^\sigma$ or $Y = X^\sigma$.

DEFINITION A.19. Let $\{X_i\}_{i\in I}$ be a finite indexed set of objects of P, and let $S = \{\{s_1, s_1'\}, \{s_2, s_2'\}, \dots, \{s_k, s_k'\}\}$ be a set of pairwise disjoint pairs of indexes of I such that each pair $\{X_{s_i}, X_{s_i'}\}$ is an inverse couple. By the naturality of all the maps generated by ϕ, ψ, and δ, it follows that S determines a unique isomorphism

$$\bigotimes_{i\in I} X_i \longrightarrow \bigotimes_{i\in I\setminus\bigcup S} X_i,$$

which we call the *contraction* defined by S and denote $\sigma(S)$.

Consider again a finite indexed set $\{X_i\}_{i\in I}$ of objects of P, and let $S = \{\{s_1, s_1'\}, \{s_2, s_2'\}, \dots, \{s_k, s_k'\}\}$ and $T = \{\{t_1, t_1'\}, \{t_2, t_2'\}, \dots, \{t_k, t_k'\}\}$ be two disjoint sets of pairwise disjoint pairs of indexes of I such that, for each i, each pair $\{X_{s_i}, X_{s_i'}\} = \{X_{t_i}, X_{t_i'}\}$ is an inverse couple. From these conditions we can conclude that (i) the graph with vertices $S \cup T$ and edges the set $\{\{x, y\} \mid x \cap y \neq \emptyset\}$ is bipartite and (ii) the connected components $C \subseteq \mathrm{P}(S \cup T)$ are either cycles or chains. We let C_0 be the set of cycles, C_1 the odd chains, and $C_2 = C_2^S \cup C_2^T$ the even chains. The set of even chains both starts and ends in either S or T. To each chain or cycle $c \in C$, there corresponds a unique inverse couple $\{X(c), X^\sigma(e)\}$.

DEFINITION A.20. Let $\{X_i\}_{i\in I}$, S, and T be as before. We say that a one-to-one mapping $\beta\colon C_2^T \to C_2^S$ is a *perfect matching* if, for each chain $e \in C_2^T$, the inverse couple corresponding to e is the same as the inverse couple corresponding to $\beta(e)$. For any perfect matching β, we define the mapping $\tilde\beta\colon I \setminus \bigcup S \to I \setminus \bigcup T$ as follows. If $i \in I \setminus \left(\bigcup S \cup \bigcup T\right)$ then we let $\tilde\beta(i) = i$; if $i \in x \in o \in C_1$ then we let $\tilde\beta(i)$ be the unique index $j \in \bigcup o \setminus \bigcup T$; if $i \in x \in e \in C_2^T$ then we let $\tilde\beta(i)$ be the unique index $j \in \bigcup \beta(e) \setminus \bigcup T$ for which $X_i = X_j$. Since $\tilde\beta$ is one-to-one and since $X_i = X_{\tilde\beta(i)}$, we obtain an isomorphism that we give the same name:

$$\bigotimes_{i\in I\setminus\bigcup S} X_i \xrightarrow{\ \tilde\beta\ } \bigotimes_{i\in I\setminus\bigcup T} X_i.$$

Since for any object X we have $\varepsilon(X) = \varepsilon(X^\sigma)$, it follows that the map ε is well-defined on $S \cup T$.

DEFINITION A.21. We define ε on the connected components of $S \cup T$ as follows:

$$\varepsilon(c) = \begin{cases} \varepsilon(x) & \text{for any } x \in c \text{ if } c \in C_i \text{ and } \sharp c \equiv i \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Finally, we define $\varepsilon(S, T) = \prod_{c \in C} \varepsilon(c)$.

THEOREM A.22. *With notation as in Definitions A.20 and A.21, for any perfect matching β, the following diagram commutes:*

$$\begin{tikzcd} & \bigotimes_{i \in I} X_i \arrow[dl, "\sigma(S)"'] \arrow[dr, "\sigma(T)"] & \\ \bigotimes_{i \in I \setminus \bigcup S} X_i \arrow[rr, "\varepsilon(S,T)\tilde{\beta}"] & & \bigotimes_{i \in I \setminus \bigcup T} X_i . \end{tikzcd}$$

Proof. By naturality we can reduce the general case to three special ones. The first case is when $C = C_0 = \{c\}$ and $\sharp c = 2k$. We can further assume that $I = \{1, 2, \ldots, 2k\}$, $X_i = X$ for $1 \le i \le k$, $X_i = X^\sigma$ for $k + 1 \le i \le 2k$, $S = \{\{1, k\}, \{2, k+1\}, \ldots, \{k, 2k\}\}$, and $T = \{\{2, k\}, \{3, k+1\}, \ldots, \{k, 2k-1\}, \{1, 2k\}\}$. Let τ be the permutation defined by $\tau(i) = i + 1$ for $1 \le i \le k - 1$ and $\tau(k) = 1$. Then τ determines an isomorphism $\psi(\tau)$ of the tensor product, and the following diagram commutes:

$$\begin{tikzcd} \bigotimes_{i \in I} X_i \arrow[r, "\psi(\tau)"] \arrow[d, "\sigma(S)"'] \arrow[dr, "\sigma(T)"] & \bigotimes_{i \in I} X_i \arrow[d, "\sigma(S)"] \\ 1 \arrow[r, "\varepsilon(c)"] & 1. \end{tikzcd}$$

The other cases are that of a single odd chain and that of two even chains; we leave the proof of these cases to the reader. $\square$

B. Letter from A. Grothendieck

Buffalo
May 19, 1973

Dear Finn Knudsen,

Mumford sent me your notes on the determinant of perfect complexes, asking me to write you some comments, if I have any. Indeed I do have several—except for the obvious one that it is nice to have written up with details at least *one* full construction of that damn functor! I did not enter into the technicalities of your construction, which perhaps will allow [me] to get a better comprehension of the main result itself. The main trouble with your presentation seems to me that the bare statement of the main result looks rather mysterious and not “natural” at all, despite your claim on page 3b! The mysterious character is of course included in the alambicated sign of definition 1.1. Here two types of criticism come to mind:

1) The sign looks complicated—are there not simpler sign conventions for getting a nice theory of $\det^*$ and its variance? It seems to me that Deligne wrote down a system that really did look natural at every stage—however he never wrote down the explicit construction, as far as I know, and the chap who had undertaken to do so gave up in disgust after a year or two of letting the question lie around and rot!

2) Even granted that your conventions are as simple or simpler than other ones, the very fact that they are so alambicated and technical calls for an elucidation, somewhat of the type you give on page 3b with those ε_i’s. That is, one would like to *define* first what any theory of $\det^*$ should be (with conventions of sign as yet unspecified), stating say something like a *uniqueness theorem* for every given system of signs chosen for canonical isomorphisms, and moreover *characterizing* those systems of sign conventions which allow for an existence theorem—which will include the existence of at least one such system of signs. If one has good insight into all of them, it will be a matter of taste and convenience for the individual mathematician (or the situation he has to deal with in any instance) to make his own choice!

A second point is the introduction of such evidently superfluous assumptions like working on Noetherian (!) schemes, whereas the construction is clearly so general as to work, say, over any ringed space and even ringed topos—and of course it will be needed in this generality, for instance on analytic spaces, or on schemes with groups of automorphisms acting, etc. It’s just a question of some slight extra care in the writing up. It is clear in any case that the question reduces to defining $\det^*$ for strictly perfect complexes (i.e. which are free of finite type in every degree), and for homotopy classes of homotopy equivalences between such complexes, as well as for short exact sequences of such complexes. (NB! One may wish to deal, more generally, in the Illusie spirit, with strictly perfect complexes filtered—by a filtration which is finite but possibly not of level two—by subcomplexes with strictly perfect quotients.) Now this allows [us] to restate the whole thing in a more general setting, which could make the theory more transparent, namely:

An additive category $\mathcal{C}$ (say free (or projective) modules of finite type over a commutative ring A) is given, as well as a category $\mathcal{P}$ which is a groupoid, endowed with an operation $\otimes$ together with associativity, unity and commutativity data, satisfying the usual compatibilities (see for instance Saavedra’s thesis in Springer’s lecture notes) and with all objects “invertible”. In the example for $\mathcal{C}$, we take for $\mathcal{P}$ invertible $\mathbb{Z}$-graded modules over A, with tensor product, the commutative law $L \otimes L' \simeq L' \otimes L$ involving the Koszul sign $(-1)^{dd'}$ where d and d' are the degrees of L and L' respectively. We are interested in functors (or a given functor) $f\colon (\mathcal{C}, \mathrm{isom}) \to \mathcal{P}$, together with a functorial isomorphism $f(M+N) \simeq f(M) \otimes f(N)$, compatible with the associativity and commutativity data (cf. Saavedra for this notion of a $\otimes$); for sintance, in the example chosen, we take $f(M) = \det^*(M)$, the determinant module where $*$ stands for the degree which we put on the determinant module (our convention will be to put the degree equal to the rank of M, which will imply that our functor is indeed compatible with the commutativity data). It can be shown (this was done by a North Vietnamese mathematician, Sinh Hoang Xuan) that given $\mathcal{C}$ (indeed any associative and commutative $\otimes$-category would do), there exists a universal way of sending $\mathcal{C}$ to $\mathcal{P}$ as above—in the case considered, this category can be called the category of "stable" projective modules over A, and its main invariants (isomorphism classes of objects, and automorphisms of the unit object) are just the invariants $\mathrm{K}^0(A)$ and $\mathrm{K}^1(A)$ of myself and Dieudonné-Bass; but this existence of a universal situation is irrelevant for the technical problem to come. Now consider the category $K = \mathrm{K}^b(\mathcal{C})$, of bounded complexes of $\mathcal{C}$, up to homotopy. It is a triangulated category,† and as such we can define the notion of a $\otimes$-functor from K into $\mathcal{P}$; it's first of all a $\otimes$-functor for the additive structure of K (the internal composition of K being $\otimes$), but with moreover an extra structure … giving isomorphisms $g(M) \simeq g(M') \otimes g(M'')$ whenever we have an exact triangle $M' \to M \to M'' \to M'$. This should of course satisfy various conditions, such as functoriality with respect to the triangle, case of split exact triangle ($M = M' \otimes M''$), case of the triangle obtained by completing a quasi-isomorphism $M' \to M$, and possibly also a condition of compatibility in the case of an exact triangle of triangles. (I guess Deligne wrote down the reasonable axioms some day; it may be more convenient to work with the filtered K-categories of Illusie, using of course finite filtrations that split in the present context). Of course if we have such a $g\colon K \to \mathcal{P}$, taking its "restriction" to $\mathcal{C}$ we get an $f\colon \mathcal{C} \to \mathcal{P}$. The beautiful statement to prove would then be that conversely, every given f extends, uniquely up to isomorphism, to a g, in other terms, that the restriction functor from the category of g's to the category of f's is an equivalence. The whole care, for such a statement, will of course be to give the right set of "sign conventions" for defining admissible g's (that is, compatibilities between the two or three structures on the set of $g(M)$'s—which in fact all can be reduced to giving the isomorphisms attached to exact triangles). In this general context, the group of signs ± 1 is replaced by the subgroup of elements of order 2 of the group $\mathrm{K}^1(\mathcal{P}) = \mathrm{Aut}(1_{\mathcal{P}})$ (which is always a commutative group). The "sign map" $n \to (-1)^n$ from the group of degrees to the group of signs is replaced here by a canonical map $\mathrm{K}^0(\mathcal{P})$($=$ group of isomorphism classes of $\mathcal{P}$) $\to \mathrm{K}^1(\mathcal{P})$, associating to every L in $\mathcal{P}$ the symmetry automorphism of $L \otimes L$ (viewed as coming from an automorphism of the unit object by tensoring with $L \otimes L$). What puzzles me a little is that apparently, you have not been able to define g in

† Be careful that one must take the term "triangulated category" in a slightly more precise sense than in Verdier's notes, the "category of triangles" being something more precise than a mere category of distinguished diagrams in K. We have a functor from the former to the latter, but it is not even a faithful one. (Illusie's treatment in terms of filtered complexes, in his Springer lecture notes, is a good reference.) It is only with respect to the category of "true" triangles that the isomorphism $g(M) \simeq g(M') \otimes g(M'')$ will be functorial. For instance, if we have an *automorphism* of a triangle, inducing u, u', and u'' upon M, M', and M'', then functoriality is expressed by the relation $\det u = \det u' \det u''$ (which implies, replacing u by $\mathrm{id} + tu$ with t an indeterminate, that $\mathrm{Tr}\, u = \mathrm{Tr}\, u' + \mathrm{Tr}\, u''$), but this relation may become *false* if we are not careful to take automorphisms of true triangles instead of taking mere automorphisms of diagrams.

terms intrinsic to the triangulated category $K = \mathrm{K}^b(\mathcal{C})$—the signs you introduce in 1.1 do depend on the actual complexes, not only on their homotopy classes. I guess the whole trouble comes from the order in which we write any given tensor product in $\mathcal{P}$, in describing $\det^*(M^\bullet)$ we had to choose such an order rather arbitrarily, and it is passing from one such to another that involves "signs".

If $\mathcal{C}$ is an *abelian* category, there should be a variant of the previous theory, putting in relations on the $\otimes$-functors $f : \mathcal{C} \to \mathcal{P}$ together with the extra structure of isomorphisms $f(M) \simeq f(M') \otimes f(M'')$ for all short exact sequences $0 \to M' \to M \to M'' \to 0$ satisfying a few axioms, and $\otimes$-functors $g : \mathrm{D}^b(\mathcal{C}) \to \mathcal{P}$. There should also be higher dimensional analogues involving $\mathcal{P}$'s that are n-categories instead of mere 1-categories, and hence involving (implicitly at least) the higher K-invariants $\mathrm{K}^i(\mathcal{C})$ $(i \geq 0)$. But of course, first of all the case of the relation between $\mathcal{C}$ and $\mathrm{K}^b(\mathcal{C})$ in the simplest case should be elucidated!

I am finishing this letter at the forum where I have no typewriter. I hope you can read the handwriting!

Best wishes,
A. Grothendieck

References

[D] P. Deligne, *Le déterminant de la cohomologie,* Current trends in arithmetical algebraic geometry (Arcata, CA, 1985), Contemp. Math., 67, pp. 93–177, Amer. Math. Soc., Providence, RI, 1987.

[G] P. Gabriel, *Des catégories abélièn,* Bull. Soc. Math. France 90 (1962), 323–448.

[Ke] G. Kelly, *On Mac Lane's conditions for coherence of natural associativities, commutativities, etc.,* J. Algebra 1 (1964), 397–402.

[KM] F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves I. Preliminaries on "det" and "Div",* Math. Scand. 39 (1976), 19–55.

[L] S. Mac Lane, *Natural associativity and commutativity,* Rice Univ. Stud. 49 (1963), 28–46.

[Q] D. Quillen, *Higher algebraic K-theory I,* Lecture Notes in Math., 341, pp. 85–147, Springer-Verlag, Berlin, 1973.

[R] A. Ranicki, *The algebraic theory of torsion I: Foundations,* Algebraic and geometric topology (New Brunswick, NJ, 1983), Lecture Notes in Math., 1126, pp. 199–237, Springer-Verlag, New York, 1985.

[S] N. Saavedra, *Catégories tannakiennes,* Lecture Notes in Math., 265, Springer-Verlag, New York, 1982.

[TT] R. W. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories,* The Grothendieck Festschrift vol. III (P. Cartier et al., eds.), Progr. Math., 88, pp. 247–435, Birkhäuser, Boston, 1990.

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