Michigan Math. J. 50 (2002)

Some Applications of Bruhat–Tits Theory to Harmonic Analysis on the Lie Algebra of a Reductive p-adic Group

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1. Introduction

In recent years, the questions of interest in the study of harmonic analysis on reductive p-adic groups have required very precise versions of what were previously qualitative results (see e.g. [19; 20; 25; 26; 27]). This paper began as an attempt to prove precise versions of some results of Fiona Murnaghan that relate the character of a supercuspidal representation to the Fourier transform of an elliptic orbital integral [15; 16; 17; 18]. In order to properly formulate these results, it was necessary to develop a "uniform" way to express both the support of invariant distributions and the local constancy of functions. We present here the product of this effort.

Let F denote a field with discrete valuation. We assume that F is complete with perfect residue field $\mathfrak{f}$. Let G be the group of F-rational points of a reductive, connected, linear algebraic group defined over F, and let $\mathfrak{g}$ denote its Lie algebra. Let $\mathcal{B}$ denote the Bruhat–Tits building of G.

Recall that, for $x \in \mathcal{B}$ and $r \in \mathbf{R}$, Allen Moy and Gopal Prasad defined a lattice $\mathfrak{g}_{x,r}$ of $\mathfrak{g}$. In Section 3 we explore the relationship between the lattices $\mathfrak{g}_{x,r}$ and $\mathcal{N}$, the set of nilpotent elements in $\mathfrak{g}$. For every real number r we construct the open, closed, G-invariant subset

$$\mathfrak{g}_r := \bigcup \mathfrak{g}_{x,r},$$

where the union is taken over the points in $\mathcal{B}$. The sets $\mathfrak{g}_r$ can be used to describe the support of invariant distributions on $\mathfrak{g}$. We show that

$$\mathfrak{g}_r = \bigcap (\mathfrak{g}_{x,r} + \mathcal{N}),$$

where the intersection is taken over the points in $\mathcal{B}$. This equality provides some intuition for the ubiquity of the nilpotent set in harmonic analysis. We prove that the sets $\mathfrak{g}_r$ behave well with respect to parabolic descent. That is, if P is a parabolic subgroup of G with Levi decomposition $P = MN$ and Lie algebras $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$, then

$$\mathfrak{m} \cap \mathfrak{g}_r = \mathfrak{m}_r.$$

Received May 17, 2001. Revision received January 31, 2002.
The first-named author was partially supported by a University of Akron Faculty Research Grant (#1470).

We also show that, given a result of [6], analogous statements can be made for the filtrations of G defined by Moy and Prasad.

Under the assumption that $\mathfrak{f}$ is finite, we examine the relation between the Fourier transform and the sets $\mathfrak{g}_r$. We prove that the Fourier transform of a locally constant, compactly supported, complex-valued function with support in $\mathfrak{g}_r$ can be written as a finite sum of functions each of which is translation invariant with respect to a lattice $\mathfrak{g}_{y,(-r)^+}$ for some y in $\mathcal{B}$. Finally, we show that if an M-invariant distribution on $\mathfrak{m}$ has a local expansion on $\mathfrak{m}_r$, then the distribution of $\mathfrak{g}$ obtained by induction has a local expansion on $\mathfrak{g}_r$.

Much of the material in this paper appeared in the second author's thesis [5].

We are indebted to many people. We thank Robert Kottwitz for his beautiful lectures on the harmonic analysis of reductive p-adic groups presented at the University of Chicago during 1995–96. We thank Gopal Prasad for providing one of our proofs of Lemma 2.4.1. We thank the referee for many helpful comments. We have benefited from conversations with Robert Kottwitz, Allen Moy, Fiona Murnaghan, Amritanshu Prasad, Gopal Prasad, Paul Sally, Jr., and Jiu-Kang Yu.

2. Preliminaries

2.1. The Basics

Let F denote a field with nontrivial discrete valuation and residue field $\mathfrak{f}$. We assume that F is complete and that $\mathfrak{f}$ is perfect. Fix a uniformizing element ϖ in F. We let R denote the ring of integers of F and $\wp = \varpi R$ its prime ideal. Let ν denote a valuation on F, normalized so that $\nu(F^\times) = \mathbf{Z}$. We will also denote by ν the unique extension of this valuation to any algebraic extension of F. Fix an additive character $\Lambda\colon F^+ \to \mathbf{C}^\times$ with conductor $\wp$. Let $\mathbf{G}$ be a connected reductive algebraic group defined over F. The Lie algebra of $\mathbf{G}$ will be denoted by $\boldsymbol{\mathfrak{g}}$.

For any field extension E of F, let $\mathbf{G}(E)$ denote the group of E-rational points of $\mathbf{G}$ and let $\boldsymbol{\mathfrak{g}}(E)$ denote the vector space of E-rational points of $\boldsymbol{\mathfrak{g}}$. We will let $G = \mathbf{G}(F)$ and $\mathfrak{g} = \boldsymbol{\mathfrak{g}}(F)$.

Let Ad denote the adjoint representation of $\mathbf{G}$ on $\boldsymbol{\mathfrak{g}}$ and of G on $\mathfrak{g}$. We will often write ${}^gX$ instead of $\mathrm{Ad}(g)X$ and ${}^gh$ instead of $\mathrm{Int}(g)h = ghg^{-1}$. We will use a similar notation for the coadjoint action of G on the linear dual $\mathfrak{g}^*$ of $\mathfrak{g}$. For any subset S of G (or $\mathfrak{g}$ or $\mathfrak{g}^*$) and any subgroup H of G, let

$$^{H}S := \{{}^{h}s \mid h \in H \text{ and } s \in S\}.$$

If S is a set and $n \in \mathbf{N}$, then $\mathrm{M}_n(S)$ denotes the set of $n \times n$ matrices with entries in S. As a set, we will always realize $\boldsymbol{\mathfrak{gl}}_n(F)$ as $\mathrm{M}_n(F)$ and $\mathrm{GL}_n(F)$ as

$$\{X \in \mathrm{M}_n(F) \mid \det(X) \neq 0\}.$$

2.1.1. BRUHAT–TITS BUILDING. Let $\mathcal{B} = \mathcal{B}(G) = \mathcal{B}(\mathbf{G}, F)$ denote the (enlarged) Bruhat–Tits building of G. Every maximal F-split torus $\mathbf{S}$ in $\mathbf{G}$ has an associated apartment $\mathcal{A}(\mathbf{S}, F) \subset \mathcal{B}$. For an F-Levi subgroup $\mathbf{M}$ of $\mathbf{G}$, we identify

$\mathcal{B}(\mathbf{M}, F)$ in $\mathcal{B}(\mathbf{G}, F)$. There is not a canonical way to do this, but every natural embedding of $\mathcal{B}(\mathbf{M}, F)$ in $\mathcal{B}(\mathbf{G}, F)$ has the same image.

For $x, y \in \mathcal{B}(G)$, let $[x, y]$ denote the geodesic in $\mathcal{B}(G)$ from x to y.

2.1.2. Filtrations of Tori. Let $\mathbf{T}$ be a torus defined over F, and let E be an extension of F. Let $\mathbf{X}^*(\mathbf{T})$ denote the group of characters of $\mathbf{T}$. Let $\mathbf{T}(E)_0$ denote the parahoric subgroup of $\mathbf{T}(E)$. The torus $\mathbf{T}(E)$ and its Lie algebra $\mathfrak{t}(E) = \mathrm{Lie}(\mathbf{T}(E))$ have natural filtrations, defined as follows. For any $r \in \mathbf{R}$, let

$$\mathfrak{t}(E)_r := \{H \in \mathfrak{t}(E) \mid \nu(d\chi(H)) \geq r \text{ for all } \chi \in \mathbf{X}^*(\mathbf{T})\}.$$

For any $r \geq 0$, let

$$\mathbf{T}(E)_r = \{t \in \mathbf{T}(E)_0 \mid \nu(\chi(t) - 1) \geq r \text{ for all } \chi \in \mathbf{X}^*(\mathbf{T})\}.$$

Note that, in general, $T_0 = \mathbf{T}(F)_0$ need not be the maximal bounded subgroup of $T = \mathbf{T}(F)$. For example, if T is the set of norm-1 elements of a ramified quadratic extension of F, then the maximal bounded subgroup of T is T, but T/T_0 has two elements. In particular, parahoric subgroups behave poorly with respect to (ramified) base change.

2.1.3. Some Fixed Notation. The following notation will be used throughout the remainder of the paper. Fix a maximal unramified extension F^{unr} of F. Let R^{unr} denote the ring of integers of F^{unr} and let $\mathfrak{F}$ denote the residue field of F^{unr}. Let L/F^{unr} be the minimal Galois extension such that $\mathbf{G}$ is L-split, and let $\ell = [L : F^{\mathrm{unr}}]$.

Let $\mathbf{S}$ be a maximal F-split torus of $\mathbf{G}$. Let $\mathbf{T}$ be a maximal F^{unr}-split F-torus of $\mathbf{G}$ that contains $\mathbf{S}$. (Such a torus exists, by [4].) The centralizer $\mathbf{Z}$ of $\mathbf{T}$ in $\mathbf{G}$ is a maximal torus, defined over F.

Let $\mathcal{A}$ be the apartment of $\mathbf{T}(F^{\mathrm{unr}})$ in $\mathcal{B}(\mathbf{G}, F^{\mathrm{unr}})$. Note that we may identify $\mathcal{A}(\mathbf{S}, F)$ (resp., $\mathcal{B}$) with the $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed points of $\mathcal{A}$ (resp., $\mathcal{B}(\mathbf{G}, F^{\mathrm{unr}})$). Let $\mathbf{\Phi}$ be the set of roots of $\mathbf{G}$ relative to $\mathbf{T}$ and F^{unr}, and let $\mathbf{\Psi}$ be the set of affine roots of $\mathbf{G}$ relative to $\mathbf{T}$, F^{unr}, and our choice of valuation. Fix an alcove (i.e., an affine chamber) $C \subset \mathcal{A}$ so that the $\mathrm{Gal}(K/k)$-fixed points of $\bar{C}$ contain an alcove of $\mathcal{B}$. Then C determines a basis Δ for $\mathbf{\Psi}$. As usual, $\psi \in \mathbf{\Psi}$ is positive ($\psi > 0$) if ψ can be written as a nonnegative integral linear combination of elements in Δ and is negative ($\psi < 0$) if $-\psi$ is positive. Note that $\psi \in \mathbf{\Psi}$ is positive if and only if $\psi|_C > 0$.

For any $b \in \mathbf{\Phi}$, let $\mathbf{U}_b$ denote the corresponding root group and let $\mathfrak{u}_b$ denote its Lie algebra. Both $\mathbf{U}_b$ and $\mathfrak{u}_b$ are defined over F^{unr}.

For each root $b \in \mathbf{\Phi}$, there is an extension L_b/F^{unr} in L such that the root-group quotient $\mathbf{U}_b(F^{\mathrm{unr}})/\mathbf{U}_{2b}(F^{\mathrm{unr}})$ is isomorphic to the additive group of L_b. (If b is not multipliable, then write $\mathbf{U}_{2b} = \{1\}$.) For any $\psi \in \mathbf{\Psi}$ of gradient $b \in \mathbf{\Phi}$, let $\ell_\psi = \ell_b = [L_b : F^{\mathrm{unr}}]^{-1}$.

2.2. The Moy–Prasad Filtrations

Following [13], one can associate to any point x in the building $\mathcal{B}$ of G a parahoric subgroup G_x of G, a filtration $\{G_{x,r}\}_{r \geq 0}$ of the parahoric, and a filtration $\{\mathfrak{g}_{x,r}\}_{r \in \mathbf{R}}$

of the Lie algebra $\mathfrak{g}$ of G. Although $\mathfrak{f}$ is assumed to be finite in [13], there is no difficulty in extending their definitions to our setting. We state the definitions.

For $\psi \in \mathbf{\Psi}$, let $\dot{\psi} \in \mathbf{\Phi}$ denote the gradient of ψ. For each $\psi \in \mathbf{\Psi}$ we can define a bounded subgroup $\mathbf{U}_\psi(F^{\mathrm{unr}})$ of the root group $\mathbf{U}_{\dot{\psi}}(F^{\mathrm{unr}})$ and a lattice $\mathfrak{u}_\psi(F^{\mathrm{unr}})$ of the root space $\mathfrak{u}_{\dot{\psi}}(F^{\mathrm{unr}})$. Indeed, choose $x \in \mathcal{A}$ such that $\psi(x) = 0$. We define $\mathbf{U}_\psi(F^{\mathrm{unr}}) := \mathbf{U}_{\dot{\psi}}(F^{\mathrm{unr}}) \cap \mathrm{stab}_{\mathbf{G}(F^{\mathrm{unr}})}(x)$. Let $\mathcal{G}$ denote the R^{unr}-group scheme associated to $\mathrm{stab}_{\mathbf{G}(F^{\mathrm{unr}})}(x)$ (see [4]). The group of R^{unr}-rational points of $\mathcal{G}$ is $\mathrm{stab}_{\mathbf{G}(F^{\mathrm{unr}})}(x)$. Let $L(\mathcal{G})$ denote the Lie algebra of $\mathcal{G}$; this is a lattice in $\mathfrak{g}(F^{\mathrm{unr}})$. We define $\mathfrak{u}_\psi(F^{\mathrm{unr}}) := \mathfrak{u}_{\dot{\psi}}(F^{\mathrm{unr}}) \cap L(\mathcal{G})$. If $\psi_1, \psi_2 \in \mathbf{\Psi}$ such that $\dot{\psi}_1 = \dot{\psi}_2$ and $\psi_1 \geq \psi_2$, then $\mathbf{U}_{\psi_1}(F^{\mathrm{unr}}) \subset \mathbf{U}_{\psi_2}(F^{\mathrm{unr}})$ and $\mathfrak{u}_{\psi_1}(F^{\mathrm{unr}}) \subset \mathfrak{u}_{\psi_2}(F^{\mathrm{unr}})$.

For $x \in \mathcal{A}(\mathbf{T}, F^{\mathrm{unr}}) \subset \mathcal{B}(\mathbf{G}, F^{\mathrm{unr}})$ and $r \in \mathbf{R}_{\geq 0}$, let $\mathbf{G}(F^{\mathrm{unr}})_{x,r}$ denote the subgroup of $\mathbf{G}(F^{\mathrm{unr}})$ generated by $\mathbf{Z}(F^{\mathrm{unr}})_r$ and the $\mathbf{U}_\psi(F^{\mathrm{unr}})$ for which $\psi(x) \geq r$. Similarly, for $r \in \mathbf{R}$, we define the lattice $\mathfrak{g}(F^{\mathrm{unr}})_{x,r}$ of $\mathfrak{g}(F^{\mathrm{unr}})$ by

$$\mathfrak{g}(F^{\mathrm{unr}})_{x,r} := \mathfrak{z}(F^{\mathrm{unr}})_r + \sum_{\substack{\psi \in \mathbf{\Psi} \\ \psi(x) \geq r}} \mathfrak{u}_\psi(F^{\mathrm{unr}}),$$

where $\mathfrak{z}$ is the Lie algebra of $\mathbf{Z}$. Suppose $x \in \mathcal{A}(\mathbf{S}, F)$. In this case x can be realized as a $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed point of $\mathcal{A}(\mathbf{T}, F^{\mathrm{unr}})$. For $r \in \mathbf{R}_{\geq 0}$ we define the subgroup $G_{x,r} \subset G$ to be the group of $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed points of $\mathbf{G}(F^{\mathrm{unr}})_{x,r}$, and, for $r \in \mathbf{R}$, we define $\mathfrak{g}_{x,r} \subset \mathfrak{g}$ to be the lattice of $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed points of $\mathfrak{g}(F^{\mathrm{unr}})_{x,r}$.

We list a few basic properties of the Moy–Prasad filtrations. (For a proof see [1; 13; 21].) For notational convenience, we will assume $r \in \mathbf{R}_{\geq 0}$ whenever we discuss objects in G.

PROPOSITION 2.2.1. *The Moy–Prasad filtrations have the following properties. Fix $x \in \mathcal{B}(G)$ and $r \in \mathbf{R}$.*

(a) *For any $g \in G$, let gx be the image of x under the action of G on $\mathcal{B}(\mathbf{G}, F)$. Then $\mathrm{Int}(g)G_{x,r} = G_{gx,r}$ and $\mathrm{Ad}(g)\mathfrak{g}_{x,r} = \mathfrak{g}_{gx,r}$. More generally, if $\tau \in \mathrm{Aut}_F(\mathbf{G})$, then τ induces an action on $\mathcal{B}(G)$ and also on $\mathfrak{g}$ (via $d\tau$). In particular, we have $d\tau(\mathfrak{g}_{x,r}) = \mathfrak{g}_{\tau(x),r}$ and $\tau(G_{x,r}) = G_{\tau(x),r}$.*
(b) *We have $\varpi\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+1}$.*
(c) *If $\mathbf{M}$ is an F-Levi subgroup of $\mathbf{G}$, $\mathfrak{m}$ is the Lie algebra of $M = \mathbf{M}(F)$, and $x \in \mathcal{B}(\mathbf{M}, F)$, then $\mathfrak{g}_{x,r} \cap \mathfrak{m} = \mathfrak{m}_{x,r}$ and $G_{x,r} \cap M = M_{x,r}$.*

As a notational convenience, we write $G_{x,r^+} = \bigcup_{s>r} G_{x,s}$ and $\mathfrak{g}_{x,r^+} = \bigcup_{s>r} \mathfrak{g}_{x,s}$.

Moy and Prasad also define filtration lattices $\{\mathfrak{g}^*_{x,r}\}$ in the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ by

$$\mathfrak{g}^*_{x,r} = \{\chi \in \mathfrak{g}^* \mid \chi(\mathfrak{g}_{x,(-r)^+}) \subset \wp\}.$$

We write $\mathfrak{g}^*_{x,r^+} = \bigcup_{s>r} \mathfrak{g}^*_{x,s}$. These lattices in $\mathfrak{g}^*$ satisfy statements analogous to those in Proposition 2.2.1.

For a $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed point $x \in \mathcal{B}(\mathbf{G}, F^{\mathrm{unr}})$, it is common to denote the parahoric $G_{x,0}$ and its subgroup $G_{x,0^+}$ by G_x and G_x^+, respectively. The quotient G_x/G_x^+ is the group of $\mathfrak{f}$-points of a connected reductive $\mathfrak{f}$-group G_x. Similarly, if

$H \subset \mathcal{B}(G)$ is a facet, then G_H denotes the parahoric subgroup of G associated to H. (Recall that $G_x = G_H$ for all $x \in H$.) We caution the reader that, in general, $G_x \neq \mathrm{stab}_G(x)$ and $G_H \neq \mathrm{stab}_G(H)$ (see e.g. [23, Sec. 3.12]).

2.3. *Optimal Points*

In [13, Sec. 6.1], *optimal* points are defined to be certain elements of $\mathcal{B}$ that have nice properties with respect to the Moy–Prasad filtrations of $\mathfrak{g}^*$. For the time being, we shall call these points $\mathfrak{g}^*$-*optimal*. In this subsection we define $\mathfrak{g}$-optimal points, which have analogous properties with respect to the filtrations of $\mathfrak{g}$. We then show that we may assume the set of $\mathfrak{g}^*$-optimal points to be a subset of the set of $\mathfrak{g}$-optimal points. This result has been independently observed by Shu-Yen Pan.

Let

$$\Sigma = \{\psi \in \boldsymbol{\Psi} \mid \psi > 0 \text{ and } \psi - 1 < 0\}.$$

This is a finite set. For each nonempty and $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-invariant subset $\mathfrak{S} \subset \Sigma$, we can choose a point $x_{\mathfrak{S}} \in \bar{C}$ such that:

(i) $\min_{\psi \in \mathfrak{S}} \psi(x_{\mathfrak{S}}) \geq \min_{\psi \in \mathfrak{S}} \psi(y)$ for all $y \in \bar{C}$;
(ii) $\psi(x_{\mathfrak{S}})$ is rational for all $\psi \in \boldsymbol{\Psi}$; and
(iii) $x_{\mathfrak{S}}$ is $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-invariant.

The existence of such a point follows by making the same arguments as those found in [13, Sec. 6.1]. For each nonempty and $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-invariant subset $\mathfrak{S} \subset \Sigma$, fix a choice of $x_{\mathfrak{S}} \in \bar{C}$ satisfying conditions (i)–(iii) and let $\mathcal{O}$ be the finite set $\{x_{\mathfrak{S}}\}$. A point $x \in \mathcal{B}$ is said to be $\mathfrak{g}$-*optimal for* $\mathfrak{S}$ if it is G-conjugate to $x_{\mathfrak{S}}$ and simply $\mathfrak{g}$-*optimal* if it is G-conjugate to some point in $\mathcal{O}$. The definition of $\mathfrak{g}^*$-optimal is the same, except that condition (i) is replaced by:

(i′) $\min_{\psi \in \mathfrak{S}}(\psi(x_{\mathfrak{S}}) - (1 - \ell_{\psi})) \geq \min_{\psi \in \mathfrak{S}}(\psi(y) - (1 - \ell_{\psi}))$ for all $y \in \bar{C}$.

LEMMA 2.3.1. *Let $\mathfrak{S}'$ be a nonempty and* $\mathrm{Gal}(F^{\mathrm{unr}}/F)$*-invariant subset of Σ. Then there exists a nonempty and* $\mathrm{Gal}(F^{\mathrm{unr}}/F)$*-invariant subset $\mathfrak{S}$ of Σ such that, for every $x \in \bar{C}$, x satisfies conditions* (i′), (ii), *and* (iii) *for $\mathfrak{S}'$ if and only if x satisfies conditions* (i)–(iii) *for $\mathfrak{S}$.*

Proof. We first define a subset $\mathfrak{S}$ of Σ. One may assume without loss of generality that distinct elements of $\mathfrak{S}'$ have distinct gradients. Let

$$S = \{\psi \in \mathfrak{S}' \mid \psi + \ell_{\psi} \notin \Sigma\}.$$

Note that S is $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-invariant. Let

$$\mathfrak{S} = \begin{cases} \{\psi + \ell_{\psi} \mid \psi \in \mathfrak{S}' \setminus S\} & \text{if } S \subsetneqq \mathfrak{S}', \\ \{\psi + \ell_{\psi} - 1 \mid \psi \in \mathfrak{S}'\} & \text{if } S = \mathfrak{S}'. \end{cases}$$

Then $\mathfrak{S}$ is a nonempty and $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-invariant subset of Σ. It only remains to show that x satisfies condition (i) for the set $\mathfrak{S}$ if and only if x satisfies condition (i′) for $\mathfrak{S}'$.

Suppose $S \subsetneq \mathfrak{S}'$. Since $\psi - (1 - \ell_\psi) > 0$ for $\psi \in S$, we have for all $y \in \bar{C}$ that

$$\min_{\psi \in \mathfrak{S}'} (\psi(y) - (1 - \ell_\psi)) = \min_{\psi \in \mathfrak{S}' \setminus S} (\psi(y) - (1 - \ell_\psi)) = \min_{\psi \in \mathfrak{S}} \psi(y) - 1,$$

so x satisfies (i) for $\mathfrak{S}$ if and only if x satisfies (i$'$) for $\mathfrak{S}'$.

Now suppose $S = \mathfrak{S}'$. Then, for $y \in \bar{C}$,

$$\min_{\psi \in \mathfrak{S}'} (\psi(y) - (1 - \ell_\psi)) = \min_{\psi \in \mathfrak{S}} \psi(y),$$

so x satisfies (i) for $\mathfrak{S}$ if and only if x satisfies (i$'$) for $\mathfrak{S}'$. □

From this lemma we may assume that the set of $\mathfrak{g}^*$-optimal points is a subset of the $\mathfrak{g}$-optimal points. Therefore, it makes sense to call a point x *optimal* if it is G-conjugate to a point in $\mathcal{O}$.

We say that $r \in \mathbf{Q}$ is an *optimal number* if there is an optimal point x such that $\mathfrak{g}_{x,r} \neq \mathfrak{g}_{x,r^+}$. Since every optimal point is conjugate to a point in $\mathcal{O}$ and since $\mathcal{O}$ is a finite set, the set of optimal numbers is discrete.

The following lemma has been extracted from the proof of [13, Prop. 6.3].

LEMMA 2.3.2. *If $y \in \mathcal{B}(G)$ and $r \in \mathbf{R}$, then there exist optimal points $x, z \in \mathcal{B}(G)$ such that*

$$\mathfrak{g}_{x,r} \subset \mathfrak{g}_{y,r} \subset \mathfrak{g}_{z,r}.$$

Proof. We first show the existence of z. We may and do assume that y is a $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-invariant point of $\bar{C}$. Define $\mathfrak{S}'$ by

$$\mathfrak{S}' := \{\psi \in \boldsymbol{\Psi} \mid \psi(y) \geq r\}.$$

Let n be the least integer such that $\psi + n > 0$ for all $\psi \in \mathfrak{S}'$. Define the nonempty and $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-invariant subset $\mathfrak{S}$ of Σ by

$$\mathfrak{S} := \{\psi + n \mid \psi \in \mathfrak{S}'\} \cap \Sigma.$$

Let $z = x_{\mathfrak{S}} \in \bar{C}$. Since $z \in \mathcal{B}$, we need only show that

$$\mathfrak{g}(F^{\mathrm{unr}})_{y,r} \subset \mathfrak{g}(F^{\mathrm{unr}})_{z,r}.$$

In order to prove this, it is enough to show that $\psi(z) \geq r$ for all $\psi \in \mathfrak{S}'$. But if $\psi \in \mathfrak{S}'$, then

$$\psi(z) + n \geq \min_{\phi \in \mathfrak{S}} \phi(z) \geq \min_{\phi \in \mathfrak{S}} \phi(y).$$

Since $\mathfrak{S}$ is a finite set, there is a $\psi' \in \mathfrak{S}'$ such that $\psi(z) \geq \psi'(y)$; but $\psi'(y) \geq r$.

This same argument applied to $\mathfrak{g}^*$ shows that, for all $t \in \mathbf{R}$, there exists an $x \in \mathcal{O}$ such that $\mathfrak{g}^*_{y,t} \subset \mathfrak{g}^*_{x,t}$.

There exists an $\varepsilon > 0$ such that, for all $s \in (r - \varepsilon, r)$, we have $\mathfrak{g}_{y,r} = \mathfrak{g}_{y,s} = \mathfrak{g}_{y,s^+}$. Fix $s \in (r - \varepsilon, r)$. From the previous paragraph, there exists an $x \in \mathcal{O}$ such that $\mathfrak{g}^*_{y,-s} \subset \mathfrak{g}^*_{x,-s}$. Thus, since $r > s$, we have

$$\mathfrak{g}_{x,r} \subset \mathfrak{g}_{x,s^+} \subset \mathfrak{g}_{y,s^+} = \mathfrak{g}_{y,r}.$$ □

COROLLARY 2.3.3. *If $y \in \mathcal{B}$ and $r \in \mathbf{R}$, then there exist optimal points $x, z \in \mathcal{B}$ such that, for all affine roots ψ,*

$$\psi(x) \geq r \implies \psi(y) \geq r,$$
$$\psi(y) \geq r \implies \psi(z) \geq r.$$

2.4. A Result about Alcoves and Parabolic Subgroups

Suppose that P is the group of F-rational points of a parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ defined over F. Let M denote the group of F-rational points of an F-Levi factor $\mathbf{M}$ of $\mathbf{P}$. Denote by N the unipotent radical of P. Let $\bar{P} = M\bar{N}$ be the parabolic opposite $P = MN$. When $\mathbf{P}$ is a minimal parabolic subgroup, both results in this subsection follow from standard results about spherical buildings.

We provide two proofs for our first result (the second proof is due to Gopal Prasad).

LEMMA 2.4.1. *Suppose $x \in \mathcal{B}(M)$. If $C' \subset \mathcal{B}(G)$ is an alcove such that $x \in \bar{C}'$, then there exists an $n \in N \cap G_x$ such that $nC' \subset \mathcal{B}(M)$.*

REMARK 2.4.2. It is sufficient to show that there exists a $p \in P \cap G_x$ such that $pC' \subset \mathcal{B}(M)$.

Proof. Without loss of generality, we suppose that $x \in \mathcal{A}(\mathbf{S}, F) \subset \mathcal{B}(M)$ and that P is a minimal parabolic subgroup of G.

Let $\mathbf{X}_*(\mathbf{S})$ denote the set of 1-parameter subgroups of $\mathbf{S}$. Let D denote the vector chamber in $\mathbf{X}_*(\mathbf{S}) \otimes \mathbf{R}$ corresponding to N (see [3]). Because D is open, there exists an alcove $C_1 \subset \mathcal{A}(\mathbf{S}, F)$ such that $x \in \bar{C}_1$ and $(x + D) \cap C_1 \neq \emptyset$. Let $\mathcal{I}$ denote the Iwahori subgroup G_{C_1}. Note that $\mathcal{I} \subset G_x$.

Let $\mathcal{A}_1 \subset \mathcal{B}(G)$ be an apartment containing both C_1 and C'. Choose $g \in \mathcal{I}$ so that $g\mathcal{A}_1 = \mathcal{A}(\mathbf{S}, F)$. In particular, $gC' \subset \mathcal{A}(\mathbf{S}, F)$, $x \in g\bar{C}'$, and $gC' \cap (C_1 - D) \neq \emptyset$. Write $g = \bar{n}mn$ with $n \in \mathcal{I} \cap N$, $m \in \mathcal{I} \cap M$, and $\bar{n} \in \mathcal{I} \cap \bar{N}$. Note that if $z \in C_1 - D$ then $\bar{n}^{-1}z = z$. Therefore, since $gC' \cap (C_1 - D)$ is open, we have

$$gC' = \bar{n}mnC' = mnC'$$

with $mn \in \mathcal{I} \cap P \subset G_x \cap P$. □

Proof (Prasad). Without loss of generality, we suppose that x is a $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed point of $\mathcal{A} \subset \mathcal{B}(\mathbf{M}, F^{\mathrm{unr}})$ and that C' is the set of $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed points of a facet C'' in $\mathcal{B}(\mathbf{G}, F^{\mathrm{unr}})$. (Note that if $\mathbf{G}$ is residually quasi-split over F, then C'' is an alcove in $\mathcal{B}(\mathbf{G}, F^{\mathrm{unr}})$.)

Here $\mathbf{G}(F^{\mathrm{unr}})_x/\mathbf{G}(F^{\mathrm{unr}})_x^+$ is the group of $\mathfrak{F}$-rational points of a reductive connected group G defined over $\mathfrak{f}$. Similarly, the image of $\mathbf{G}(F^{\mathrm{unr}})_{C''}$ (resp. $\mathbf{G}(F^{\mathrm{unr}})_x \cap \mathbf{P}(F^{\mathrm{unr}})$, resp. $\mathbf{S}(F^{\mathrm{unr}}) \cap \mathbf{G}(F^{\mathrm{unr}})_x$) in $\mathsf{G}(\mathfrak{F})$ is the group of $\mathfrak{F}$-rational points of a minimal parabolic subgroup B (resp. parabolic subgroup P, resp. maximal $\mathfrak{f}$-split torus S) of G defined over $\mathfrak{f}$. Since B and P are parabolic subgroups of G that are defined over $\mathfrak{f}$, there exists [2, Prop. 20.7] a maximal $\mathfrak{f}$-split torus

$\mathsf{S}' \subset \mathsf{B} \cap \mathsf{P}$. Moreover, since S and S' are maximal $\mathfrak{f}$-split tori of P, there exists a $\bar{p} \in \mathsf{P}(\mathfrak{f})$ such that ${}^{\bar{p}}\mathsf{S}' = \mathsf{S}$. Choose $p \in G_x \cap P$ so that its image in $\mathsf{P}(\mathfrak{f})$ is $\bar{p}$. Let $\mathbf{T}' = {}^{p^{-1}}\mathbf{T}$. It follows from [23, Sec. 3.6.1] that $C'' \subset \mathcal{A}(\mathbf{T}', F^{\mathrm{unr}})$ and so $pC' \subset \mathcal{A}^{\mathrm{Gal}(F^{\mathrm{unr}}/F)} \subset \mathcal{B}(M)$. □

The following result is due to Allen Moy and Fiona Murnaghan [12]. We present here a different proof.

COROLLARY 2.4.3 [12]. *If C' is an alcove in $\mathcal{B}(G)$, then there exists an $n \in N$ such that $nC' \subset \mathcal{B}(M)$.*

Proof. Suppose $x \in \bar{C}'$ is special (see e.g. [23, Sec. 1.9]). Let S' be the group of F-rational points of a maximal F-split torus $\mathbf{S}'$ such that $x \in \mathcal{A}(\mathbf{S}', F)$. Choose $g \in G$ such that ${}^{g}S' \subset P$. From the Iwasawa decomposition ($G = PG_x$) we can write $g = ph$ with $p \in P$ and $h \in G_x$. Thus

$$x = hx \in \mathcal{A}({}^{h}\mathbf{S}', F)$$

and

$${}^{h}S' \subset {}^{p^{-1}}P = P.$$

There exists an $n_1 \in N$ such that ${}^{n_1 \cdot h}S' \subset M$. Since $n_1 x \in \mathcal{A}({}^{n_1 \cdot h}\mathbf{S}', F) \subset \mathcal{B}(M)$, by Lemma 2.4.1 there exists an $n_2 \in G_{n_1 x} \cap N$ such that $n_2 n_1 C' \subset \mathcal{B}(M)$. □

2.5. *Nilpotent Elements*

Let $\mathbf{X}_*^F(\mathbf{G})$ denote the set of 1-parameter subgroups of $\mathbf{G}$ defined over F. Call an element $X \in \mathfrak{g}$ *nilpotent* if there is some $\lambda \in \mathbf{X}_*^F(\mathbf{G})$ such that $\lim_{t \to 0} {}^{\lambda(t)}X = 0$. Let $\mathcal{N}$ denote the set of nilpotent elements. A more usual definition is that an element is nilpotent if the Zariski closure of its G-orbit contains zero. Let $\mathcal{N}''$ denote the set of elements in $\mathfrak{g}$ that are nilpotent in this sense, and let $\mathcal{N}'$ denote the set of elements X in $\mathfrak{g}$ such that the p-adic closure of the G-orbit of X contains zero. It is clear that $\mathcal{N} \subseteq \mathcal{N}' \subseteq \mathcal{N}''$. By a theorem of Kempf [11, Cor. 4.3], $\mathcal{N} = \mathcal{N}''$ when F is perfect.

LEMMA 2.5.1. *If F is perfect or $\mathfrak{f}$ is finite, then $\mathcal{N} = \mathcal{N}'$.*

Proof. From [11, Cor. 4.3] we may assume that $\mathfrak{f}$ is finite. It is clear that $\mathcal{N} \subset \mathcal{N}'$.

Let $M = C_{\mathbf{G}}(\mathbf{S})(F)$ and choose a minimal parabolic $P \subset G$ containing M. Then P has a Levi decomposition $P = MN$. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the associated Lie algebras. Let $\Phi = \Phi(\mathbf{G}, \mathbf{S}, F)$ denote the set of F-roots of $\mathbf{S}$ with respect to $\mathbf{G}$ and let $\Phi^+ \subset \Phi$ denote the set of positive F-roots in Φ with respect to P. Let $x \in \mathcal{A}(\mathbf{S}, F)$ be a special point for G.

Suppose $X \in \mathcal{N}'$. It will be sufficient to show that $X \in {}^{G_x}\mathfrak{n}$.

We have the Cartan decomposition $G = G_x S^+ \omega G_x$, where

$$S^+ = \{s \in \mathbf{S}(F) : |\alpha(s)| \leq 1 \text{ for all } \alpha \in \Phi^+\}$$

and ω is a finite subset of M. Since zero is in the p-adic closure of the G-orbit of X, there exists a sequence $\{g_i \in G\}$ such that

$$^{g_i}X \in \mathfrak{g}_{x,i}$$

for all positive integers i.

Without loss of generality, $g_i = k_i' a_i w k_i$ for $k_i, k_i' \in G_x$, $a_i \in S^+$, and fixed $w \in \omega$. Since $^{k_i'}\mathfrak{g}_{x,i} = \mathfrak{g}_{x,i}$, we may and do assume that $g_i = a_i w k_i$. Let $\bar{P} = M\bar{N}$ be the parabolic opposite P with Lie algebras $\bar{\mathfrak{p}} = \mathfrak{m} + \bar{\mathfrak{n}}$. Then we have the direct sum decomposition $\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{n}$. Write $^{k_i}X = \bar{Z}_i + Y_i + Z_i$, where $\bar{Z}_i \in \bar{\mathfrak{n}}$, $Y_i \in \mathfrak{m}$, and $Z_i \in \mathfrak{n}$. Then $^{g_i}X \to 0$ implies $^{a_i w}\bar{Z}_i \to 0$ and $^{a_i w}Y_i \to 0$. This implies $\bar{Z}_i \to 0$ and $Y_i \to 0$. Therefore, since G_x is compact, there exists a $k \in G_x$ such that $^kX \in \mathfrak{n}$. □

REMARK 2.5.2. For $\mathfrak{g}^*$ we let $\mathcal{N}^*$, $\mathcal{N}^{*\prime}$, and $\mathcal{N}^{*\prime\prime}$ denote the analogues of $\mathcal{N}$, $\mathcal{N}'$, and $\mathcal{N}''$, respectively. The preceding proof can be modified to show that $\mathcal{N}^* = \mathcal{N}^{*\prime}$ when $\mathfrak{f}$ is finite or F is perfect. It is known that $\mathcal{N}^*$ and $\mathcal{N}^{*\prime\prime}$ need not be equal in positive characteristic. For example, Gopal Prasad pointed out to us that $\mathcal{N}^* \neq \mathcal{N}^{*\prime\prime}$ for $\mathrm{SL}_2(F)$ when F has characteristic 2.

3. Some Results for Moy–Prasad Filtrations of $\mathfrak{g}$

The results on $\mathfrak{g}$ in this section do not rely on the structure of $\mathfrak{g}$ as a Lie algebra. Therefore, with appropriate changes, they are all valid for the filtrations of $\mathfrak{g}^*$ and, indeed, for the filtrations of $\hat{\mathfrak{g}}$, the Pontrjagin dual of $\mathfrak{g}$. (Note that $\hat{\mathfrak{g}}_{x,r}$ consists of those characters of $\mathfrak{g}$ whose restriction to $\mathfrak{g}_{x,(-r)^+}$ is trivial.) Moreover, there are versions of these results for G (see Section 3.7).

3.1. Statement of the Main Result

DEFINITION 3.1.1. For $r \in \mathbf{R}$,

$$\mathfrak{g}_r := \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r}.$$

The main results of this section are summarized in the following theorem.

THEOREM 3.1.2. *Suppose $r \in \mathbf{R}$.*

(1) $\mathfrak{g}_r = \bigcap_{x \in \mathcal{B}(G)} (\mathfrak{g}_{x,r} + \mathcal{N})$.
(2) *If P is a parabolic subgroup of G with Levi decomposition $P = MN$ and Lie algebras $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$, then $\mathfrak{g}_r \cap \mathfrak{m} = \mathfrak{m}_r$.*

The original proof of this theorem in [5] relied on several propositions in [13; 14], which in turn relied on a fundamental result of Kempf [11]. The proof here has removed the reliance on [11]. Part (1) of the theorem is Lemma 3.4.2 and part (2) is Lemma 3.5.3.

3.2. Asymptotic Results on the Filtrations

In [9, Lemma 2.4] it is shown that, when $\mathfrak{f}$ is finite, the $\mathrm{GL}_n(F)$-orbit of $\varpi^m \mathrm{M}_n(R)$ is contained in $\varpi^m \mathrm{M}_n(R) + \mathcal{N}$. Under these same conditions on $\mathfrak{f}$, it is also known (see e.g. [8, Lemma 12.2]) that, for any compact set $\omega \subset \mathfrak{g}$, there exists a lattice $\mathcal{L} \subset \mathfrak{g}$ such that ${}^G\omega \subset \mathcal{L} + \mathcal{N}$. The Moy–Prasad filtration lattices allow us simultaneously to extend the first result and refine the second without requiring that $\mathfrak{f}$ be finite.

LEMMA 3.2.1. *Let $x, y \in \mathcal{B}$, and let $r \in \mathbf{R}$. Then $\mathfrak{g}_{x,r} \subset \mathfrak{g}_{y,r} + \mathcal{N}$.*

Proof. Since $\mathfrak{g}_{x,r} \subset \mathfrak{g}_{y,r} + \mathcal{N}$ if and only if $\mathfrak{g}_{gx,r} \subset \mathfrak{g}_{gy,r} + \mathcal{N}$ for $g \in G$, we may assume that x and y are $\mathrm{Gal}(F^{\mathrm{unr}}/F)$-fixed points of $\mathcal{A}$.

Choose $\vec{v} \in \mathbf{X}_*(\mathbf{S}) \otimes \mathbf{R}$ such that $x = y + \vec{v}$. Let P be a minimal parabolic subgroup determined by $\vec{v}$. That is, P has a Levi decomposition $P = MN$ such that $\mathcal{A}(\mathbf{S}, F) \subset \mathcal{B}(M)$ and, for all F-roots α of $\mathbf{S}$ that are positive with respect to N, we have $\langle \alpha, \vec{v} \rangle \geq 0$.

Let $\bar{P} = M\bar{N}$ denote the parabolic opposite P and let $\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{n}$ denote the associated Lie algebras. We have

$$\mathfrak{g}_{x,r} = (\mathfrak{g}_{x,r} \cap \bar{\mathfrak{n}}) + (\mathfrak{g}_{x,r} \cap \mathfrak{m}) + (\mathfrak{g}_{x,r} \cap \mathfrak{n}) \subset \bar{\mathfrak{n}} + \mathfrak{g}_{y,r} \subset \mathcal{N} + \mathfrak{g}_{y,r}. \qquad \square$$

COROLLARY 3.2.2. *For $r \in \mathbf{R}$ and $x \in \mathcal{B}(G)$, we have $\mathfrak{g}_r \subset \mathfrak{g}_{x,r} + \mathcal{N}$.*

DEFINITION 3.2.3. For $r \in \mathbf{R}$, we define

$$\mathfrak{g}_{r^+} := \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r^+}.$$

REMARK 3.2.4. From Lemma 2.3.2 we can write

$$\mathfrak{g}_r = \bigcup {}^G\mathfrak{g}_{x,r},$$

where the union is over a finite set of optimal points (independent of r). Therefore, $\mathfrak{g}_r = \mathfrak{g}_{r^+}$ unless r is an optimal number. (Jiu-Kang Yu has observed that the converse is often false; consider, for example, $\mathrm{Sp}_4(F)$ and the optimal number $1/3$.) Note that for all $r \in \mathbf{R}$ we have $\mathfrak{g}_{r^+} = \bigcup_{s>r} \mathfrak{g}_s$. Thus, for all $r \in \mathbf{R}$ there exist $s, t \in \mathbf{R}$ with $s > r > t$ such that $\mathfrak{g}_s = \mathfrak{g}_{r^+} \subset \mathfrak{g}_r = \mathfrak{g}_{t^+}$.

DEFINITION 3.2.5. For $y \in \mathcal{B}(G)$ and $s \in \mathbf{R}$, a coset $\Xi \in \mathfrak{g}_{y,s}/\mathfrak{g}_{y,s^+}$ is *degenerate* if and only if $\Xi \cap \mathcal{N} \neq \emptyset$.

One purpose of the following corollary is to give a new proof of [13, Prop. 6.3]. The proof here does not rely on [11].

COROLLARY 3.2.6 (of Lemma 3.2.1 and its proof). *Fix $y \in \mathcal{B}(G)$ and $s \in \mathbf{R}$. The coset $\Xi \in \mathfrak{g}_{y,s}/\mathfrak{g}_{y,s^+}$ is degenerate if and only if $\Xi \subset \mathfrak{g}_{s^+}$.*

Proof. "$\Leftarrow$" If $\Xi \subset \mathfrak{g}_{s^+}$, then by Corollary 3.2.2 and Remark 3.2.4 we have $\Xi \subset \mathfrak{g}_{y,s^+} + \mathcal{N}$.

"$\Rightarrow$" If $\Xi = \mathfrak{g}_{y,s^+}$ then there is nothing to prove, so assume that $\Xi = X + \mathfrak{g}_{y,s^+}$ with $X \in \mathcal{N} \cap (\mathfrak{g}_{y,s} \setminus \mathfrak{g}_{y,s^+})$. Fix $r > s$. Since $X \in \mathcal{N}$, there exists an $x \in \mathcal{B}(G)$ such that $X \in \mathfrak{g}_{x,r}$. For all points z on the geodesic $[x, y]$ that are sufficiently near but not equal to y, we have $\mathfrak{g}_{y,s^+} \subset \mathfrak{g}_{z,s^+}$ and $X \in \mathfrak{g}_{z,s^+}$. (The second statement is valid because $r > s$.) Thus,

$$X + \mathfrak{g}_{y,s^+} \subset \mathfrak{g}_{z,s^+} \subset \mathfrak{g}_{s^+}. \qquad \square$$

REMARK 3.2.7. From this proof it follows that, if $\Xi \in \mathfrak{g}_{y,s}/\mathfrak{g}_{y,s^+}$ is degenerate, then there exist an alcove $C' \subset \mathcal{B}(G)$ and a $z \in \bar{C}'$ such that $y \in \bar{C}'$ and $\Xi \subset \mathfrak{g}_{z,s^+}$. Proposition 6.3 of [13] now follows from Corollary 3.2.6 and Lemma 2.3.2

REMARK 3.2.8. Suppose $x \in \mathcal{B}(G)$ and $r \in \mathbf{R}$. If $\mathfrak{g}_r = \mathfrak{g}_{r^+}$, then it follows from Corollary 3.2.6 that every coset of $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ is degenerate. In particular, every coset of $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ is degenerate unless r is an optimal number.

3.3. A Depth Function for $\mathfrak{g}$

In this subsection we define a G-invariant function $\mathrm{d}\colon \mathfrak{g} \to \mathbf{Q} \cup \{\infty\}$ that measures the depth of elements of $\mathfrak{g}$ with respect to the Moy–Prasad filtrations.

The next lemma shows that the $\mathfrak{g}_r$ are related to the Moy–Prasad filtrations in the sense of [8, Sec. 2 and Lemma 5.6].

LEMMA 3.3.1. *For all $y \in \mathcal{B}(G)$ and $r \in \mathbf{R}$,*

$$\varpi \mathfrak{g}_{y,r} \subset \varpi \mathfrak{g}_r \subset {}^G(\mathfrak{g}_{y,r}).$$

Proof. We may suppose that $y \in \bar{C}$. It is enough to show that, for all $x \in \bar{C}$,

$$\mathfrak{g}(F^{\mathrm{unr}})_{x,r+1} \subset \mathfrak{g}(F^{\mathrm{unr}})_{y,r}.$$

Suppose this is not true. Then there exists an affine root ψ such that

$$\psi(x) \geq r + 1 \quad \text{and} \quad \psi(y) < r.$$

Hence $\psi(x) - \psi(y) > 1$, which contradicts the fact that both x and y lie in $\bar{C}$. $\square$

LEMMA 3.3.2.

$$\mathcal{N}' = \bigcap_{r \in \mathbf{R}} \mathfrak{g}_r.$$

Proof. Suppose $X \in \mathcal{N}'$. Then ${}^G X$ intersects $\mathfrak{g}_{x,r}$ for all $x \in \mathcal{B}(G)$ and all $r \in \mathbf{R}$, so $X \in \bigcap_{r \in \mathbf{R}} \mathfrak{g}_r$. Conversely, fix $y \in \mathcal{B}(G)$. If $X \in \bigcap_{r \in \mathbf{R}} \mathfrak{g}_r$, then $X \in \bigcap_{r \in \mathbf{R}} ({}^G\mathfrak{g}_{y,r})$ by Lemma 3.3.1. Thus $X \in \mathcal{N}'$. $\square$

COROLLARY 3.3.3. *The elements of the set $\{\mathfrak{g}_r \mid r \in \mathbf{R}\}$ form a neighborhood basis of G-invariant open neighborhoods of $\mathcal{N}'$.*

Proof. This follows from Lemma 3.3.1 and Lemma 3.3.2. $\square$

The following result, or something similar, has also been proved by Allen Moy.

LEMMA 3.3.4. *Let $X \in \mathfrak{g} \setminus \mathcal{N}'$. Then there exists a unique number* $\mathrm{d}(X) \in \mathbf{R}$ *such that* $X \in \mathfrak{g}_{\mathrm{d}(X)} \setminus \mathfrak{g}_{\mathrm{d}(X)^+}$.

Proof. By Lemma 3.3.2 there exists an $s \in \mathbf{R}$ such that $X \notin \mathfrak{g}_s$. Fix $x \in \mathcal{B}(G)$; then there exists an $r \in \mathbf{R}$ such that $X \in \mathfrak{g}_{x,r}$. Note that $r < s$ and $X \in \mathfrak{g}_r \setminus \mathfrak{g}_s$. Since the set of optimal numbers between r and s is finite, the result follows from Remark 3.2.4. □

Note that, for $X \in \mathfrak{g} \setminus \mathcal{N}'$, $\mathrm{d}(X)$ is an optimal number. Thus $\mathrm{d}(X) \in \mathbf{Q}$.

DEFINITION 3.3.5. If $X \in \mathcal{N}'$, then $\mathrm{d}(X) := \infty$.

DEFINITION 3.3.6. For $X \in \mathfrak{g}$, we call $\mathrm{d}(X)$ the *depth* of X.

LEMMA 3.3.7. *The map* $\mathrm{d}\colon \mathfrak{g} \to \mathbf{Q} \cup \{\infty\}$ *sending X to* $\mathrm{d}(X)$ *is locally constant on $\mathfrak{g} \setminus \mathcal{N}'$. More precisely, if $X \in \mathfrak{g}_{x,\mathrm{d}(X)}$ for some $x \in \mathcal{B}$, then* d *is constant on* $X + \mathfrak{g}_{x,\mathrm{d}(X)^+}$.

Proof. Suppose that $\mathrm{d}(X) = s < \infty$. Then there exists a point $x \in \mathcal{B}(G)$ for which $X \in \mathfrak{g}_{x,s} \setminus \mathfrak{g}_{x,s^+}$. Choose $Y \in \mathfrak{g}_{x,s^+}$. We claim that $\mathrm{d}(X + Y) = s$.

Since $X+Y \in \mathfrak{g}_{x,s}$, we have that $\mathrm{d}(X+Y) \geq s$. If $\mathrm{d}(X+Y) > s$ then, by Corollary 3.2.2 and Remark 3.2.4, we have $X + Y \in \mathfrak{g}_{x,s^+} + \mathcal{N}$ and so $X \in \mathcal{N} + \mathfrak{g}_{x,s^+}$. But then from Corollary 3.2.6 we have $\mathrm{d}(X) > s$, a contradiction. □

LEMMA 3.3.8. *Assume that F has characteristic* 0. *If $X \in \mathfrak{g}$ has Jordan decomposition $X = X_s + X_n$, then* $\mathrm{d}(X) = \mathrm{d}(X_s)$.

Proof. Fix $z \in \mathcal{B}(G)$ such that $X_s \in \mathfrak{g}_{z,\mathrm{d}(X_s)} \setminus \mathfrak{g}_{z,\mathrm{d}(X_s)^+}$. Because zero is in the p-adic closure of the $C_G(X_s)$-orbit of X_n, there exists a $g \in C_G(X_s)$ such that ${}^gX_n \in \mathfrak{g}_{z,\mathrm{d}(X_s)^+}$. Therefore,

$$X = {}^{g^{-1}}({}^gX) = {}^{g^{-1}}(X_s + {}^gX_n) \in {}^{g^{-1}}(\mathfrak{g}_{z,\mathrm{d}(X_s)}) = \mathfrak{g}_{g^{-1}z,\mathrm{d}(X_s)}.$$

Let $y = g^{-1}z$. Then $X \in \mathfrak{g}_{y,\mathrm{d}(X_s)} \subset \mathfrak{g}_{\mathrm{d}(X_s)}$, which implies that $\mathrm{d}(X) \geq \mathrm{d}(X_s)$. Note that $X_n \in \mathfrak{g}_{y,\mathrm{d}(X_s)^+}$.

On the other hand, if there is a point $x \in \mathcal{B}(G)$ such that $X \in \mathfrak{g}_{x,\mathrm{d}(X_s)^+}$, then $X \in \mathfrak{g}_{y,\mathrm{d}(X_s)^+} + \mathcal{N}$, which implies that $X_s \in \mathfrak{g}_{y,\mathrm{d}(X_s)^+} + \mathcal{N}$. But then Corollary 3.2.6 yields $X_s \in \mathfrak{g}_{\mathrm{d}(X_s)^+}$, a contradiction. □

3.4. *G-Domains*

DEFINITION 3.4.1. A set $V \subset \mathfrak{g}$ is called a *G-domain* provided that V is G-invariant, open, and closed.

The main results of this section were inspired by ideas from the study of GL_n. For this paragraph, let $G = \mathrm{GL}_n(F)$ and $\mathfrak{g} = \mathrm{M}_n(F)$. Let P be a proper parabolic subgroup of G with Levi decomposition $P = MN$ and Lie algebras $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$. Suppose that $\mathfrak{k} = \mathrm{M}_n(R)$ and that $\mathfrak{m}$ is in standard form (i.e., $\mathfrak{m} = \prod \mathrm{M}_{n_i}(F)$ is embedded in the usual way). If $V = {}^G\mathfrak{k}$ then it is clear that V is open and

G-invariant. It is also true that V is closed. Therefore, V is a G-domain in $\mathfrak{g}$. Furthermore, $V \cap \mathfrak{m}$ is an M-domain in $\mathfrak{m}$ and equals ${}^{M}(\mathfrak{k} \cap \mathfrak{m})$.

The following results extend these ideas to arbitrary G.

LEMMA 3.4.2. *For all $r \in \mathbf{R}$,*

$$\mathfrak{g}_r = \bigcap_{x \in \mathcal{B}(G)} (\mathfrak{g}_{x,r} + \mathcal{N}).$$

Proof. Corollary 3.2.2 implies that the left-hand side is a subset of the right. Let us suppose that X is contained in the right-hand side but not the left and then derive a contradiction.

If $X \notin \mathfrak{g}_r$, then it follows that $\mathrm{d}(X) < r$ and there exists a $y \in \mathcal{B}(G)$ such that $X \in \mathfrak{g}_{y,\mathrm{d}(X)} \cap (\mathfrak{g}_{y,r} + \mathcal{N})$. Since $\mathfrak{g}_{y,r} \subset \mathfrak{g}_{y,\mathrm{d}(X)^+}$, we have that $X + \mathfrak{g}_{y,\mathrm{d}(X)^+}$ is degenerate in $\mathfrak{g}_{y,\mathrm{d}(X)}/\mathfrak{g}_{y,\mathrm{d}(X)^+}$. But then Corollary 3.2.6 implies that $X \in \mathfrak{g}_{\mathrm{d}(X)^+}$, a contradiction. □

COROLLARY 3.4.3. *If $r \in \mathbf{R}$, then $\mathfrak{g}_r$ is a G-domain.*

REMARK 3.4.4. In fact, $\mathfrak{g}_r$ is $\mathrm{Aut}_F(\mathbf{G})$-invariant.

Proof. Because $\mathfrak{g}_r$ is G-invariant and open, we need only show that $\mathfrak{g}_r$ is closed. Since the complement of $\mathfrak{g}_{x,r} + \mathcal{N}$ is the (disjoint) union of open cosets, the set $\mathfrak{g}_{x,r} + \mathcal{N}$ is closed. Our result now follows from Lemma 3.4.2. □

EXAMPLE 3.4.5. If $G = \mathrm{GL}_n(F)$, then $\mathfrak{g}_0 = {}^{G}\mathfrak{k}$ (in the notation introduced prior to Lemma 3.4.2).

EXAMPLE 3.4.6. If $G = \mathrm{SL}_2(F)$, we have

$$\mathfrak{g}_0 = {}^{G}\begin{pmatrix} R & R \\ R & R \end{pmatrix} \cup {}^{G}\begin{pmatrix} R & \wp^{-1} \\ \wp & R \end{pmatrix}$$

and

$$\mathfrak{g}_{1/2} = {}^{G}\begin{pmatrix} \wp & R \\ \wp & \wp \end{pmatrix},$$

where, for example, $\left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right)$ is interpreted to mean the set of matrices in $\mathfrak{sl}_2(F)$ with entries in R. Note that, up to scaling, these are the only two G-domains of the form $\mathfrak{g}_r$ that occur in $\mathfrak{sl}_2(F)$.

COROLLARY 3.4.7. *$\mathcal{N}'$ is closed (in the p-adic topology).*

Proof. This is immediate from Corollary 3.4.3 and Lemma 3.3.2. □

3.5. Parabolic Descent

We now will show that the $\mathfrak{g}_r$ behave well with respect to parabolic descent.

Suppose that P is the group of F-rational points of a parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ defined over F. Let M denote the group of F-rational points of an F-Levi factor

M of **P**. Denote by N the unipotent radical of P, and let $\bar{P} = M\bar{N}$ be the parabolic opposite $P = MN$. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ and $\bar{\mathfrak{p}} = \mathfrak{m} + \bar{\mathfrak{n}}$ be the associated Lie algebras. Recall that $\mathfrak{g}$ has the direct sum decomposition

$$\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{n}.$$

If $X \in \mathfrak{g}$, then X can be written uniquely as $X_{\bar{\mathfrak{n}}} + X_{\mathfrak{m}} + X_{\mathfrak{n}}$, where $X_{\bar{\mathfrak{n}}} \in \bar{\mathfrak{n}}$, $X_{\mathfrak{m}} \in \mathfrak{m}$, and $X_{\mathfrak{n}} \in \mathfrak{n}$. Let $\mathcal{N}_{\mathfrak{m}}$ denote the set of nilpotent elements in $\mathfrak{m}$.

The following result of Moy and Prasad is [13, Prop. 4.7]. The proof presented here is independent of [11].

PROPOSITION 3.5.1 [13]. *Suppose that $x \in \mathcal{B}(M) \subset \mathcal{B}(G)$, $r \in \mathbf{R}$, and $X \in \mathfrak{g}_{x,r} \setminus \mathfrak{g}_{x,r^+}$. If $(X + \mathfrak{g}_{x,r^+}) \cap \mathcal{N} \neq \emptyset$, then there exists an element $n \in N \cap G_x$ such that*

$$(({}^nX)_{\mathfrak{m}} + \mathfrak{m}_{x,r^+}) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset.$$

Proof. Since $(X + \mathfrak{g}_{x,r^+}) \cap \mathcal{N} \neq \emptyset$, Remark 3.2.7 shows that there exist an alcove $C' \subset \mathcal{B}(G)$ and a $z \in \bar{C}'$ such that $x \in \bar{C}'$ and $(X + \mathfrak{g}_{x,r^+}) \subset \mathfrak{g}_{z,r^+}$. It follows from Lemma 2.4.1 that there exists an $n \in G_x \cap N$ such that $nz \in \mathcal{B}(M)$. This implies that

$${}^nX + \mathfrak{g}_{x,r^+} = {}^n(X + \mathfrak{g}_{x,r^+}) \subset \mathfrak{g}_{nz,r^+}.$$

From this it follows that $({}^nX)_{\mathfrak{m}} + \mathfrak{m}_{x,r^+} \subset \mathfrak{m}_{nz,r^+}$, and by Corollary 3.2.6 we are done. □

COROLLARY 3.5.2. *Suppose that $x \in \mathcal{B}(M)$, $r \in \mathbf{R}$, and $X \in \mathfrak{m}_{x,r} \setminus \mathfrak{m}_{x,r^+}$. Then*

$$(X + \mathfrak{g}_{x,r^+}) \cap \mathcal{N} \neq \emptyset \iff (X + \mathfrak{m}_{x,r^+}) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset.$$

Proof. Since $\mathcal{N}_{\mathfrak{m}} \subset \mathcal{N}$ and $X + \mathfrak{m}_{x,r^+} \subset X + \mathfrak{g}_{x,r^+}$, it is clear that

$$(X + \mathfrak{m}_{x,r^+}) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset \implies (X + \mathfrak{g}_{x,r^+}) \cap \mathcal{N} \neq \emptyset.$$

Now suppose that $(X + \mathfrak{g}_{x,r^+}) \cap \mathcal{N} \neq \emptyset$. From Proposition 3.5.1 there exists an $n \in N \cap G_x$ such that $(({}^nX)_{\mathfrak{m}} + \mathfrak{m}_{x,r^+}) \cap \mathcal{N}_{\mathfrak{m}} \neq \emptyset$. However, ${}^nX = X + Z$, where Z is an element of $\mathfrak{n}$. □

Recall that $\mathfrak{m}_r = \bigcup_{x \in \mathcal{B}(M)} \mathfrak{m}_{x,r}$.

LEMMA 3.5.3. *For $r \in \mathbf{R}$,*

$$\mathfrak{g}_r \cap \mathfrak{m} = \mathfrak{m}_r.$$

Proof. It is clear that $\mathfrak{m}_r \subset \mathfrak{g}_r \cap \mathfrak{m}$.

Suppose that $X \in \mathfrak{g}_r \cap \mathfrak{m}$ and $X \notin \mathfrak{m}_r$. We will derive a contradiction. By Remark 3.2.4 there exists an $s < r$ such that $X \in \mathfrak{m}_s \setminus \mathfrak{m}_{s^+}$; hence there exists an $x \in \mathcal{B}(M)$ such that $X \in \mathfrak{m}_{x,s} \setminus \mathfrak{m}_{x,s^+}$.

Since $X \in \mathfrak{g}_r \subset \mathfrak{g}_{s^+} \subset \mathfrak{g}_{x,s^+} + \mathcal{N}$, Corollary 3.5.2 implies that $X \in \mathfrak{m}_{x,s^+} + \mathcal{N}_{\mathfrak{m}}$. Thus, by Corollary 3.2.6 we have $X \in \mathfrak{m}_{s^+}$, a contradiction. □

REMARK 3.5.4. From Remark 3.2.4 and Lemma 3.5.3 it follows that $\mathfrak{m}_{r^+} = \mathfrak{g}_{r^+} \cap \mathfrak{m}$.

COROLLARY 3.5.5.

$$\mathcal{N}' \cap \mathfrak{m} = \mathcal{N}'_{\mathfrak{m}}.$$

Proof.

$$\mathcal{N}' \cap \mathfrak{m} = \left(\bigcap \mathfrak{g}_r\right) \cap \mathfrak{m} = \bigcap (\mathfrak{g}_r \cap \mathfrak{m}) = \bigcap \mathfrak{m}_r = \mathcal{N}'_{\mathfrak{m}}. \qquad \square$$

COROLLARY 3.5.6. *Let* $\mathrm{d}_{\mathfrak{m}}$ *denote the depth function* (*see Section 3.3*) *on* $\mathfrak{m}$. *Then, for all* $X \in \mathfrak{m}$, *we have*

$$\mathrm{d}_{\mathfrak{m}}(X) = \mathrm{d}(X).$$

3.6. An Alternate Description

The results of Section 3.5 were first proved using different techniques. These techniques achieved slightly more general results at the expense of some mild restrictions on $\mathbf{G}$ and F. Since these results provide a more intuitive understanding of the $\mathfrak{g}_r$, we briefly describe them here. For a subset V of $\mathfrak{g}$, we let $V^{\text{s.s.}}$ denote the set of semisimple elements in V.

Suppose that every maximal F-torus of $\mathbf{G}$ splits over a tamely ramified extension and that $\mathfrak{g}^{\text{s.s.}}$ is dense in $\mathfrak{g}$. Under these hypotheses, we have

$$\mathfrak{g}_r^{\text{s.s.}} = \bigcup_{\mathbf{T}'} \mathfrak{t}'(F)_r,$$

where the union is taken over all maximal F-tori $\mathbf{T}'$ in $\mathbf{G}$ and where $\mathfrak{t}'(F)$ is the Lie algebra of $\mathbf{T}'(F)$. So, roughly speaking, the $\mathfrak{g}_r$ can be interpreted in terms of the valuations of eigenvalues. From this result it follows, under the stated hypothesis on $\mathbf{G}$ and F, that if $\mathbf{M}$ is any reductive subgroup of $\mathbf{G}$ and if $\mathfrak{m}$ denotes the Lie algebra of $\mathbf{M}(F)$ then

$$\mathfrak{m}_r = \mathfrak{g}_r \cap \mathfrak{m}.$$

It also follows from this result that $\mathfrak{g}_r^{\text{s.s.}}$ is stable. That is, if $X \in \mathfrak{g}_r^{\text{s.s.}}$ and $Y \in {}^{\mathbf{G}(\bar{F})}X \cap \mathfrak{g}$ (here $\bar{F}$ denotes an algebraic closure of F), then $Y \in \mathfrak{g}_r^{\text{s.s.}}$.

Unfortunately, all of these results can fail when the hypotheses on $\mathbf{G}$ are not satisfied. For example, they both fail for $\mathrm{PGL}_2(F)$ when F has residual characteristic 2. (Consider the case when $\mathbf{M}$ is a maximal ramified elliptic torus of $\mathrm{PGL}_2(F)$.) Perhaps one could realize analogous results by defining the filtrations on tori and their Lie algebras in a different manner.

3.7. Analogous Results for Moy–Prasad Filtrations of the Group

In this subsection we discuss analogues of the previous results.

3.7.1. NOTATION AND STATEMENT OF THE DEPTH-0 RESULT.

DEFINITION 3.7.1. For $r \in \mathbf{R}_{\geq 0}$ we define

$$G_r := \bigcup_{x \in \mathcal{B}(G)} G_{x,r}$$

and

$$G_{r^+} := \bigcup_{x\in\mathcal{B}(G)} G_{x,r^+}.$$

REMARK 3.7.2. The reasoning of Remark 3.2.4 shows that, for all $r \in \mathbf{R}_{\geq 0}$, we have $G_r = G_{r^+}$ unless r is an optimal number. In particular, for all $r \in \mathbf{R}_{\geq 0}$ there exists an $s > r$ such that $G_{r^+} = G_s$, and for each $r \in \mathbf{R}_{>0}$ there exists a $t \in \mathbf{R}_{\geq 0}$ with $t < r$ such that $G_{t^+} = G_r$.

Suppose P is a parabolic subgroup of G with a Levi decomposition $P = MN$. Let $\bar{P} = M\bar{N}$ denote the parabolic opposite P. If $x \in \mathcal{B}(M)$ and $r > 0$, then $G_{x,r}$ has an Iwahori factorization with respect to $P = MN$. That is, for $g \in G_{x,r}$ we can (uniquely) write

$$g = g_{\bar{N}} \cdot g_M \cdot g_N,$$

where $g_{\bar{N}} \in \bar{N}_{x,r} := G_{x,r} \cap \bar{N}$, $g_M \in M_{x,r} = G_{x,r} \cap M$, and $g_N \in N_{x,r} := G_{x,r} \cap N$.

DEFINITION 3.7.3. Call an element $g \in G$ *unipotent* if there exists a $\lambda \in \mathbf{X}_*^F(\mathbf{G})$ such that $\lim_{t\to 0} {}^{\lambda(t)}g = 1$. Let $\mathcal{U}$ denote the set of unipotent elements, and let $\mathcal{U}'$ denote the set of elements g in G such that the p-adic closure of the G-orbit of g contains the identity.

LEMMA 3.7.4. *If F is perfect or if $\mathfrak{f}$ is finite, then $\mathcal{U} = \mathcal{U}'$.*

Proof. See the proof of Lemma 2.5.1; the direct sum decomposition of $\mathfrak{g}$ will be replaced by the Iwahori decomposition of $G_{x,i}$. □

From [6] we have the following result.

THEOREM 3.7.5.

$$G_0 = \bigcap_{x\in\mathcal{B}(G)} G_x\mathcal{U}$$

and, if M is a Levi subgroup of G, then

$$G_0 \cap M = M_0.$$

3.7.2. ASYMPTOTIC RESULTS. Special cases of the first lemma may be found in [22, Prop. 1.3.2] and [24, Lemma 1.2.8].

LEMMA 3.7.6. *If $x, y \in \mathcal{B}(G)$ and $r > 0$, then $G_{x,r} \subset G_{y,r}\mathcal{U}$.*

REMARK 3.7.7. Since $\mathcal{U}$ is G-invariant, we have $G_{y,r}\mathcal{U} = \mathcal{U}G_{y,r} = G_{y,r}\mathcal{U}G_{y,r}$.

Proof. See the proof of Lemma 3.2.1. □

COROLLARY 3.7.8. *If $x \in \mathcal{B}(G)$ and $r \in \mathbf{R}_{\geq 0}$, then $G_r \subset G_{x,r}\mathcal{U}$.*

Proof. If $r = 0$, this is covered by Theorem 3.7.5. If $r > 0$, this follows from Lemma 3.7.6. □

DEFINITION 3.7.9. Suppose $y \in \mathcal{B}(G)$ and $s \in \mathbf{R}_{\geq 0}$. A coset $\Xi \in G_{y,s}/G_{y,s^+}$ is called *degenerate* provided that $\Xi \cap \mathcal{U} \neq \emptyset$.

COROLLARY 3.7.10. *Suppose $y \in \mathcal{B}(G)$ and $s \in \mathbf{R}_{\geq 0}$. A coset $\Xi \in G_{y,s}/G_{y,s^+}$ is degenerate if and only if $\Xi \subset G_{s^+}$.*

Proof. See the proof of Corollary 3.2.6. □

REMARK 3.7.11. As in Remark 3.2.7, if $\Xi \in G_{y,s}/G_{y,s^+}$ is degenerate then there exist an alcove $C' \subset \mathcal{B}(G)$ and a point $z \in \bar{C}'$ such that $y \in \bar{C}'$ and $\Xi \subset G_{z,s^+}$.

3.7.3. A DEPTH FUNCTION ON G.

LEMMA 3.7.12. *If $y \in \mathcal{B}(G)$ and $r \in \mathbf{R}_{\geq 0}$, then*

$$G_{(r+1)} \subset {}^{G}G_{y,r} \subset G_r .$$

Proof. See the proof of Lemma 3.3.1. □

LEMMA 3.7.13.

$$\mathcal{U}' = \bigcap_{r \geq 0} G_r .$$

Proof. See the proof of Lemma 3.3.2. □

LEMMA 3.7.14. *If $g \in G_0 \setminus \mathcal{U}'$, then there is a unique number $\mathrm{d}_G(g) = \mathrm{d}(g) \geq 0$ such that $g \in G_{\mathrm{d}(g)} \setminus G_{\mathrm{d}(g)^+}$.*

Proof. See the proof of Lemma 3.3.4. □

DEFINITION 3.7.15. If $g \in \mathcal{U}'$, then $\mathrm{d}(g) := \infty$.

DEFINITION 3.7.16. For $g \in G_0$, we will call $\mathrm{d}(g)$ the *depth* of g.

LEMMA 3.7.17. *The function $\mathrm{d} \colon G_0 \to \mathbf{R}$ is locally constant on $G_0 \setminus \mathcal{U}'$. More precisely, if $g \in G_{x,\mathrm{d}(X)}$ for some $x \in \mathcal{B}$, then d is constant on $gG_{x,\mathrm{d}(X)^+}$.*

Proof. See the proof of Lemma 3.3.7. □

LEMMA 3.7.18. *Assume that F has characteristic 0. Let $g \in G$ have Jordan decomposition $g = su = us$, where s is semisimple and u is unipotent. If $g \in G_0$, then $s \in G_0$ and $\mathrm{d}(g) = \mathrm{d}(s)$. If $g \notin G_0$, then $s \notin G_0$.*

Proof. If $s \in G_0$, then (arguing as in the proof of Lemma 3.3.8) we have $g \in G_0$ and $\mathrm{d}(s) = \mathrm{d}(g)$. Therefore, we need only show that if $g \in G_0$ then $s \in G_0$. Define

$$\mathcal{S} := \{z \in \mathcal{B}(G) \mid gz = z\}.$$

Since $g \in G_0$, by [6, Lemma 4.2.1] we have $\mathcal{S} = \{z \in \mathcal{B}(G) \mid g \in G_z\}$. The set $\mathcal{S}$ is a closed, convex, and u-stable subset of $\mathcal{B}(G)$. Since u has a fixed point in $\mathcal{B}(G)$ and since $\mathcal{B}(G)$ has nonpositive curvature [3, Prop. 3.2.4], there exists a point $z \in \mathcal{S}$ such that $uz = z$. Consequently, from [6, Lemma 4.2.1] we have $u \in G_z$. Thus $s = gu^{-1} \in G_z$. □

3.7.4. G-DOMAINS.

DEFINITION 3.7.19. A subset $S \subset G$ is called a *G-domain* provided that S is G-invariant, open, and closed.

LEMMA 3.7.20. *If $r \in \mathbf{R}_{>0}$, then*

$$G_r = \bigcap_{x \in \mathcal{B}(G)} (G_{x,r}\mathcal{U}).$$

Proof. Fix $r > 0$. Corollary 3.7.8 tells us that the left-hand side is a subset of the right-hand side. Let us suppose that g is an element of the right-hand side but not the left and then derive a contradiction.

From Theorem 3.7.5, we know that $g \in G_0$. Since $g \notin G_r$, we have $0 \leq \mathrm{d}(g) < r$. The proof now mimics the proof of Lemma 3.4.2. □

COROLLARY 3.7.21. *For all $r \in \mathbf{R}_{\geq 0}$, we have that G_r is a G-domain.*

Proof. See the proof of Corollary 3.4.3. □

COROLLARY 3.7.22. *$\mathcal{U}'$ is closed.*

Proof. This follows from Corollary 3.7.21 and Lemma 3.7.13. □

3.7.5. PARABOLIC DESCENT. Suppose P is a parabolic subgroup of G with a Levi decomposition $P = MN$. Let $\mathcal{U}_M$ denote the set of unipotent elements in M.

PROPOSITION 3.7.23. *Suppose $x \in \mathcal{B}(M)$ and $r \in \mathbf{R}_{\geq 0}$. If $g \in G_{x,r} \setminus G_{x,r^+}$ represents a degenerate coset of $G_{x,r}/G_{x,r^+}$, then there exists an $n \in N \cap G_x$ such that $({}^n g)_M \cdot M_{x,r^+} \cap \mathcal{U}_M \neq \emptyset$.*

Proof. See the proof of Proposition 3.5.1. Note that, in the notation of that proof, we have ${}^n g \in G_{nz,r^+}$ and $nz \in \mathcal{B}(M)$, so $({}^n g)_M$ makes sense. □

COROLLARY 3.7.24. *Suppose $x \in \mathcal{B}(M)$ and $r \in \mathbf{R}_{\geq 0}$. If $m \in M_{x,r} \setminus M_{x,r^+}$, then*

$$m \cdot G_{x,r^+} \cap \mathcal{U} \neq \emptyset \iff m \cdot M_{x,r^+} \cap \mathcal{U}_M \neq \emptyset.$$

Proof. See the proof of Corollary 3.5.2. □

LEMMA 3.7.25. *For all $r \in \mathbf{R}_{>0}$ we have $G_r \cap M = M_r$.*

Proof. Fix $r > 0$. By Theorem 3.7.5 we have that $M_r \subset G_r \cap M \subset M_0$. The proof now proceeds as in the proof of Lemma 3.5.3. □

Recall that Theorem 3.7.5 was the identical result for $r = 0$.

COROLLARY 3.7.26.

$$\mathcal{U}'_M = \mathcal{U}' \cap M.$$

COROLLARY 3.7.27. *Let* d_M *denote the depth function on* M. *For all* $m \in M_0$,

$$\mathrm{d}_M(m) = \mathrm{d}(m).$$

4. Invariant Distributions on the Lie Algebra

From now on we assume that $\mathfrak{f}$ is a finite field.

After reviewing the Fourier transform, we introduce a special space of functions. This space is then used to examine some questions concerning G-invariant distributions on $\mathfrak{g}$. All of the results in this section remain valid when the roles of $\mathfrak{g}$ and $\mathfrak{g}^*$ are interchanged.

4.1. Review of the Fourier Transform

Suppose that V is a finite-dimensional vector space over F with vector space dual V^*. As usual, let $C_c^\infty(V)$ denote the space of locally constant and complex-valued functions on V with compact support. Let dv be a Haar measure on V. For any $f \in C_c^\infty(V)$, we define the *Fourier transform* $\hat{f} \in C_c^\infty(V^*)$ of f by

$$\hat{f}(\chi) := \int_V dv\, f(v) \cdot \Lambda(\chi(v))$$

for $\chi \in V^*$. Let $d\chi$ be a Haar measure on V^*. For $f \in C_c^\infty(V^*)$ we exploit the natural identification of V^{**} with V and define the Fourier transform $\hat{f} \in C_c^\infty(V)$ by

$$\hat{f}(v) := \int_{V^*} d\chi\, f(\chi) \cdot \Lambda(\chi(v))$$

for $v \in V$. We normalize our measures so that, for $v \in V$ and $f \in C_c^\infty(V)$,

$$\hat{\hat{f}}(v) = f(-v).$$

A *distribution* on V is a linear function from $C_c^\infty(V)$ to $\mathbf{C}$. For any distribution T on V, the Fourier transform of T is the distribution $\hat{T}$ on V^* given by

$$\hat{T}(f) = T(\hat{f})$$

for all $f \in C_c^\infty(V)$.

Suppose that $\mathcal{H}$ is a subspace of $C_c^\infty(V)$. If T is a distribution on V, then $\mathrm{res}_{\mathcal{H}}\, T$ will denote the restriction of T to $\mathcal{H}$.

Suppose a group H acts on V. For $v \in V$ and $h \in H$, we denote by ${}^h v$ the image of v under the action of h. If $f \in C_c^\infty(V)$ and $h \in H$, we define $f^h \in C_c^\infty(V)$ by

$$f^h(v) := f({}^h v)$$

for $v \in V$. If T is a distribution on V and $h \in H$, then the distribution ${}^h T$ is defined by

$${}^h T(f) := T(f^h)$$

for $f \in C_c^\infty(V)$. The distribution T is said to be H-invariant if ${}^h T = T$ for all $h \in H$. Suppose ω is a closed H-invariant subset of V. We let $J_H(\omega)$ denote the set of H-invariant distributions on V with support in ω.

Because the Haar measures on $\mathfrak{g}$ and $\mathfrak{g}^*$ are G-invariant, we have that $T \in J_G(\mathfrak{g})$ if and only if $\hat{T} \in J_G(\mathfrak{g}^*)$.

If $\mathcal{M} \subset \mathcal{L}$ are lattices in V, then $C(\mathcal{L}/\mathcal{M}) \subset C_c^\infty(V)$ denotes the set of functions on V with support in $\mathcal{L}$ that are invariant with respect to $\mathcal{M}$. The Moy–Prasad filtration lattices are defined so that, if $x, y \in \mathcal{B}(G)$, $r, s \in \mathbf{R}$, and $\mathfrak{g}_{y,s} \subset \mathfrak{g}_{x,r}$, then $f \in C(\mathfrak{g}_{x,r}/\mathfrak{g}_{y,s}) \subset C_c^\infty(\mathfrak{g})$ if and only if $\hat{f} \in C(\mathfrak{g}^*_{y,(-s)^+}/\mathfrak{g}^*_{x,(-r)^+}) \subset C_c^\infty(\mathfrak{g}^*)$. Our normalization of measures implies that

$$\operatorname{meas}_{\mathfrak{g}}(\mathfrak{g}_{x,r}) \cdot \operatorname{meas}_{\mathfrak{g}^*}(\mathfrak{g}^*_{x,(-r)^+}) = 1.$$

4.2. *An Interesting Space of Functions*

The results of this subsection are not formally stated (but do appear) in [5]. Similar statements can be made on the group [6] (where the Fourier transform is realized as an operator-valued Fourier transform on the admissible dual of G).

Suppose that $v \in \mathbf{R}$. Recall that for $r \in \mathbf{R}$ we have $\mathfrak{g}_{r^+} = \bigcup_{s>r} \mathfrak{g}_s$.

DEFINITION 4.2.1.

$$\mathcal{D}_v := \{f \in C_c^\infty(\mathfrak{g}) : \operatorname{supp}(\hat{f}) \subset \mathfrak{g}^*_{(-v)^+}\}.$$

REMARK 4.2.2. $\mathcal{D}_v$ is G-invariant.

LEMMA 4.2.3.

$$\mathcal{D}_v = \sum_{x \in \mathcal{B}(G)} C_c(\mathfrak{g}/\mathfrak{g}_{x,v}).$$

REMARK 4.2.4. By this notation we mean the following. If

$$f \in \sum_{x \in \mathcal{B}(G)} C_c(\mathfrak{g}/\mathfrak{g}_{x,v}),$$

then there exists an $N \in \mathbf{N}$ such that $f = \sum_{i=1}^N f_i$ and $f_i \in C_c(\mathfrak{g}/\mathfrak{g}_{x_i,v})$ for some $x_i \in \mathcal{B}(G)$.

Proof. It is clear that the right-hand side is a subset of the left-hand side.

Suppose that $f \in \mathcal{D}_v$. Since $f \in C_c^\infty(\mathfrak{g})$, we have $\hat{f} \in C_c^\infty(\mathfrak{g}^*)$. Consequently, there exist an $M \in \mathbf{N}$ and points $x_j \in \mathcal{B}(G)$ for $1 \leq j \leq M$ such that

$$\operatorname{supp}(\hat{f}) \subset \bigcup_{j=1}^M \mathfrak{g}^*_{x_j,(-v)^+}.$$

Since

$$C_c^\infty\left(\bigcup_{j=1}^M \mathfrak{g}^*_{x_j,(-v)^+}\right) = \sum_{j=1}^M C_c^\infty(\mathfrak{g}^*_{x_j,(-v)^+})$$

(this is a general fact about finite collections of closed and open subsets of a topological space), the lemma follows. $\square$

REMARK 4.2.5. For a Levi subgroup M of G with Lie algebra $\mathfrak{m} \subset \mathfrak{g}$, we define $\mathcal{D}_v^M \subset C_c^\infty(\mathfrak{m})$ similarly.

Let x_0 be a special point in $\mathcal{B}$, and let $K = G_{x_0}$. Then, in the language of Harish-Chandra, K is a compact open subgroup of Bruhat–Tits [7, Thm. 5, p. 16]. Indeed, in the context of the Moy–Prasad filtrations, it is clear that $\mathfrak{g}_{x_0,r}$ is well adapted in the sense of [8, Sec. 10.2]. Moreover, for any proper parabolic subgroup P of G, we have $G = PK$. Let dk be the normalized Haar measure on K. If P has Levi decomposition $P = MN$, let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the corresponding Lie algebras. Let $\bar{P} = M\bar{N}$ be the parabolic opposite P and let $\bar{\mathfrak{n}}$ denote the Lie algebra of $\bar{N}$.

DEFINITION 4.2.6. For $f \in C_c^\infty(\mathfrak{g})$, define $f_P \in C_c^\infty(\mathfrak{m})$ by

$$f_P(Y) := \int_{\mathfrak{n}} dZ \int_K dk\, f({}^k(Y+Z))$$

for $Y \in \mathfrak{m}$.

DEFINITION 4.2.7. For $V \subset \mathfrak{g}$ we define $V^\perp \subset \mathfrak{g}^*$ by

$$V^\perp := \{\lambda \in \mathfrak{g}^* \mid \lambda|_V = 0\}.$$

REMARK 4.2.8. We identify $\mathfrak{m}^*$ with $(\mathfrak{n} \oplus \bar{\mathfrak{n}})^\perp$.

DEFINITION 4.2.9. For $f \in C_c^\infty(\mathfrak{g}^*)$, define $f_P \in C_c^\infty(\mathfrak{m}^*)$ by

$$f_P(\mu) := \int_{\mathfrak{p}^\perp} d\lambda \int_K dk\, f({}^k(\mu+\lambda))$$

for $\mu \in \mathfrak{m}^*$.

REMARK 4.2.10. For any G-domain $\nu \subset \mathfrak{g}$, we have $\mathfrak{p} \cap \nu = (\mathfrak{m} \cap \nu) + \mathfrak{n}$. Thus, if f has support in $\mathfrak{g}_r$, then f_P has support in $\mathfrak{m}_r = \mathfrak{g}_r \cap \mathfrak{m}$ (by Lemma 3.5.3).

COROLLARY 4.2.11. *The constant-term map $f \mapsto f_P$ takes $\mathcal{D}_v$ into $\mathcal{D}_v^M$.*

Proof. Recall [8, Lemma 1.4] that the following diagram commutes:

$$\begin{array}{ccc} C_c^\infty(\mathfrak{g}) & \xrightarrow{(\cdot)\hat{}} & C_c^\infty(\mathfrak{g}^*) \\ {\scriptstyle (\cdot)_P}\downarrow & & \downarrow{\scriptstyle (\cdot)_P} \\ C_c^\infty(\mathfrak{m}) & \xrightarrow{(\cdot)\hat{}} & C_c^\infty(\mathfrak{m}^*). \end{array}$$

It is therefore sufficient to show that $\mathrm{supp}((\hat{f})_P) \subset \mathfrak{m}^*_{(-v)^+}$. Since $f \in \mathcal{D}_v$ we have $\mathrm{supp}(\hat{f}) \subset \mathfrak{g}^*_{(-v)^+}$, and by Remark 4.2.10 we have $\mathrm{supp}((\hat{f})_P) \subset \mathfrak{m}^*_{(-v)^+}$. □

REMARK 4.2.12. Corollary 4.2.11 is another example of depth preservation (see the discussion in Section 4.3).

4.3. Induced Distributions

By way of introduction, we first look at the situation on the group. Suppose that π is an irreducible admissible representation of G. Following [13], we associate a nonnegative number $\rho(\pi)$—the depth of π—to π. In [13] one finds the following.

CONJECTURE 4.3.1 (Hales, Moy, and Prasad). *The Harish-Chandra–Howe local expansion for the character Θ_π of π is valid for all regular $g \in G_{x,\rho(\pi)^+}$ for any point x in $\mathcal{B}(G)$.*

In [14] it is shown that parabolic induction preserves depth. Let us make this statement more precise. Suppose that $P = MN$ is a parabolic subgroup of G with unipotent radical N and a Levi factor M. If σ is an admissible irreducible representation of M and if π is an irreducible subquotient of the induced representation $\operatorname{Ind}_{MN}^G \sigma$, then $\rho(\pi) = \rho(\sigma)$. (This result does not depend on whether or not our induction is normalized.)

Given the previous discussion, it is natural to ask: If we assume that the local character expansion for Θ_σ is valid on $M_{\rho(\sigma)^+}$, is the local character expansion of Θ_π valid on $G_{\rho(\sigma)^+}$?

Since the exponential map is not defined everywhere that it would have to be defined in order to answer this question, we will instead consider the analogous question for G-invariant distributions on the Lie algebra.

Let $\mathfrak{m}$ denote the Lie algebra of M. If $\theta \in J_M(\mathfrak{m})$, then we can define (see e.g. [8, Lemma 1.12]) a G-invariant distribution $\operatorname{Ind}_P^G \theta \in J_G(\mathfrak{g})$ by

$$(\operatorname{Ind}_P^G \theta)(f) := \theta(f_P)$$

for all $f \in C_c^\infty(\mathfrak{g})$. We define an induction operation from $J_M(\mathfrak{m}^*)$ to $J_G(\mathfrak{g}^*)$ in a similar fashion.

LEMMA 4.3.2.

$$\operatorname{Ind}_P^G(J_M(\mathcal{N}_\mathfrak{m})) \subset J_G(\mathcal{N}).$$

Proof. This follows immediately from the following fact. If $f \in C_c^\infty(\mathfrak{g})$ and $\operatorname{supp}(f) \cap \mathcal{N} = \emptyset$, then $\operatorname{supp}(f_P) \cap \mathcal{N}_\mathfrak{m} = \emptyset$. □

REMARK 4.3.3. For more information about the image of $\operatorname{Ind}_P^G(J_M(\mathcal{N}_\mathfrak{m}))$ in $J_G(\mathcal{N})$, see [10].

REMARK 4.3.4. Moreover, we have that if $r \in \mathbf{R}$ and $f \in C_c^\infty(\mathfrak{g})$ such that $\operatorname{supp}(f) \cap \mathfrak{g}_r = \emptyset$, then $\operatorname{supp}(f_P) \cap \mathfrak{m}_r = \emptyset$.

DEFINITION 4.3.5. A distribution $D \in J_G(\mathfrak{g})$ has a *local expansion* on $\mathfrak{g}_r$ if and only if there exists a $T \in J_G(\mathcal{N}^*)$ such that $D(f) = \hat{T}(f)$ for all $f \in C_c^\infty(\mathfrak{g}_r)$.

COROLLARY 4.3.6. *If $\theta \in J_M(\mathfrak{m})$ has a local expansion on $\mathfrak{m}_r$, then $\operatorname{Ind}_P^G \theta$ has a local expansion on $\mathfrak{g}_r$.*

Proof. By hypothesis, there exists a distribution $T \in J_M(\mathcal{N}_\mathfrak{m}^*)$ such that $\theta(f) = \hat{T}(f)$ for all $f \in C_c^\infty(\mathfrak{m}_r)$. Thus, for $f \in C_c^\infty(\mathfrak{g}_r)$ we have

$$
\begin{aligned}
(\operatorname{Ind}_P^G \theta)(f) = \theta(f_P) = \hat{T}(f_P) \quad &\text{(by Remark 4.2.10)} \\
= T((f_P)\hat{}\,) = T((\hat{f})_P) \quad &\text{(as in the proof of Corollary 4.2.11)} \\
= (\operatorname{Ind}_P^G T)(\hat{f}) = (\operatorname{Ind}_P^G T)\hat{}\,(f).&
\end{aligned}
$$

However, by Lemma 4.3.2 we have $\operatorname{Ind}_P^G T \in J_G(\mathcal{N}^*)$. □

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