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CR Maps and Point Lie Transformations

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1. Introduction

This paper concerns the following well-known result (see Chern and Moser [3]) of geometric complex analysis: Any biholomorphic map between two real analytic Levi nondegenerate hypersurfaces in $\mathbb{C}^{n+1}$ $(n > 0)$ is uniquely determined by its 2-jet at fixed point. Moser's proof is based on his general theory of normal forms for Levi nondegenerate hypersurfaces. We present a new geometric approach to the problem that allows us to deduce Moser's result from a general assertion concerning point Lie transformations of certain second-order PDE systems. The Segre family of a Levi nondegenerate hypersurface is a general solution of such a system, and every biholomorphism of such a hypersurface is a Lie symmetry of this system. Our approach is mostly inspired by the ideas of Webster [10] as well as the works of Diederich and Webster [5] and Diederich and Fornæss [4].

Our main result is the following.

Theorem 1.1. *Any holomorphic point transformation between two holomorphic completely integrable systems $D^{(2)}u = F(x, u, D^{(1)}u)$ and $D^{(2)}u = \hat{F}(x, u, D^{(1)}u)$ with one dependent variable and n independent variables is determined by its 2-jet at a fixed point. The set of all such transformations can be parameterized by at most $n^2 + 4n + 3$ complex parameters.*

The terminolgy will be explained in the next section. The infinitesimal version of this theorem has been established by the author in [9].

We stress that PDE systems defining Segre families of real analytic hypersurfaces form a highly special subclass of PDE systems considered in our result. From this point of view, the study of point transformations of PDE systems is a substantially more general problem. We hope that our approach will be useful for both the CR geometry and the geometry of differential equations.

2. Preliminaries

In this section we establish a correspondence between the geometry of real analytic CR structures and completely integrable PDE systems.

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A. Jet Bundles, Point Transformations, and Prolongations

Denote by $J^r(n,m)$ the manifold of r-jets of holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ and by $j^r_p(\phi)$ the r-jet of a map ϕ at a point p. Let $x=(x_1,\dots,x_n)$ and $u=(u^1,\dots,u^m)$ be complex coordinates in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. We define the natural coordinates on $J^r(n,m)$ as follows. Set $u^{(1)}=(u^1_1,\dots,u^1_n,\dots, u^m_1,\dots,u^m_n)$, ..., $u^{(s)}=(u^j_\alpha)$ with $j=1,\dots,m$ and $\alpha=(\alpha_1,\dots,\alpha_s)$, $\alpha_1\le\alpha_2\le\dots\le\alpha_s$. The chart $j^r_p(\phi)\mapsto(x_j,u^k,u^{(1)},\dots,u^{(r)})$ is defined by $x_j=p$, $u=\phi(p)$,

$$u^j_\alpha=\frac{\partial^s\phi^j}{\partial x_{\alpha_1}\dots\partial x_{\alpha_s}}(p),\quad 1\le s\le r,\ \alpha=(\alpha_1,\dots,\alpha_s),\ \alpha_1\le\dots\le\alpha_s.$$

When $m=1$ we write simply $u^{(1)}=(u_1,\dots,u_n)$ and so forth.

A (germ of) biholomorphism $f\colon\mathbb{C}^n\times\mathbb{C}^m\to\mathbb{C}^n\times\mathbb{C}^m$, $f\colon(x,u)\mapsto(x^*,u^*)$ of the base space lifts canonically to a fiber-preserving biholomorphism $f^{(r)}$: $J^r(n,m)\to J^r(n,m)$. If $u=\phi(x)$ is a holomorphic function near p with $q=\phi(p)$, let $u^*=\phi^*(x^*)$ be its image under f understood in the following sense: the graph of ϕ^* is the image of the graph of ϕ under f near the point $(p^*,q^*)=f(p,q)$. Then the jet $j^r_{p^*}(\phi^*)$ is by definition the image of $j^r_p(\phi)$ under $f^{(r)}$. The map $f^{(r)}$ is called the *r-prolongation* of f. The prolongation is defined only if f takes the graph of any holomorphic function $u=u(x)$ near p to the graph of some function $u^*=u^*(x^*)$. We call biholomorphic maps satisfying this condition *point* (*Lie*) *transformations* of the base. A prolongation of any order of a point transformation can be computed quite explicitly by recurrence (see e.g. [1]). For the case $m=1$, setting $x^*=X(x,u)=(X^1,\dots,X^n)$ and $u^*=U(x,u)$ yields

$$\begin{pmatrix}u^*_1\\ \vdots\\ u^*_n\end{pmatrix}=\begin{pmatrix}U_1\\ \vdots\\ U_n\end{pmatrix}=A^{-1}\begin{pmatrix}D_1U\\ \vdots\\ D_nU\end{pmatrix}\tag{1}$$

and

$$\begin{pmatrix}u^*_{i_1\dots i_{k-1}1}\\ \vdots\\ u^*_{i_1\dots i_{k-1}n}\end{pmatrix}=\begin{pmatrix}U_{i_1\dots i_{k-1}1}\\ \vdots\\ U_{i_1\dots i_{k-1}n}\end{pmatrix}=A^{-1}\begin{pmatrix}D_1U_{i_1\dots i_{k-1}}\\ \vdots\\ D_nU_{i_1\dots i_{k-1}}\end{pmatrix},\tag{2}$$

where $A=(D_iX^j)_{i,j=1,\dots,n}$ (i denotes a row) and D_i is the total derivative operator:

$$D_i=\frac{\partial}{\partial x_i}+u_i\frac{\partial}{\partial u}+\sum_j u_{ij}\frac{\partial}{\partial u_j}+\cdots$$

(for the convenience of notation in this formula we follow the convention $U_{i_ki_s}=U_{i_si_k}$ and $u_{i_ki_s}=u_{i_si_k}$ for any indices).

B. Transformations of PDE Systems

In this paper we deal with holomorphic completely integrable second-order PDE systems $(\mathcal{S})$ of the form $D^{(2)}u=F(x,u,D^{(1)}u)$ with n independent variables x

and one dependent variable u (here and below we denote by $D^{(j)}h$ a vector function formed by all partial derivatives of f of order j). The *integrability condition* means that a distribution on the tangent bundle of the jet space $J^1(n,1)$ defined by the differential forms $du_i - \sum_j F_{ij}(x,u,u^{(1)})dx_j$ and $du - \sum_i u_i dx_i$ is completely integrable. This class of PDE systems was first studied by Chern [2] from the point view of the general theory of G-structures. In the special case $n = 1$, the integrability condition always holds and we simply get a general second-order ODE.

Let $(\mathcal{S})$ and $(\hat{\mathcal{S}})$ be two systems of this class. Then they define naturally complex subvarieties $(\mathcal{S})_2$ and $(\hat{\mathcal{S}})_2$ in the jet space $J^2(n,1)$ obtained by replacing the derivatives of dependent variables with the corresponding natural coordinates on $J^2(n,1)$. Explicitly, $(\mathcal{S})_2$ is defined by $u_{ij} = F_{ij}(x,u,u^{(1)})$ and $(\hat{\mathcal{S}})_2$ is defined by $u_{ij} = \hat{F}_{ij}(x,u,u^{(1)})$. A point transformation f is called *a point transformation between* $(\mathcal{S})$ *and* $(\hat{\mathcal{S}})$ if it takes the graph of any solution of $(\mathcal{S})$ to the graph of a solution of $(\hat{\mathcal{S}})$. The integrability condition implies (by the Frobenius theorem) that, for any point $P \in (\mathcal{S})_2$ with the projection $p \in \mathbb{C}^n$, there exists a holomorphic solution $u = u(x)$ of the system $(\mathcal{S})$ such that $P = j^2_p(u)$. Using this fact, it is easy to prove that f is a point transformation between our systems if and only if the 2-prolongation $f^{(2)}$ of f is a biholomorphism between $(\mathcal{S})_2$ and $(\hat{\mathcal{S}})_2$ (see [6]). In the case where $(\mathcal{S})$ coincides with $(\hat{\mathcal{S}})$, a pointwise transformation is just the classical Lie symmetry of $(\mathcal{S})$.

C. Segre Varieties and PDE Systems

Let M be a real analytic Levi nondegenerate hypersurface in $\mathbb{C}^{n+1}$ through the origin. We use the notation $Z = (z,w) \in \mathbb{C}^n \times \mathbb{C}$ for coordinates in $\mathbb{C}^{n+1}$.

After a biholomorphic change of coordinates in a neighborhood of the origin, M is given by the equation $\left\{r = w + \bar{w} + \sum_{j=1}^n \varepsilon_j z_j \bar{z}_j + R(Z,\bar{Z}) = 0\right\}$, where $\varepsilon_j = 1$ or -1 and $R = o(|Z|^2)$. For every point $\zeta = (\eta_1,\dots,\eta_n,\omega)$, the corresponding Segre variety $Q(\zeta)$ is defined by $w + \omega + \sum_{j=1}^n \varepsilon_j z_j \eta_j + R(Z,\zeta) = 0$. If we consider the variables $x_j = z_j$ as independent and the variable $w = u(x)$ as the dependent one, then this equation can be rewritten in the form

$$u + \omega + \sum_{j=1}^n \varepsilon_j x_j \eta_j + R(x,\zeta) = 0 \qquad (*)$$

(after an application of the implicit function theorem in order to remove u from R). Taking the derivatives in x_k, we obtain

$$u_{x_k} + \varepsilon_k \eta_k + R_{x_k}(x,\zeta) = 0, \quad k = 1,\dots,n. \qquad (**)$$

The equations $(*)$ and $(**)$, together with the implicit function theorem, imply that $\zeta = \zeta(x,u,u_{x_1},\dots,u_{x_n})$ is a holomorphic function; taking again the partial derivatives in x_j in $(**)$, we obtain the following completely overdetermined second-order holomorphic PDE system:

$$u_{x_j x_k} = F_{jk}(x,u,D^{(1)}u), \quad j,k = 1,\dots,n. \qquad (\mathcal{S}_M)$$

We point out that this system necessarily satisfies the integrability condition. (This follows immediately from the representation $(*)$ of the general solution of $(\mathcal{S}_M)$ that the distribution defined by the corresponding differential forms on the tangent bundle of $J^1(n,1)$ is completely integrable and so satisfies the Frobenius condition.)

If f is a local biholomorphism of M, then $f(Q(\zeta)) = Q(\bar{f}(\bar{\zeta}))$. This property of biholomorphic invariance of the Segre varieties means that any biholomorphism of M transforms the graph of a solution of $(\mathcal{S}_M)$ to the graph of another solution; that is, any biholomorphism of M is a Lie symmetry of $(\mathcal{S}_{\mathcal{M}})$. Thus, the study of biholomorphisms of real analytic Levi nondegenerate hypersurfaces can be reduced to the study of symmetries of holomorphic completely integrable PDE systems (with one dependent variable). However, the systems corresponding to Segre families form a very special subclass between completely integrable systems, since the coefficients of any defining function of a real hypersurface satisfy additional conjugation relations due to the fact that the defining function r is real-valued.

3. Study of Point Transformations

In this section we present our method of study of point transformations between completely integrable systems. We suppose that we are in the setting of Theorem 1.1.

Let f be a point transformation between two holomorphic completely integrable systems $(\mathcal{S})$ and $(\hat{\mathcal{S}})$. Let $f\colon (x,u) \mapsto (x^*,u^*)$ be given by $x^* = X(x,u)$ and $u^* = U(x,u)$. First of all, we can assume that f is defined in a neighborhood of the origin, $f(0) = 0$, and, moreover, that the tangent map $f'(0)$ is the identity. Indeed, we always can obtain this situation by a translation and a linear change of coordinates in the space of variables (x,u). All our consideration will be in a suitable neighborhood of the origin.

The integrability condition implies that $f^{(2)}$ takes $(\mathcal{S}_2)$ to $(\hat{\mathcal{S}}_2)$.

Our main idea is to use the relation $(f^{(2)})^{-1}((\hat{\mathcal{S}})_2) = (\mathcal{S})_2$. The left side is defined by $U_{ij} = \hat{F}(X,U,U^{(1)})$, where we use the notation $U^{(1)} = (U_1,\dots,U_n)$. Substituting the recurrence expressions (2) for U_{ij} and multiplying by the matrix A yields

$$D_iU_j = \Phi_{ij}(x,u,u^{(1)},X,U,D^{(1)}X,U^{(1)}). \tag{3}$$

Applying the total derivative operator in (1), we have

$$\begin{pmatrix} D_iU_1 \\ \vdots \\ D_iU_n \end{pmatrix} = (D_iA^{-1})\begin{pmatrix} D_1U \\ \vdots \\ D_nU \end{pmatrix} + A^{-1}\begin{pmatrix} D_iD_1U \\ \vdots \\ D_iD_nU \end{pmatrix}. \tag{4}$$

This implies the following description of D_iU_j.

LEMMA 3.1. *Every D_iU_j is a rational function of the form*

$$P_{ij}/Q = (u^{(1)},u^{(2)},X,U,D^{(1)}X,D^{(1)}U,D^{(2)}X,D^{(2)}U).$$

The denominator Q is equal to $(\det A)^2$. *Every numerator P_{ij} is a polynomial in $u^{(1)}$ and $u^{(2)}$ and has the form $P_{ij} = \sum_{|\alpha|\leq N} A^{\alpha}_{ij}[u^{(1)}]^{\alpha} + B_{ij}$, where the B_{ij} are terms of nonzero degree in $u^{(2)}$ and N is a universal integer constant. Coefficients A^{α}_{ij} are second-order differential expressions of the form*

$$\sum_{k,l} a^{\alpha}_{ijkl}(x,u)U_{x^k u^l} + \sum_{k,l,s} b^{\alpha}_{ijkls}(x,u)X^s_{x^k u^l} + C, \tag{5}$$

where C denotes the terms with partial derivatives of f of order ≤ 1. The coefficients $a^{\alpha}_{ijkl}(x,u)$ and $b^{\alpha}_{ijkls}(x,u)$ are polynomials in $(D^{(1)}X)(x,u)$ and $(D^{(1)}U)(x,u)$.

Proof. The expression for Q is clear; $\det A$ does not vanish in a neighborhood of the origin since $f'(0) = \mathrm{id}$. We also have

$$D_i D_j U = U_{x_i x_j} + U_{x_i u}u_j + U_{x_j u}u_i + U_{uu}u_i u_j + \cdots,$$

where we have dropped the terms containing $u^{(2)}$. Furthermore, the elements of A^{-1} have the form $h_{ij}/\det A$, where h_{ij} are polynomials in $D^{(1)}X, D^{(1)}U, u^{(1)}$, so the elements of $D_i A^{-1}$ have a representation of the form (5). Since $D_j U = U_{x_j} + u_j U_u$, we get (5). We point out that $N = \max\{n, 3\}$, but we do not need this. □

We substitute the obtained representation $D_i U_j = P_{ij}/Q$ into (3) and multiply both sides by Q. Next we substitute $F_{ij}(x,u,u^{(1)})$ instead of u_{ij} at every term of B_{ij} and put them to the right side. We thus obtain

$$\sum_{\alpha} A^{\alpha}_{ij}[u^{(1)}]^{\alpha} = \Psi_{ij}(x,u,u^{(1)}, X, U, D^{(1)}X, D^{(1)}U).$$

Expanding the right side in a power series in $u^{(1)}$ and comparing the coefficients, we obtain the equations

$$A^{\alpha}_{ij} = \psi^{\alpha}_{ij}(x,u,X,U,D^{(1)}X, D^{(1)}U). \tag{6}$$

In view of the previously given representation for A^{α}_{ij}, this last system can be rewritten in the form

$$MD^{(2)}f = \Phi(x,u,f,D^{(1)}f), \tag{7}$$

where M is a matrix whose elements are polynomials in $D^{(1)}f$. We point out that this is a PDE system with $n+1$ independent variables (x,u) and $n+1$ dependent variables $f = (X,U)$.

Recall some simple PDE notions. The *k-prolongation* of a PDE system is the system obtained by application of all partial derivatives of order k (with respect to independent variables) at every equation. Clearly, the k-prolongation of our system (7) is a system that is linear with respect to the derivatives of order $k+2$ of f. If—after an application of the Cramer rule to a subsystem of the k-prolongation—this subsystem can be rewritten in the form $D^{(k+2)}f = h(x,f,\ldots,D^{(k+1)}f)$ with a function h analytic near a point $P \in J^{k+1}(n+1,n+1)$ with the canonical coordinates $x(P) = p$, $u(P) = \hat{u}$, and $u^j_i(P) = \hat{u}^j_i, \ldots, u^{(k+1)}(P) = \hat{u}^{(k+1)}$, then we say that the k-prolongation has *the trivial principal symbol* at P. It follows by the chain rule that in this case there exists at most one solution of (7) that is holomorphic near p and satisfies $j_p^{k+1}(f) = P$, namely, $f(p) = \hat{u}$, $D^{(1)}f(p) = \hat{u}^{(1)}, \ldots, D^{(k+1)}f(p) = \hat{u}^{(k+1)}$. Hence, in order to prove that any solution of (7) is uniquely determined by its finite-order jet at the origin, it is enough to show that, for some k, the k-prolongation of this system has the trivial principal symbol.

Recall that we suppose that $f(0) = 0$ and the tangent map $f'(0)$ is the identity. Hence the following statement concludes the proof of the main theorem.

PROPOSITION 3.2. *The* 1*-prolongation of the system* (7) *has the trivial principal symbol at the point $P \in J^1(n+1, n+1)$ with the canonical coordinates $x(P) = 0$, $u(P) = 0$, and $u_i^j(P) = \delta_{ij}$* (*the Kroneker symbol*).

The proof is given in the next section.

4. Computations and Proof of the Main Result

We begin with the special case where $n = 1$. A direct computation shows that the system (7) has the form

$$\begin{aligned} U_{xx}X_x - U_xX_{xx} &= \varphi_1, \\ 2U_{xu}X_x + U_{xx}X_u - U_uX_{xx} - 2U_xX_{xu} &= \varphi_2, \\ 2U_{xu}X_u - 2U_uX_{ux} + U_{uu}X_x - U_xX_{uu} &= \varphi_3, \\ U_{uu}X_u - U_uX_{uu} &= \varphi_4, \end{aligned}$$

and its 1-prolongation has the trivial symbol (at P).

However, for $n > 1$ the direct computation of elements of the matrix M becomes extremely cumbersome. Fortunately, we do not need to know the explicit expression for M in order to study the 1-prolongation of (7). Let us explain this reduction.

Considering M as a matrix function in (x, u), set $M_{\mathrm{id}} = M(0)$. Thus M_{id} is a matrix with *constant* elements obtained by evaluating the corresponding elements of M (which are polynomials in $D^{(1)}f$) under the substitution $X^i_{x_j} = \delta_{ij}$, $U_{x_j} = 0$, $U_u = 1$ (recall that we study our system near the point P defined in Proposition 3.2). Consider the *reduced system*

$$M_{\mathrm{id}}D^{(2)}f = 0. \tag{8}$$

We need the following simple statement.

LEMMA 4.1. *Suppose that the principal symbol of the* 1*-prolongation of the reduced system* (8) *is trivial at P. Then the* 1*-prolongation of* (7) *has the trivial principal symbol at P as well.*

REMARK. Of course, since the system (8) has constant coefficients, the principal symbol of its prolongation is trivial at P if and only if it is trivial at any other point. But we stress that M_{id} is obtained by the evaluation of M at P.

Proof. The lemma follows from our assumption that the 1-prolongation of (8) contains a subsystem equivalent to $D^{(3)}f = 0$. Hence, by continuity, the determinant of the corresponding subsystem of (7) does not vanish near P and the Cramer rule can be applied, which proves the lemma. □

REMARK. We point out that the system (8) is often called the principal symbol of the system (7) (see e.g. [7] for precise definitions). Formally it is more appropriate to define the principal symbol on the corresponding jet bundle; in our case this would lead to useless complications of the terminology, so we continue to call it "the reduced system".

For instance, in the special case $n = 1$ the reduced system has the form

$$U_{xx} = 0, \quad 2U_{xu} - X_{xx} = 0, \quad -2X_{ux} + U_{uu} = 0, \quad X_{uu} = 0. \tag{9}$$

Taking first-order partial derivatives in these equations, we immediately obtain a third-order system equivalent to $D^{(3)}f = 0$ that is the 1-prolongation of (9) and has the trivial principal symbol. So if f satisfies $f(0) = 0$ and $f'(0) = \mathrm{id}$, then its Taylor expansion at the origin is uniquely determined by two second-order partial derivatives of f: $X_{xx}(0)$ and $U_{uu}(0)$ (other second-order derivatives are determined by (9)). For an arbitrary point transformation f we must add six complex parameters assigning values to $f(0)$ and $D^{(1)}f(0)$. Consequently, the set of all point transformations between two second-order ODEs may be parameterized by at most eight complex parameters. In the general case these parameters are not independent because (6) imposes additional restrictions.

The crucial observation that substantially simplifies computations is that we may compute the matrix M_{id} of the reduced system *directly* without an explicit expression for M in (7). As an example consider the case $n = 2$.

An application of the recursive formula leads to the equalities

$$U_1 = P/\Delta, \qquad U_2 = R/\Delta,$$

where

$$\begin{aligned}\Delta = \det A &= (X^1_{x_1}X^2_{x_2} - X^1_{x_2}X^2_{x_1}) + (X^1_u X^2_{x_2} - X^1_{x_2}X^2_u)u_1\\ &\quad + (X^1_{x_1}X^2_u - X^1_u X^2_{x_1})u_2,\\ P &= (X^2_{x_2}U_{x_1} - X^2_{x_1}U_{x_2}) + (X^2_{x_2}U_u - X^2_u U_{x_2})u_1 + (X^2_u U_{x_1} - X^2_{x_1}U_u)u_2,\\ R &= (X^1_{x_1}U_{x_2} - X^1_{x_2}U_{x_1}) + (X^1_u U_{x_2} - X^1_{x_2}U_u)u_1 + (X^1_{x_1}U_u - X^1_u U_{x_1})u_2.\end{aligned}$$

Hence

$$D_iU_1 = (1/\Delta^2)((D_iP)\Delta - P(D_i\Delta)), \qquad D_iU_2 = (1/\Delta^2)((D_iQ)\Delta - Q(D_i\Delta)).$$

Because we are interested in the reduced system only, we can simplify further computations and proceed using the following rules:

(1) the conditions $X^i_{x_j} = \delta_{ij}$, $U_{x_j} = 0$, and $U_u = 1$ are used every time that we compute a coefficient near a second-order derivative of f;

(2) we do not compute and do not write the terms containing $u^{(2)}$, since they have no influence on the reduced system;

(3) we do not compute and do not write the terms containing the partial derivatives of f of order ≤ 1 (i.e., the terms denoted by C in (5)).

The result of such a computation is denoted by $D_i U_j|_{j_0^1(f)=\mathrm{id}}$. In other words, we have

$$D_i U_j|_{j_0^1(f)=\mathrm{id}} = \sum_\alpha \Bigg(\sum_{k,l} a^\alpha_{ijkl}(0) U_{x^k u^l} + \sum_{k,l,s} b^\alpha_{ijkls}(0) X^s_{x^k u^l} \Bigg) [u^{(1)}]^\alpha \tag{10}$$

in the notation of (5), where we consider the coefficients a, b as functions in (x, u) and take their values at the origin.

We thus obtain

$$\begin{aligned} D_1U_1|_{j_0^1(f)=\mathrm{id}} &= U_{x_1x_1} + (2U_{x_1u} - X^1_{x_1x_1})u_1 + (2X^1_{x_1u} - U_{uu})u_1^2 \\ &\quad - X^2_{x_1x_1}u_2 - 2X^2_{x_1u}u_1u_2 - X^1_{uu}u_1^3 - X^2_{uu}u_1^2u_2, \\ D_1U_2|_{j_0^1(f)=\mathrm{id}} &= U_{x_1x_2} + (U_{x_2u} - X^1_{x_1x_2})u_1 + (-X^2_{x_1x_2} + U_{x_1u})u_2 \\ &\quad - X^1_{x_2u}u_1^2 + (U_{uu} - X^2_{x_2u} - X^1_{x_1u})u_1u_2 \\ &\quad - X^2_{x_1u}u_2^2 - X^1_{uu}u_1^2u_2 - X^2_{uu}u_1u_2^2, \\ D_2U_2|_{j_0^1(f)=\mathrm{id}} &= U_{x_2x_2} + (2U_{x_2u} - X^2_{x_2x_2})u_2 - (2X^2_{x_2u} - U_{uu})u_2^2 \\ &\quad - X^1_{x_2x_2}u_1 - 2X^1_{x_2u}u_1u_2 - X^2_{uu}u_2^3 - X^1_{uu}u_1u_2^2. \end{aligned}$$

Therefore, the reduced system has the form

$$\begin{gathered} U_{x_ix_j} = 0 \ \ (i, j = 1, 2), \quad X^1_{uu} = X^1_{x_2u} = X^1_{x_2x_2} = 0, \quad X^2_{uu} = X^2_{x_1u} = X^2_{x_1x_1} = 0, \\ 2U_{x_1u} - X^1_{x_1x_1} = 0, \quad U_{uu} - 2X^1_{x_1u} = 0, \quad U_{x_2u} - X^1_{x_1x_2} = 0, \\ U_{x_1u} - X^2_{x_1x_2} = 0, \quad 2U_{x_2u} - X^2_{x_2x_2} = 0, \quad U_{uu} - 2X^2_{x_2u} = 0, \end{gathered}$$

where we drop the equation $U_{uu} - X^2_{x_2u} - X^1_{x_1u} = 0$ since it is a linear combination of others.

Now a direct verification shows that the 1-prolongation of this system has the trivial symbol and so the proposition is established when $n = 2$. Moreover, quite similarly to the previous example, we see that the space of all point transformations is parameterized by at most fifteen complex parameters.

We consider now the general case with arbitrary n. We proceed by following the conventions (1), (2), and (3) and by using (4). We point out that we can avoid any computation of A^{-1}. Indeed, we have the identity $D_i A^{-1} = -A^{-1}(D_i A)A^{-1}$; because A and $D_i U$ do not contain second-order derivatives of f of order 2, the contribution of the terms

$$(D_i A^{-1}) \begin{pmatrix} D_1 U \\ \vdots \\ D_n U \end{pmatrix}$$

to $M_{\rm id}$ will be equal to

$$-D_i A \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Using our convention, one may also easily compute the contribution of the terms $D_i A$ and $D_i D_j U$ to $M_{\rm id}$ (clearly the matrix factor A^{-1} can be dropped). We have

$$\begin{aligned} D_j U_s|_{j_0^1(f)=\rm id} &= \left(-\sum_{k=1}^n X^k_{x_j x_s} u_k + u_s U_{x_j u} + u_j U_{x_s u}\right) \\ &\quad + \left(-u_s \sum_{k=1}^n X^k_{x_j u} u_k - u_j \sum_{k=1}^n X^k_{x_s u} u_k + u_j u_s U_{uu}\right) \\ &\quad - u_j u_s \sum_{k=1}^n X^k_{uu} u_k + U_{x_j x_s}. \end{aligned}$$

We distinguish two cases: $j = s$ and $j < s$. We obtain

$$\begin{aligned} D_s U_s|_{j_0^1(f)=\rm id} &= U_{x_s x_s} + u_s(2U_{x_s u} - X^s_{x_s x_s}) - \sum_{k=1, k\neq s}^n X^k_{x_s x_s} u_k \\ &\quad + u_s^2(U_{uu} - 2X^s_{x_s u}) - 2\sum_{k=1, k\neq s}^n X^k_{x_s u} u_s u_k - \sum_{k=1}^n X^k_{uu} u_k u_s^2 \end{aligned}$$

and

$$\begin{aligned} D_j U_s|_{j_0^1(f)=\rm id} &= U_{x_j x_s} + u_s(U_{x_j u} - X^s_{x_j x_s}) + u_j(U_{x_s u} - X^j_{x_j x_s}) \\ &\quad - \sum_{k=1, k\neq j,s}^n X^k_{x_j x_s} u_k + u_s u_j(U_{uu} - X^s_{x_s u} - X^j_{x_j u}) \\ &\quad - u_s \sum_{k=1, k\neq j}^n X^k_{x_j u} u_k - u_j \sum_{k=1, k\neq s}^n X^k_{x_s u} u_k - u_j u_s \sum_{k=1}^n X^k_{uu} u_k. \end{aligned}$$

So the principal symbol of the reduced system is defined by the following PDE system:

$$\begin{aligned} &(\mathrm{I}) \quad U_{x_j x_s} = 0,\ 1 \le j \le s \le n, \\ &(\mathrm{II}) \quad X^k_{uu} = 0, \\ &(\mathrm{III}) \quad X^k_{x_j x_s} = 0,\ 1 \le j \le s \le n,\ k \neq j, s, \\ &(\mathrm{IV}) \quad X^k_{x_s u} = 0,\ k \neq s, \\ &(1) \quad 2U_{x_s u} - X^s_{x_s x_s} = 0, \\ &(2) \quad U_{uu} - 2X^s_{x_s u} = 0, \\ &(3) \quad U_{x_j u} - X^s_{x_j x_s} = 0,\ j \neq s, \end{aligned}$$

where we write linearly independent equations only.

Let us show that the principal symbol of the 1-prolongation of this system is trivial.

From (I) we have $U_{x_j x_s t} = 0$ for $t \in \{x_j, u\}$. From (II) it follows that $X^k_{uut} = 0$, and (IV) implies $X^k_{x_s ut} = 0$ for $k \neq s$; furthermore, (III) gives $X^k_{x_j x_s t} = 0$ for $1 \leq j \leq s \leq n$ and $k \neq j, s$.

Taking the derivatives in x_j in (1) and in x_p in (3), we get $X^s_{x_s x_i x_j} = 0$ for any i, j. Hence, $X^s_{x_i x_j x_p} = 0$ for any i, j, p. Taking the derivative in x_k in (2), we obtain $U_{uux_k} = 0$ and similarly $U_{uuu} = 0$, which implies $D^{(3)}U = 0$. From (1) we have $X^s_{x_s x_s t} = 0$; from (3) we have $X^s_{x_j x_s t} = 0$ for $s \neq j$ and hence $X^k_{x_i x_j t} = 0$, so $D^{(3)}X = 0$. Therefore, the principal symbol of the 1-prolongation is trivial. This completes the proof of Proposition 3.2.

Thus, the Taylor expansion of f at the origin is determined by its second-order part. In order to conclude the proof of Theorem 1.1, we observe that the reduced system just described contains $(n^3 + 4n^2 + 3n)/2$ independent equations with $(n+1)^2(n+2)/2$ unknown variables (second derivatives of f). But f satisfies $f(0) = 0$ and $f'(0) = \mathrm{id}$, so f depends on at most $n+1$ parameters. For an arbitrary f we have to add $(n+1) + (n+1)^2$ parameters corresponding to $f(0)$ and $f'(0)$. So, in general, f is uniquely determined by at most $n^2 + 4n + 3$ parameters. This completes the proof of the theorem.

We conclude the paper with two additional remarks.

REMARK 1. The main result still holds with the same proof in the case where $m \geq 1$; that is, our PDE system has several dependent variables. The maximal number of parameters in this case is equal to $(n + m + 2)(n + m)$. This estimate is precise, since the equality is realized for $F = \hat{F} \equiv 0$.

REMARK 2. Our method can be applied in a substantially more general situation: where the PDE systems considered in our main theorem have some additional nonlinear first-order equations. Such systems describe Segre families of real submanifolds of higher codimension (for more details see [8]).

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