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Some Applications of Bruhat–Tits Theory to Harmonic Analysis on a Reductive p-adic Group

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1. Introduction

Let k denote a field with nontrivial discrete valuation. We assume that k is complete with perfect residue field $\mathfrak{f}$. Let G denote the group of k-rational points of a reductive, connected, linear algebraic group $\mathbf{G}$ defined over k and let $\mathfrak{g}$ denote its Lie algebra. Let $\mathcal{B}(G)$ denote the Bruhat–Tits building of G.

The basic tools of harmonic analysis on $\mathfrak{g}$ are invariant distributions and the Fourier transform. In [1; 12] the formalism of Moy and Prasad [17; 18] is used to develop a "uniform" way to describe both the support of invariant distributions and how certain important spaces of functions behave with respect to the Fourier transform. The purpose of this paper is to prove the group analogues of these results. As discussed below, these results are more difficult to obtain than their Lie algebra counterparts.

We begin by studying a relationship between the structure of G and the geometry associated to the displacement function on $\mathcal{B}(G)$. Fix $g \in G$. In Section 3.1 we associate to g a Levi subgroup, M_g. We then show that either g fixes a point in $\mathcal{B}(G)$ or there is a line in $\mathcal{B}(G)$ on which g acts by nontrivial translation, but not both. (A line in a building is a 1-dimensional affine subspace of an apartment.) This result (Corollary 3.1.5) uses the nonpositive curvature of $\mathcal{B}(G)$, and it is the basis for many of the results of the paper. Define the *displacement function* d_g on $\mathcal{B}(G)$ by setting $\mathrm{d}_g(x)$ equal to the distance g moves x. We show that the set of elements in $\mathcal{B}(G)$ where d_g assumes its minimum value is nonempty. It follows [7, Chap. II] that the subset of $\mathcal{B}(G)$ where d_g assumes its minimum value can be characterized as either the set of g-fixed points in $\mathcal{B}(G)$ or the union of lines in $\mathcal{B}(G)$ on which g acts by nontrivial translation. We then show that if ℓ is a line on which g acts by nontrivial translation, then the Levi subgroup M_g is equal to the Levi subgroup of G naturally associated to ℓ.

Suppose $r \geq 0$. We next obtain group analogues of the results on $\mathfrak{g}$ of [12, Sec. 1.6] (see also [1, Sec. 3.1]); these results are used to describe the support of invariant distributions. We show that

$$\bigcup_{x \in \mathcal{B}(G)} G_{x,r} = \bigcap_{x \in \mathcal{B}(G)} G_{x,r} \cdot \mathcal{U}. \tag{†}$$

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Here $\mathcal{U}$ denotes the set of unipotent elements, and $G_{x,r}$ is the Moy–Prasad filtration subgroup of G defined in [17; 18]. The chief obstacle to obtaining these results was that for G the "depth" at a point $x \in \mathcal{B}(G)$ can take only nonnegative values. For the Lie algebra, if $X \in \mathfrak{g}$ and $x \in \mathcal{B}(G)$, then there exists an $r \in \mathbb{R}$ such that X belongs to the Moy–Prasad filtration lattice $\mathfrak{g}_{x,r}$. However, for $h \in G$ and $x \in \mathcal{B}(G)$ we can speak about the "depth" of h at x only if h belongs to the parahoric subgroup associated to x. The key is to show that our problems can be reduced to the "depth zero" (or parahoric) situation by using the nonpositive curvature of $\mathcal{B}(G)$. Once this is accomplished, the proofs mirror the proofs for the Lie algebra. We also show that the set defined by equation (†) behaves well with respect to parabolic descent. That is, for a parabolic subgroup P of G with Levi decomposition $P = MN$, we have

$$M \cap \Bigl(\bigcup_{x \in \mathcal{B}(G)} G_{x,r} \Bigr) = \bigcup_{x \in \mathcal{B}(M)} M_{x,r}.$$

Under the assumption that $\mathfrak{f}$ is finite, we consider the interplay between representations of G, their depth, and functions on G. The main result in this section is a group analogue of [1, Lemma 4.2.3]. We begin by redefining the depth of a representation in a way that is independent of the Lie algebra. (This is necessary because the usual definition [17; 18] requires that $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+} \cong G_{x,r}/G_{x,r^+}$ for $r > 0$; Jiu-Kang Yu was the first to notice that this is not always true.) For this definition of depth, we verify the usual facts about depth. For example, we show that the depth of a smooth representation is rational and that parabolic induction preserves depth. Finally, we study how the local constancy of a function on G is related to the depths of the smooth irreducible representations occurring in the function's Plancherel support. For example, we show that a complex-valued, locally constant, compactly supported function of G that has Plancherel support in the smooth irreducible representations of G that possess Iwahori-fixed vectors is necessarily a finite sum of functions each of which is right-invariant with respect to some Iwahori subgroup of G.

Some of the results presented in Section 3 are well known. For example, under the hypotheses that $\mathfrak{f}$ is finite and $\mathbf{G}$ is semisimple, variants of some of the results in Section 3 occur in [14]. Since most of Section 3 was motivated by Allen Moy's suggestion that there exists a geometric interpretation for the displacement results of [2; 16], it is not surprising that some of the results of Section 3 occur in [2; 16]. In any case, the proofs presented here are different and more general than those that occur in these other sources; we require this extra generality.

I thank Philip Kutzko and Gopal Prasad for allowing me to use their proofs (of Lemma 5.4.4 and of Lemmas 3.13, 3.2.1, and 3.4.2, respectively). I thank Robert Kottwitz for his many helpful suggestions. In particular, one of his comments evolved into Definition 5.3.1; this definition (and its Lie algebra analogue) has greatly simplified our approach to various homogeneity problems. I thank Gopal Prasad for his encouragement and many suggestions, particularly regarding Section 3. I thank the referee for many helpful comments. This paper has benefited from discussions with Jeff Adler, Jahwan Kim, Robert Kottwitz, Philip Kutzko, Allen Moy, Amritanshu Prasad, Gopal Prasad, Alan Roche, Paul J. Sally, Jr., and Jiu-Kang Yu. It is a pleasure to thank all of these people.

2. Notation

2.1. Basic Notation

Let k denote a field with nontrivial discrete valuation ν. We denote by ν the unique extension of ν to any algebraic extension of k. We assume that k is complete and that the residue field $\mathfrak{f}$ is perfect. Denote the ring of integers of k by R and the prime ideal by $\wp$. Fix a uniformizing element ϖ. Then $\wp = \varpi R$ and $\mathfrak{f} = R/\wp$. Let K denote a fixed maximal unramified extension of k.

Let $\mathbf{G}$ be a connected, reductive, linear algebraic group defined over k. We let $G = \mathbf{G}(k)$, the group of k-rational points of $\mathbf{G}$. We denote by $\boldsymbol{\mathfrak{g}}$ the Lie algebra of $\mathbf{G}$. We let $\mathfrak{g} = \boldsymbol{\mathfrak{g}}(k)$, the vector space of k-rational points of $\boldsymbol{\mathfrak{g}}$. We use $\mathcal{D}G$ to denote the group of k-rational points of the derived group of $\mathbf{G}$.

Let L be the minimal Galois extension of K such that $\mathbf{G}$ is L-split. As in [18], we normalize ν by requiring $\nu(L^\times) = \mathbb{Z}$.

If $g, h \in G$, then ${}^g h = ghg^{-1}$. If $S \subset G$, we let ${}^G S$ denote the set $\{{}^g s \mid g \in G$ and $s \in S\}$. If $h \in G$ then we write ${}^G h$ for ${}^G\{h\}$, the G-orbit of h.

An element $h \in G$ is *unipotent* provided that there exists a $\lambda \in \mathbf{X}_*^k(\mathbf{G})$ such that $\lim_{t\to 0} {}^{\lambda(t)}h = 1$. Let $\mathcal{U}$ denote the set of unipotent elements in G. It is more usual to say that an element is unipotent if the Zariski closure of its G-orbit contains 1. Let $\mathcal{U}''$ denote the set of elements in G that are unipotent in this sense. We let $\mathcal{U}'$ denote the set of elements in G that contain the identity in the p-adic closure of their G-orbit. It follows that $\mathcal{U} \subseteq \mathcal{U}' \subseteq \mathcal{U}''$. From [15] we have that $\mathcal{U} = \mathcal{U}''$ if k is perfect. From [1, Lemma 3.7.4] it follows that, if k is perfect or $\mathfrak{f}$ is finite, then $\mathcal{U} = \mathcal{U}'$.

If a group H acts on a set S, then S^H denotes the set of H-fixed points of S.

2.2. Apartments, Buildings, and Associated Notation

Let $\mathcal{B}(G) = \mathcal{B}(\mathbf{G}, k)$ denote the (enlarged) Bruhat–Tits building of G. Let $\mathcal{B}^{\mathrm{red}}(G)$ denote the reduced Bruhat–Tits building of G; that is, $\mathcal{B}^{\mathrm{red}}(G) = \mathcal{B}(\mathcal{D}G)$.

We let $\mathrm{dist}\colon \mathcal{B}(G) \times \mathcal{B}(G) \to \mathbb{R}_{\geq 0}$ denote a (nontrivial) G-invariant distance function as discussed in [22, Sec. 2.3]. For $x, y \in \mathcal{B}(G)$, let $[x, y]$ denote the geodesic in $\mathcal{B}(G)$ from x to y and let $(x, y]$ denote $[x, y] \setminus \{x\}$. We define (x, y) and $[x, y)$ similarly.

For a k-Levi subgroup $\mathbf{M}$ of $\mathbf{G}$, we identify $\mathcal{B}(\mathbf{M}, k)$ in $\mathcal{B}(\mathbf{G}, k)$. There is not a canonical way to do this, but every natural embedding of $\mathcal{B}(\mathbf{M}, k)$ in $\mathcal{B}(\mathbf{G}, k)$ has the same image. For $\Omega \subset \mathcal{B}(G)$, we let $\mathrm{stab}_G(\Omega)$ denote the stabilizer of Ω in G.

Given a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$, we have the torus $S = \mathbf{S}(k)$ in G and the corresponding apartment $\mathcal{A}(S) = \mathcal{A}(\mathbf{S}, k)$ in $\mathcal{B}(G)$. We let $\langle \cdot, \cdot \rangle$ denote the extension of the perfect pairing between $\mathbf{X}^*(\mathbf{S})$ and $\mathbf{X}_*(\mathbf{S})$ to a pairing of $\mathbf{X}^*(\mathbf{S}) \otimes \mathbb{R}$ and $\mathbf{X}_*(\mathbf{S}) \otimes \mathbb{R}$.

We let $\Phi(S) = \Phi(\mathcal{A}) = \Phi(\mathbf{S}, k)$ denote the set of roots of $\mathbf{G}$ with respect to k and $\mathbf{S}$; we denote by $\Psi(S) = \Psi(\mathcal{A}) = \Psi(\mathbf{S}, k, \nu)$ the set of affine roots of $\mathbf{G}$ with respect to k, $\mathbf{S}$, and ν. If $\psi \in \Psi(\mathcal{A})$, then $\dot{\psi} \in \Phi(\mathcal{A})$ denotes the gradient of ψ.

For $\psi \in \Psi(\mathcal{A})$, let U_ψ denote the corresponding subgroup of the root group $U_{\dot{\psi}}$ (see [17, Sec. 2.4]).

2.3. *The Moy–Prasad Filtrations of G*

In [17; 18], Allen Moy and Gopal Prasad associate to a pair $(x, r) \in \mathcal{B}(G) \times \mathbb{R}_{\geq 0}$ a subgroup $G_{x,r}$ in G. Although they consider only finite $\mathfrak{f}$ in [17; 18], there is no difficulty in extending their definition to our situation (see [1, Sec. 2.2]). We will not repeat the definition here. Recall that $G_{x,r^+} := \bigcup_{s>r} G_{x,s}$.

For $x \in \mathcal{B}(G)$, we will denote the parahoric subgroup attached to x by G_x $(= G_{x,0})$, and we denote its pro-unipotent radical $G_{x,0^+}$ by G_x^+. Note that both G_x and G_x^+ depend only on the facet of $\mathcal{B}(G)$ to which x belongs. If F is a facet in $\mathcal{B}(G)$ and $x \in F$, then we define $G_F = G_x$ and $G_F^+ = G_x^+$. Recall that G_x is a subgroup of $\mathrm{stab}_G(x)$.

For $x \in \mathcal{B}(G)$, the quotient G_x/G_x^+ is the group of $\mathfrak{f}$-rational points of a connected reductive group G_x defined over $\mathfrak{f}$.

2.4. *Optimal Points*

Let $\mathcal{O}$ denote a choice of optimal points in $\mathcal{B}(G)$ (see [12, Sec. 1.4] or [1, Sec. 2.3]). The set $\mathcal{O}$ is invariant under the action of G on $\mathcal{B}(G)$ and has several other properties. For example, the set $\{r \in \mathbb{R}_{\geq 0} \mid G_{x,r} \neq G_{x,r^+},\ x \in \mathcal{O}\}$ is a discrete subset of $\mathbb{Q}$. The elements of this set are called *optimal numbers*. Also, if $(x, r) \in \mathcal{B}(G) \times \mathbb{R}_{\geq 0}$, then there exist $y, z \in \mathcal{O}$ such that

$$G_{y,r^+} \subset G_{x,r^+} \subset G_{z,r^+}.$$

This follows from [1, Cor. 2.3.3].

3. Points, Lines, and the Displacement Function

Suppose $g \in G$. The displacement function is defined by $\mathrm{d}_g(x) = \mathrm{dist}(x, gx)$ for $x \in \mathcal{B}(G)$, and it is continuous. Define $\mathrm{d}(g) := \inf_{x \in \mathcal{B}(G)} \mathrm{d}_g(x)$ and $\mathcal{B}(g) := \{x \in \mathcal{B}(G) \mid \mathrm{d}_g(x) = \mathrm{d}(g)\}$. In this section, we provide a geometric interpretation of $\mathrm{d}(g)$ and $\mathcal{B}(g)$. That this should be possible was first pointed out to me by Allen Moy.

3.1. *Lines in Apartments*

DEFINITION 3.1.1. If $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$, then a 1-dimensional affine subspace of $\mathcal{A}$ will be called a *line*.

As in [19], we can associate to an element g of G a parabolic subgroup P_g of G and a Levi subgroup M_g of P_g, as follows.

DEFINITION 3.1.2. Suppose $g \in G$. We define the subgroup

$$P_g := \{h \in G \mid \text{the sequence } \{g^i h g^{-i} \mid i \in \mathbb{N}\} \text{ is bounded}\}$$

of G. Note that P_g is the group of k-rational points of a parabolic subgroup $\mathbf{P}_g$ of $\mathbf{G}$ defined over k [19]. Let $\bar{P}_g = P_{g^{-1}}$. Then $\bar{P}_g$ is a parabolic opposite P_g, and the Levi subgroup $M_g := P_g \cap \bar{P}_g$ is the group of k-rational points of a Levi subgroup $\mathbf{M}_g$ of $\mathbf{P}_g$ defined over k.

The original statement and proof of Lemma 3.1.3 applied only to those g that belonged to a maximal k-torus of $\mathbf{G}$ that split over a tamely ramified extension. The proof here is due to Gopal Prasad.

LEMMA 3.1.3. *Suppose $g \in G$. Then either*

(1) *there exists an $x \in \mathcal{B}(G)$ such that $gx = x$, or*
(2) *there exist an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ and a line ℓ in $\mathcal{A}$ such that g acts on ℓ by nontrivial translation.*

Proof (G. Prasad). Suppose $g \in G$. Let Z denote the group of k-rational points of the maximal k-split torus in the center of $\mathbf{M}_g$. Then, according to [20], we have $\mathcal{B}(M_g) = \mathcal{B}(Z) \times \mathcal{B}^{\text{red}}(M_g)$. The action of $g \in M_g$ on $\mathcal{B}(Z)$ is given by translation. Moreover, the image of the group $\langle g \rangle$ in M_g/Z is a bounded group. Therefore, g has a fixed point in $\mathcal{B}^{\text{red}}(M_g)$. Since $\mathcal{B}(M_g) \subset \mathcal{B}(G)$, this proves the lemma. □

LEMMA 3.1.4. *Let $\mathcal{A}$ be an apartment and w a point in $\mathcal{B}(G)$. Suppose that ℓ is a line in $\mathcal{A}$ and that x_1 and x_2 are two distinct points on ℓ such that* $\operatorname{dist}(x_1, w) = \operatorname{dist}(x_2, w)$. *Then*

(1) *for all $y \in (x_1, x_2)$ we have* $\operatorname{dist}(y, w) < \operatorname{dist}(x_1, w)$, *and*
(2) *for all $y \in \ell \setminus [x_1, x_2]$ we have* $\operatorname{dist}(y, w) > \operatorname{dist}(x_1, w)$.

Proof. This is a consequence of the nonpositive sectional curvature of $\mathcal{B}(G)$ (see [22, Sec. 2.3]). □

COROLLARY 3.1.5. *Suppose $g \in G$. Exactly one of the following statements is true.*

(1) *There exists an $x \in \mathcal{B}(G)$ such that $gx = x$.*
(2) *There exist an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ and a line ℓ in $\mathcal{A}$ such that g acts on ℓ by nontrivial translation.*

Proof. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ and ℓ a line in $\mathcal{A}$. Let x be a point in $\mathcal{B}(G)$. Suppose that g both fixes x and acts by nontrivial translation on ℓ. Choose y on ℓ and $z \in (y, gy)$. We have that z lies on ℓ and $gy \in (z, gz)$. We also have $\operatorname{dist}(y, x) = \operatorname{dist}(gy, x)$ and $\operatorname{dist}(z, x) = \operatorname{dist}(gz, x)$. From Lemma 3.1.4 we conclude that $\operatorname{dist}(z, x) < \operatorname{dist}(gy, x)$ and $\operatorname{dist}(gy, x) < \operatorname{dist}(z, x)$, a contradiction. □

3.2. Some Results about Geodesics

We first show that the property of being a geodesic is a local one; this fact is used without proof in [2; 14]. I thank Gopal Prasad for explaining the proof to me.

LEMMA 3.2.1. *Suppose* $x, s, y, t \in \mathcal{B}(G)$ *such that* $s \in [x, y)$. *If* $y \in [s, t]$, *then* $y \in [x, t]$.

Proof (G. Prasad). Let $q \in [y, t]$ be the point nearest t such that the geodesic $[x, q]$ contains y. Since this is a closed condition and $y \in [y, t]$, the point q exists. If $q = t$, then we are finished.

Suppose $q \neq t$. Let C be an alcove such that $q \in \bar{C}$ and $(q, t] \cap \bar{C} \neq \emptyset$. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ containing both C and the point x. Note that the geodesic $[x, q]$ lies in $\mathcal{A}$.

Choose $q' \in (q, t] \cap \bar{C}$. Since $q' \in [q, t]$ and since $[q, t]$ lies in the geodesic $[y, t]$ that lies in the geodesic $[s, t]$, we conclude that $q' \in [s, t]$. Thus, the geodesic $[s, q]$ lies in both $[s, q']$ and $[x, q]$. Since geodesics in $\mathcal{A}$ are line segments and since $[s, q]$, $[s, q']$, and $[x, q]$ are geodesics in $\mathcal{A}$, we conclude that the geodesic $[x, q']$ contains y. Since $q' \in [y, t]$ is nearer t than q is, we have a contradiction. □

An induction argument yields the following corollary.

COROLLARY 3.2.2. *Fix* $n \in \mathbb{N}_{\geq 2}$. *If* $x_0, x_1, \dots, x_n$ *are points in* $\mathcal{B}(G)$ *such that* x_i *belongs to the geodesic* $[x_{i-1}, x_{i+1}]$ *for all* $0 < i < n$, *then* x_j *belongs to the geodesic* $[x_0, x_n]$ *for all* $0 \leq j \leq n$.

Our final result of this subsection shows that an infinite geodesic is a line.

LEMMA 3.2.3. *If* Γ *is an infinite geodesic in* $\mathcal{B}(G)$, *that is, if*

$$\Gamma = \bigcup_{n \in \mathbb{N}} [x_{-n}, x_n],$$

where $x_{\pm 1}, x_{\pm 2}, \dots$ *in* $\mathcal{B}(G)$ *such that* $[x_{-n}, x_n] \subset [x_{-1-n}, x_{n+1}]$ *for all* $n \in \mathbb{N}$ *and*

$$\min\{\operatorname{dist}(x_1, x_n), \operatorname{dist}(x_1, x_{-n})\} \to \infty,$$

then there exists an apartment $\mathcal{A}$ *in* $\mathcal{B}(G)$ *such that* $\Gamma \subset \mathcal{A}$.

Proof. This is a special case of [8, Prop. 2.8.3]. □

REMARK 3.2.4. If Γ is an infinite geodesic in an apartment $\mathcal{A}$, then Γ is a line in $\mathcal{A}$.

3.3. The Displacement Function

In [2; 16], many properties of the displacement function are derived. We will need the following property; we provide a more direct proof than that given in [2, Prop. 2.4].

LEMMA 3.3.1. *Suppose $g \in G$. If $y \in \mathcal{B}(G)$ such that $\mathrm{d}_g(y) > \mathrm{d}(g)$, then we have*

$$\mathrm{d}_g|_{(y,gy)} < \mathrm{d}_g(y) = \mathrm{d}_g(gy).$$

Proof. We have

$$\mathrm{d}_g(y) = \mathrm{dist}(y, gy) = \mathrm{dist}(gy, g^2 y) = \mathrm{d}_g(gy).$$

Thus we need only establish the inequality $\mathrm{d}_g|_{(y,gy)} < \mathrm{d}_g(y)$.

Suppose there exists a $z \in (y, gy)$ such that $\mathrm{d}_g(z) \geq \mathrm{d}_g(y)$. We will generate a contradiction. From the triangle inequality we have

$$\begin{aligned}\mathrm{d}_g(y) &= \mathrm{dist}(y, gy) = \mathrm{dist}(y, z) + \mathrm{dist}(z, gy) = \mathrm{dist}(z, gy) + \mathrm{dist}(gy, gz)\\ &\geq \mathrm{dist}(z, gz) = \mathrm{d}_g(z).\end{aligned}$$

If $\mathrm{d}_g(z) > \mathrm{d}_g(y)$, then we conclude that $\mathrm{d}_g(z) > \mathrm{d}_g(z)$, a contradiction.

If $\mathrm{d}_g(z) = \mathrm{d}_g(y)$, then we conclude that

$$\mathrm{dist}(z, gy) + \mathrm{dist}(gy, gz) = \mathrm{dist}(z, gz).$$

This implies that $g^m y$ lies on the geodesic $[g^{(m-1)}z, g^m z]$ for all $m \in \mathbb{Z}$. From Corollary 3.2.2 we conclude that

$$y \in [g^{-n}y, g^n y] \subset [g^{-(n+1)}y, g^{(n+1)}y]$$

and $\mathrm{dist}(g^{-n}y, g^n y) = 2n \cdot \mathrm{d}_g(y)$ for all $n \in \mathbb{N}$. Fix $x \in \mathcal{B}(G)$ such that $\mathrm{d}_g(x) < \mathrm{d}_g(y)$. We now argue as in the proof of [2, Prop. 2.4]. Two applications of the triangle inequality yield

$$\begin{aligned}2n \cdot \mathrm{d}_g(y) &= \mathrm{dist}(g^{-n}y, g^n y)\\ &\leq \mathrm{dist}(g^{-n}y, g^{-n}x) + \mathrm{dist}(g^{-n}x, g^n x) + \mathrm{dist}(g^n x, g^n y)\\ &\leq 2 \cdot \mathrm{dist}(x, y) + 2n \cdot \mathrm{d}_g(x)\end{aligned}$$

for all $n \in \mathbb{N}$. We therefore conclude that

$$0 < \mathrm{d}_g(y) - \mathrm{d}_g(x) \leq \mathrm{dist}(x, y)/n$$

for all $n \in \mathbb{N}$, a contradiction. □

COROLLARY 3.3.2. *Suppose $g \in G$. If $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ containing a line ℓ on which g acts by translation, then $\mathrm{d}(g)$ is equal to the distance that g translates any point on ℓ.*

For future reference, we record the following corollary.

COROLLARY 3.3.3. *Suppose $g \in G$. If there exists an $x \in \mathcal{B}^{\mathrm{red}}(G)$ such that $gx = x$, then for all $y \in \mathcal{B}^{\mathrm{red}}(G)$ we have*

$$\mathrm{d}_g|_{(y,gy)} < \mathrm{d}_g(y) = \mathrm{d}_g(gy).$$

Proof. The proof of Lemma 3.3.1 uses only the triangle inequality and the fact that being a geodesic is a local property. These both remain valid for $\mathcal{B}^{\mathrm{red}}(G)$. □

3.4. A Geometric Interpretation of $\mathrm{d}(g)$ *and* $\mathcal{B}(g)$

The proofs and results of this subsection provide a geometric context for some of the standard facts (see e.g. [14, Secs. 5, 6; 16, Secs. 5.5–5.7]) about $\mathrm{d}(g)$ and $\mathcal{B}(g)$.

LEMMA 3.4.1. *If* $g \in G$, *then* $\mathcal{B}(g) \neq \emptyset$.

Proof. From Lemma 3.1.3 we have that either g fixes a point in $\mathcal{B}(G)$ or there exist an apartment $\mathcal{A}$ and a line in $\mathcal{A}$ on which g acts by nontrivial translation. If g fixes a point, then $\mathrm{d}(g) = 0$ and $\mathcal{B}(g) \neq \emptyset$. If g acts by nontrivial translation on a line ℓ in an apartment $\mathcal{A}$, then it follows from Corollary 3.3.2 that every point on ℓ belongs to $\mathcal{B}(g)$. □

The statement and proof of the next result are due to Gopal Prasad.

LEMMA 3.4.2. *If* $g \in G$ *then, for all* $n \in \mathbb{N}$, *we have* $\mathrm{d}(g^n) = n \cdot \mathrm{d}(g)$.

Proof (G. Prasad). If $\mathrm{d}(g) = 0$, then $\mathrm{d}(g^n) = 0$. If $\mathrm{d}(g) > 0$, then from Lemma 3.1.3 there exist an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ and a line ℓ in $\mathcal{A}$ on which g acts by translation. From Corollary 3.3.2, the distance by which g translates any point on ℓ is equal to $\mathrm{d}(g)$. Consequently, the group element g^n translates any point on ℓ by a distance $n \cdot \mathrm{d}(g)$. From Corollary 3.3.2 we now conclude that $\mathrm{d}(g^n) = n \cdot \mathrm{d}(g)$. □

After presenting a definition, we describe $\mathcal{B}(g)$ when $\mathrm{d}(g) \neq 0$.

DEFINITION 3.4.3. Suppose that ℓ is a line in an apartment $\mathcal{A}$ in $\mathcal{B}(G)$. Let $\mathbf{S}$ denote the maximal k-split torus in $\mathbf{G}$ corresponding to $\mathcal{A}$. Let Φ_ℓ denote those k-roots β of $\mathbf{S}$ such that, for any affine root ψ for which $\dot{\psi} = \beta$, we have that the root hyperplane for ψ is parallel (in the Euclidean space $\mathcal{A}$) to the line ℓ; that is, Φ_ℓ is the set of roots that are perpendicular to the "direction of ℓ" under the pairing $\langle \cdot, \cdot \rangle$. Let M_ℓ denote the Levi subgroup of G generated by $C_{\mathbf{G}}(\mathbf{S})(k)$ and the root groups U_β for $\beta \in \Phi_\ell$.

LEMMA 3.4.4. *Suppose* $g \in G$ *such that* $\mathrm{d}(g) \neq 0$.

(1) *If* $x \in \mathcal{B}(G)$ *such that* $\mathrm{d}_g(x) = \mathrm{d}(g)$, *then there exist an apartment* $\mathcal{A}$ *and a line* ℓ *in* $\mathcal{A}$ *such that* x *lies on* ℓ *and* g *acts by translation on* ℓ.
(2) *If* $\mathcal{A}$ *is an apartment in* $\mathcal{B}(G)$ *and* ℓ *is a line in* $\mathcal{A}$ *on which* g *acts by translation, then* $M_\ell = M_g$ (*see Definitions 3.1.2 and 3.4.3*).
(3) *If* $\mathcal{A}$ *is an apartment in* $\mathcal{B}(G)$ *and* ℓ *is a line in* $\mathcal{A}$ *on which* g *acts by translation, then* $\mathcal{A}$ *is an apartment in* $\mathcal{B}(M_g)$ *and the image of* ℓ *in* $\mathcal{B}^{\mathrm{red}}(M_g)$ *is a* g*-fixed point*.
(4) *If* $\mathcal{A}_1$ (*resp.,* $\mathcal{A}_2$) *is an apartment in* $\mathcal{B}(G)$ *and if* ℓ_1 *is a line in* $\mathcal{A}_1$ (*resp.,* ℓ_2 *is a line in* $\mathcal{A}_2$) *on which* g *acts by translation, then there exists an apartment* $\mathcal{A}$ *containing both* ℓ_1 *and* ℓ_2.

REMARK 3.4.5. Since $\mathcal{B}(g) \neq \emptyset$, parts (1) and (4) follow from [7, Thm. II.6.8].

Proof. (1) Suppose $x \in \mathcal{B}(G)$ such that $\mathrm{d}_g(x) = \mathrm{d}(g) \neq 0$. For all $n \in \mathbb{N}$, it follows from the definition of $\mathrm{d}(g^{2n})$, the triangle inequality, and Lemma 3.4.2 that

$$\begin{aligned}\mathrm{d}(g^{2n}) &\leq \mathrm{dist}(x, g^{2n}x) = \mathrm{dist}(g^{-n}x, g^n x) \\ &\leq \sum_{i=-n}^{(n-1)} \mathrm{dist}(g^i x, g^{(i+1)}x) = 2n \cdot \mathrm{d}_g(x) = \mathrm{d}(g^{2n}).\end{aligned}$$

Consequently, we must have

$$\mathrm{dist}(g^{-n}x, g^n x) = \sum_{i=-n}^{(n-1)} \mathrm{dist}(g^i x, g^{(i+1)}x).$$

That is, for all $n \in \mathbb{N}$, the geodesic $[g^{-n}x, g^n x]$ contains the points $g^i x$ for $-n \leq i \leq n$. Consequently, we have $x \in [g^{-n}x, g^n x] \subset [g^{(-1-n)}x, g^{(n+1)}x]$ for all $n \in \mathbb{N}$ and $\mathrm{dist}(x, g^{-n}x) = \mathrm{dist}(x, g^n x) \to \infty$. By Lemma 3.2.3 there is an apartment $\mathcal{A}$ and a line ℓ in $\mathcal{A}$ such that $g^i x$ lies on ℓ for all $i \in \mathbb{Z}$. We conclude that g acts on ℓ by nontrivial translation.

(2) Suppose $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ and ℓ is a line in $\mathcal{A}$ on which g acts by nontrivial translation.

We first show that $M_\ell \subset M_g$. Let Z_ℓ denote the center of M_ℓ. From [20, proof of Lemme 2.4.16, esp. part (e)] we have that $g \in M_\ell$. The image of ℓ in $\mathcal{B}^{\mathrm{red}}(M_\ell)$ is a g-fixed point. Thus, the image of $\langle g \rangle$ in M_ℓ / Z_ℓ is a bounded group. It follows that, for all $m \in M_\ell$, the sequences $\{g^i m g^{(-i)}\}$ and $\{g^{(-i)} m g^i\}$ are bounded. That is, we have $M_\ell \subset M_g$.

Let $\mathbf{S}$ be the maximal k-split torus of $\mathbf{G}$ corresponding to $\mathcal{A}$. Since $\mathcal{A}$ is an apartment in M_ℓ and $M_\ell \subset M_g$, we have $\mathbf{S}(k) \subset M_g$. If $M_\ell \subsetneq M_g$, then there exists a root $\alpha \in \Phi(\mathbf{S}, k)$ such that $U_\alpha \subset M_g \setminus M_\ell$. In this case, the set of points on ℓ formed by looking at the intersection of ℓ with all those hyperplanes in $\mathcal{A}$ that correspond to affine roots of gradient α is infinite and discrete. Thus, there exist $x, y \in \ell$ and $r \in \mathbb{R}$ such that $(M_g)_{x,r} \neq (M_g)_{y,r}$. However, the image of ℓ in $\mathcal{B}^{\mathrm{red}}(M_g)$ is either a point or a line. Since g has a fixed point in $\mathcal{B}^{\mathrm{red}}(M_g)$, from Corollary 3.1.5 we conclude that the image of ℓ in $\mathcal{B}^{\mathrm{red}}(M_g)$ is a point. This means that for all $x, y \in \ell$ and for all $r \in \mathbb{R}$ we have $(M_g)_{x,r} = (M_g)_{y,r}$, a contradiction.

(3) Suppose $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ and ℓ is a line in $\mathcal{A}$ on which g acts by nontrivial translation. Then $\mathcal{A}$ is an apartment in $\mathcal{B}(M_\ell)$ and the image of ℓ in $\mathcal{B}^{\mathrm{red}}(M_\ell)$ is a g-fixed point. The result now follows from (2).

(4) Now suppose we have two lines ℓ_1 in $\mathcal{A}_1$ and ℓ_2 in $\mathcal{A}_2$ as in the statement of (4). By (3), $\mathcal{A}_i$ is an apartment in $\mathcal{B}(M_g)$ and the image of ℓ_i in $\mathcal{B}^{\mathrm{red}}(M_g)$ is a g-fixed point for $i = 1, 2$. From [20] we have $\mathcal{B}(M_g) = \mathcal{B}^{\mathrm{red}}(M_g) \times \mathcal{B}(Z)$, where Z is the group of k-rational points of the maximal k-split torus in the center of $\mathbf{M}_g$. We conclude that g acts on ℓ_1 (resp., ℓ_2) via translation in $\mathcal{B}(Z)$. Thus, since there is an apartment in $\mathcal{B}^{\mathrm{red}}(M_g)$ containing the images of ℓ_1 and ℓ_2, we conclude that there exists an apartment $\mathcal{A}$ in $\mathcal{B}(M_g)$ containing both ℓ_1 and ℓ_2. □

Corollary 3.4.6. *The set $\mathcal{B}(g)$ is convex.*

Proof. If $\mathrm{d}(g) = 0$, this follows from the fact that the action of G on $\mathcal{B}(G)$ takes geodesics to geodesics. If $\mathrm{d}(g) > 0$, this follows from Lemma 3.4.4. $\square$

LEMMA 3.4.7. *The function* $\mathrm{d}\colon G \to \mathbb{R}$ *is a locally constant function whose image is a discrete subset of* $\mathbb{R}$.

Proof. We first show that d is a locally constant function. Fix g in G and $x \in \mathcal{B}(g)$. Let H be the subgroup of G that fixes $[x, gx]$ pointwise. We have that H is an open subgroup of G. It follows from Corollary 3.4.6 that, for all $h \in H$ and all $y \in [x, gx]$, we have

$$\mathrm{d}_{gh}(y) = \mathrm{d}_g(y) = \mathrm{d}(g).$$

An application of Lemma 3.3.1 shows that $\mathrm{d}(gh) = \mathrm{d}(g)$ for all $h \in H$.

We now show that the image of d is a discrete subset of $\mathbb{R}$. Fix a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$ and an alcove C in the apartment $\mathcal{A}(\mathbf{S}, k)$. Since d is a class function, it follows from the Bruhat decomposition [22, Sec. 3.3.1] that the image of d is a subset of

$$\left\{ \min_{x \in \bar{C}} \mathrm{d}_n(x) \mid n \in N_{\mathbf{G}}(\mathbf{S})(k) \right\},$$

which is a discrete subset of $\mathbb{R}$. $\square$

3.5. Two Questions of Allen Moy

In [16, Secs. 5.7, 5.10], Moy poses two questions regarding the set $\mathcal{B}(g)$. These questions stem from the following result (see [16, Cors. 5.7(ii), 5.9]).

LEMMA 3.5.1. *Suppose* $g \in G$.

(1) *If* $h \in G$ *commutes with* g, *then* $h\mathcal{B}(g) \subset \mathcal{B}(g)$.
(2) *If* g *has a Jordan decomposition* $g = su$ *with* u *unipotent and* s *semisimple, then* $\emptyset \neq \mathcal{B}(s)^u \subset \mathcal{B}(g)$ *and* $\emptyset \neq \mathcal{B}(g)^u \subset \mathcal{B}(s)$.

Here $\mathcal{B}(s)^u$ denotes the u-fixed points of $\mathcal{B}(s)$ and similarly for $\mathcal{B}(g)^u$.

QUESTION 3.5.2. Suppose $s \in G$ is a semisimple element contained in the group of k-rational points of some maximal k-split torus of $\mathbf{G}$. Is it true that $\mathcal{B}(s) = \mathcal{B}(C_G(s))$?

From Lemma 3.5.1 we know that the Levi subgroup $C_G(s)$ acts on $\mathcal{B}(s)$. With some restrictions on k and $\mathbf{G}$, we can probably answer this question in the affirmative (see e.g. [13, Cor. 4.4.2]). In general, however, the answer is no. For example, suppose our field is the field $\mathbb{Q}_2$ of 2-adic numbers, $\mathbf{S}$ is a maximal k-split torus in SL_2, and $s \in \mathbf{S}(\mathbb{Z}_2)$ has distinct eigenvalues. In this case, we have that $\mathcal{B}(C_G(s))$ is the apartment in $\mathcal{B}(\mathrm{SL}_2(\mathbb{Q}_2))$ corresponding to $\mathbf{S}$. However, $\mathcal{B}(s)$ is strictly larger than $\mathcal{B}(C_G(s))$ since, for each vertex v of $\mathcal{A}(\mathbf{S}, \mathbb{Q}_2)$, s must fix the three edges adjacent to v. This is a specific example of a general phenomenon discussed in [22, Sec. 3.6.1].

QUESTION 3.5.3. If $g \in G$ has Jordan decomposition $g = su$ with u unipotent and s semisimple, then is it true that $\mathcal{B}(g)^u = \mathcal{B}(g)$?

Unfortunately, the answer to this question is almost always no. Prasad was the first to notice this—he looked at integral elements of $\mathrm{SL}_n(\mathbb{Q}_p)$ with nonintegral Jordan decompositions. Here is a more concrete example. Fix $a \in 1 + \wp$ such that $a \neq 1$. Let $b = (a - a^{-2})$ and consider the elements

$$g = \begin{pmatrix} a & a & 0 \\ 0 & a^{-2} & b \cdot \varpi^{-1} \\ 0 & 0 & a \end{pmatrix}, \quad s = \begin{pmatrix} a & a & -a \cdot \varpi^{-1} \\ 0 & a^{-2} & b \cdot \varpi^{-1} \\ 0 & 0 & a \end{pmatrix},$$

and

$$u = \begin{pmatrix} 1 & 0 & \varpi^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $\mathrm{SL}_3(k)$. We have $g \in \mathrm{SL}_3(R)$, but neither s nor u belong to $\mathrm{SL}_3(R)$. We also have $g = su = us$ with u unipotent and s semisimple.

4. Some Results for Moy–Prasad Filtrations of *G*

4.1. The Main Results

DEFINITION 4.1.1. For $r \in \mathbb{R}_{\geq 0}$, define

$$G_r := \bigcup_{x \in \mathcal{B}(G)} G_{x,r}.$$

For $r \in \mathbb{R}_{\geq 0}$, we also define $G_{r^+} := \bigcup_{x \in \mathcal{B}(G)} G_{x,r^+}$. Note that $G_{r^+} = \bigcup_{s>r} G_s$.

DEFINITION 4.1.2.

$$\mathfrak{U}_G := \bigcup_{x \in \mathcal{B}(G)} \mathrm{stab}_G(x).$$

When there is no possibility for confusion we will write $\mathfrak{U}$ for $\mathfrak{U}_G$. Note that $\mathfrak{U}$ is G-invariant and $\mathcal{U} \subset G_r \subset \mathfrak{U}$ for all $r \in \mathbb{R}_{\geq 0}$.

REMARK 4.1.3. Here is another definition of $\mathfrak{U}$ (due to G. Prasad). The set $\mathfrak{U}$ consists of those $g \in G$ for which (i) all of the eigenvalues of $\mathrm{Ad}(g)$ have modulus 1 and (ii) modulo the commutator subgroup of G, g generates a bounded subgroup. It follows that $\mathfrak{U}$ is closed (see also [21, Lemma 1]).

The following theorems state the main results of this section.

THEOREM 4.1.4.

(1) *If* $r \in \mathbb{R}_{\geq 0}$, *then* $G_r = \bigcap_{z \in \mathcal{B}(G)} G_{z,r} \cdot \mathcal{U}$.
(2) $\mathfrak{U} = \bigcap_{z \in \mathcal{B}(G)} \mathrm{stab}_G(z) \cdot \mathfrak{U}$.

THEOREM 4.1.5. *Suppose P is a parabolic subgroup of G with a Levi decomposition $P = MN$.*

(1) *If $r \in \mathbb{R}_{\geq 0}$, then $M \cap G_r = M_r$.*
(2) $M \cap \mathfrak{U} = \mathfrak{U}_M$.

In the definition of $\mathfrak{U}$, it is important to keep in mind that $\mathcal{B}(G)$ is the enlarged Bruhat–Tits building of G.

4.2. Two Results about Parahoric Subgroups

Another version of the following lemma has been proved by Eugene Kushnirsky, whose proof appears in [13, Lemma 4.5.1].

LEMMA 4.2.1. *If $y \in \mathcal{B}(G)$ and $g \in G_0 \cap \operatorname{stab}_G(y)$, then $g \in G_y$.*

Proof. There exists a $z \in \mathcal{B}(G)$ such that $g \in G_z$. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ containing y and z.

Define
$$V = \{z' \in \mathcal{A} \mid g \in G_{z'}\}.$$
Because V is a closed and nonempty set, we can choose $x \in V$ such that, for all $z' \in V$,
$$\operatorname{dist}(x, y) \leq \operatorname{dist}(z', y).$$
If $x = y$ then there is nothing to do, so we suppose that $x \neq y$ and derive a contradiction. We let F be the first facet in $\mathcal{A}$ through which $(x, y]$ passes as we move from x to y. Note that $x \in \bar{F}$ and $F \cap V = \emptyset$.

Since g fixes $[x, y] \cap F$, we have that g normalizes G_F. Let $\bar{g}$ and $\overline{G_F}$ denote the images of g and G_F in the connected reductive group $\mathsf{G}_x(\mathfrak{f})$. Since $\overline{G_F}$ is a parabolic subgroup of $\mathsf{G}_x(\mathfrak{f})$ and $\overline{G_F}$ is normalized by $\bar{g}$, we have $\bar{g} \in \overline{G_F}$ (see [10, Thm. 8.3.3; 5, Secs. 21.15–21.16]). Therefore, $g \in G_F$. In other words, $F \subset V$, a contradiction. □

The next result concerns the structure of parahoric subgroups of G.

LEMMA 4.2.2. *Suppose that P is a parabolic subgroup of G with a Levi decomposition $P = MN$. Let $\bar{N}$ denote the unipotent radical of the parabolic subgroup opposite $P = MN$. If $\mathbf{S}$ is a maximal k-split torus of $\mathbf{G}$ such that $\mathbf{S}(k) \subset M$ then, for all $x \in \mathcal{A}(\mathbf{S}, k)$, we have*
$$G_x = N_x \cdot \bar{N}_x \cdot N_x \cdot M_x,$$
where M_x is the parahoric subgroup of M associated to x, $N_x = G_x \cap N$, and $\bar{N}_x = G_x \cap \bar{N}$.

Proof. Since $\mathbf{S}(k) \subset M$, the image of $G_x \cap P$ (resp., $G_x \cap M$, $G_x \cap N$, $G_x \cap \bar{N}$) in G_x/G_x^+ is the group of $\mathfrak{f}$-rational points of a parabolic subgroup P (resp., a Levi subgroup M of P, the unipotent radical N of P, the unipotent radical $\bar{\mathsf{N}}$ of the parabolic opposite $\mathsf{P} = \mathsf{M}\mathsf{N}$) of G_x defined over $\mathfrak{f}$. From [6, Prop. 6.25] we have that

$\mathsf{G}_x(\mathfrak{f}) = \mathsf{N}(\mathfrak{f}) \cdot \bar{\mathsf{N}}(\mathfrak{f}) \cdot \mathsf{N}(\mathfrak{f}) \cdot \mathsf{M}(\mathfrak{f})$. Since G_x^+ has the Iwahori decomposition $G_x^+ = (\bar{N} \cap G_x^+) \cdot M_x^+ \cdot (N \cap G_x^+)$, the lemma follows. □

4.3. *A Proof of Theorem 4.1.4*

Lemma 4.3.1 establishes Theorem 4.1.4(2). We note that, since $\mathfrak{U}$ and $\mathcal{U}$ are G-invariant, for all $x \in \mathcal{B}(G)$ we have

$$\mathfrak{U} \cdot \mathrm{stab}_G(x) = \mathrm{stab}_G(x) \cdot \mathfrak{U} \quad \text{and} \quad G_x \cdot \mathcal{U} = \mathcal{U} \cdot G_x.$$

LEMMA 4.3.1.

$$\mathfrak{U} = \bigcap_{z \in \mathcal{B}(G)} \mathrm{stab}_G(z) \cdot \mathfrak{U}.$$

Proof. That the left-hand side is included in the right-hand side is clear. We now turn our attention to the opposite inclusion.

We will argue by contradiction. Suppose that $g \in \bigcap_{z \in \mathcal{B}(G)} \mathrm{stab}_G(z) \cdot \mathfrak{U}$ does not fix a point in $\mathcal{B}(G)$. From Lemma 3.1.3 there must exist an apartment $\mathcal{A}$ and a line ℓ in $\mathcal{A}$ such that g acts on ℓ by (nontrivial) translation.

There exists a facet F in $\mathcal{A}$ such that $F \cap \ell$ is open in ℓ. Since G acts "simplicially" on $\mathcal{B}(G)$, there exist an open subset F' of ℓ and $x \in F'$ such that $F' \subset F$ and $\mathrm{stab}_G(y) = \mathrm{stab}_G(x)$ for all $y \in F'$. By hypothesis, there exist elements $h \in \mathrm{stab}_G(x)$ and $u \in \mathfrak{U}$ such that $g = uh$. Let $w \in \mathcal{B}(G)$ be a fixed point of u. Then, for all $y \in F'$,

$$\mathrm{dist}(w, y) = \mathrm{dist}(w, uy) = \mathrm{dist}(w, uhy) = \mathrm{dist}(w, gy).$$

This contradicts Lemma 3.1.4. □

LEMMA 4.3.2.

$$G_0 = \bigcap_{z \in \mathcal{B}(G)} G_z \cdot \mathcal{U}.$$

Proof. "$\supset$" Suppose that $g \in \bigcap_{z \in \mathcal{B}(G)} G_z \cdot \mathcal{U}$. From Lemma 4.3.1, the element g must fix a point $x \in \mathcal{B}(G)$. By hypothesis, there exist $h \in G_x$ and $u \in \mathcal{U}$ such that $g = uh$. Since g and h fix x, so must u. But then Lemma 4.2.1 says that $u \in G_x$.

"$\subset$" We need to show that $G_x \subset \mathcal{U} \cdot G_y$ for $x, y \in \mathcal{B}(G)$. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ containing both x and y; let $\mathbf{S}$ be the corresponding k-split torus. Let $M = C_{\mathbf{G}}(\mathbf{S})(k)$. Let P be a minimal parabolic with a Levi decomposition MN so that the (spherical) chamber in $\mathbf{X}_*^k(\mathbf{S}) \otimes \mathbb{R}$ determined by N is invariant under translation by the vector $(y - x)$.

Let $\bar{N}$ be the unipotent radical of the parabolic opposite $P = MN$. From Lemma 4.2.2, $G_x = N_x \cdot \bar{N}_x \cdot N_x \cdot M_x$ and similarly for G_y. Because of the way we chose M and N, we have $M_x = M_y$ and $N_x \subset N_y$. Consequently, if $g \in G_x$ then there exist $n_1, n_2 \in N_x$, $\bar{n} \in \bar{N}_x$, and $m \in M_x$ such that

$$\begin{aligned} g &= n_1 \cdot \bar{n} \cdot n_2 \cdot m \\ &= {}^{n_1}\bar{n} \cdot n_1 \cdot n_2 \cdot m \\ &\in \mathcal{U} \cdot G_y. \end{aligned}$$ □

By [1, Lemma 3.7.20], Lemma 4.3.2 implies Theorem 4.1.4(1) in the case when $r > 0$. Thus, we have established Theorem 4.1.4.

4.4. Parabolic Descent

In this subsection we prove Theorem 4.1.5. Suppose that P is a parabolic subgroup of G with a Levi decomposition $P = MN$.

LEMMA 4.4.1. $\mathfrak{U} \cap M = \mathfrak{U}_M$.

Proof. The right-hand side is clearly contained in the left-hand side.

To show the opposite inclusion, we suppose there is an $m \in \mathfrak{U} \cap M$ such that $m \notin \mathfrak{U}_M$ and derive a contradiction. Since $m \notin \mathfrak{U}_M$, Lemma 3.1.3 tells us that there exist an apartment $\mathcal{A}$ in $\mathcal{B}(M)$ and a line ℓ in $\mathcal{A}$ such that m acts on ℓ by a non-trivial translation. Therefore, m acts nontrivially on ℓ and fixes a point in $\mathcal{B}(G)$, which contradicts Corollary 3.1.5. □

LEMMA 4.4.2. $G_0 \cap M = M_0$.

Proof. The right-hand side is clearly a subset of the left-hand side.

From Lemma 4.4.1 we have that $G_0 \cap M \subset \mathfrak{U}_M$. However, since $G_x \cap M = M_x$ for all $x \in \mathcal{B}(M)$, Lemma 4.2.1 tells us that $G_0 \cap \mathfrak{U}_M \subset M_0$. □

From [1, Lemma 3.7.25], Lemma 4.4.2 implies Theorem 4.1.5(1) in the case when $r > 0$. Thus, we have established Theorem 4.1.5.

5. Some Results Concerning Representations of *G*

In this section we wish to transfer many of the ideas of [1, Sec. 4] (see also [12, Chap. 2]) from $\mathfrak{g}$ to G. After introducing some additional notation (which will be used throughout the remainder of this paper), we consider the interplay between representations of G, their depth, and functions on G.

5.1. Notation

We place further restrictions upon k: we assume that $\mathfrak{f}$ is a finite field. Let dg denote a Haar measure on G.

Let $\mathcal{H} = \mathcal{H}^G = C_c^\infty(G)$ denote the space of complex-valued, compactly supported, locally constant functions on G. For a compact open subgroup H of G, let $C_c(G/H) \subset \mathcal{H}$ denote the set of those $f \in \mathcal{H}$ such that $f(gh) = f(g)$ for all $h \in H$. If H' is a compact open subgroup of G containing H, then $C(H'/H)$ denotes the set of those $f \in C_c(G/H)$ with support in H'.

Suppose that (π, V) is a finite-length, admissible, complex representation of G. For $f \in \mathcal{H}$, we define the operator-valued Fourier transform of f by $\hat{f} := \pi(f)$. Here $\pi(f) \in \mathrm{End}_{\mathbb{C}}(V)$ is the finite-rank operator defined by

$$\pi(f)v := \int_G f(g)\pi(g)v\,dg$$

for $v \in V$.

Let P be a parabolic subgroup of G with a Levi decomposition $P = MN$. Then, for a smooth irreducible representation σ of M, we denote by $\mathrm{Ind}_P^G \sigma$ the finite-length admissible representation of G obtained by normalized induction.

5.2. The Depth of a Representation

In this subsection we define the depth of a smooth representation and collect some facts about depth. The definition of depth given here is slightly different from that given by Moy and Prasad in [17]; the reliance on $\mathfrak{g}$ has been removed from their definition. The proofs of this subsection follow, to a large extent, those of [17, 18].

LEMMA 5.2.1. *Suppose that (π, V) is a smooth representation of G. Then there exists an $r(\pi) \in \mathbb{Q}$ with the following properties.*

(1) *If $(x, r) \in \mathcal{B}(G) \times \mathbb{R}_{\geq 0}$ such that $V^{G_{x,r^+}}$ is nontrivial, then $r \geq r(\pi)$.*
(2) *There exists a $y \in \mathcal{B}(G)$ such that $V^{G_{y,r(\pi)^+}}$ is nontrivial.*

Proof. Since the set of optimal numbers is a discrete subset of $\mathbb{Q}$ (see Section 2.4), we can let $r(\pi)$ be the least nonnegative optimal number for which there exists an optimal point z such that

$$V^{G_{z,r(\pi)^+}} \neq \{0\}.$$

Suppose $(x, r) \in \mathcal{B}(G) \times \mathbb{R}_{\geq 0}$ such that $V^{G_{x,r^+}}$ is nontrivial. Then there exists an optimal point y such that

$$G_{y,r^+} \subset G_{x,r^+}.$$

Thus $\{0\} \neq V^{G_{x,r^+}} \subset V^{G_{y,r^+}}$, which implies that $r \geq r(\pi)$. $\square$

Thanks to Lemma 5.2.1, the following definition makes sense.

DEFINITION 5.2.2. Suppose that (π, V) is a smooth representation of G. The *depth* $\rho(\pi)$ of π is the least nonnegative real number for which there exists an $x \in \mathcal{B}(G)$ such that $V^{G_{x,\rho(\pi)^+}}$ is nontrivial.

REMARK 5.2.3. The depth of a smooth representation is an optimal number; in particular, it is rational.

LEMMA 5.2.4. *Suppose P is a parabolic subgroup of G with a Levi decomposition $P = MN$. Suppose that (σ, W) is an irreducible smooth representation of M. If (π, V) is an irreducible subquotient of $\mathrm{Ind}_P^G(\sigma)$, then $\rho(\pi) = \rho(\sigma)$.*

Proof. For notational ease, we assume for this proof that our induction is not normalized. Because the modular character is unramified (i.e., it has depth zero), this does not affect the statement of the lemma.

Without loss of generality, we may assume that σ is an irreducible supercuspidal representation. We may also assume that π is a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$ (see e.g. [11, Thm. 6.3.7]).

Fix an alcove C in $\mathcal{B}(M) \subset \mathcal{B}(G)$. Let $x_0 \in \bar{C}$ be a special point for G.

We first show that $\rho(\pi) \geq \rho(\sigma)$. Since (π, V) has depth $\rho(\pi)$, there exist a $y \in \bar{C}$ and a nonzero function $f : G \to W$ such that:

(1) $f \in V$;
(2) $f(m \cdot n \cdot g) = \sigma(m) f(g)$ for all $m \in M$, $n \in N$, and $g \in G$; and
(3) $f(g \cdot h') = f(g)$ for all $g \in G$ and $h' \in G_{y,\rho(\pi)^+}$.

Since $G = PG_{x_0}$ (Iwasawa decomposition) and $f \neq 0$, there exists an $h \in G_{x_0}$ such that $0 \neq f(h) \in W$. Since $hx_0 = x_0 \in \mathcal{B}(M)$ and $x_0 \in h\bar{C}$, from [1, Lemma 2.4.1] there exists an $n \in G_{x_0} \cap N$ such that $nh\bar{C} \subset \mathcal{B}(M)$. Now, for all $m \in M_{nhy,\rho(\pi)^+} = M \cap G_{nhy,\rho(\pi)^+}$ we have

$$\begin{aligned}\sigma(m) f(h) = \sigma(n^{-1} \cdot m \cdot n) f(h) &= f(n^{-1} \cdot m \cdot n \cdot h)\\ &= f(h \cdot ({}^{h^{-1}n^{-1}}m)).\end{aligned}$$

But ${}^{h^{-1}n^{-1}}m \in {}^{h^{-1}n^{-1}}M_{nhy,\rho(\pi)^+} = M_{y,\rho(\pi)^+} \subset G_{y,\rho(\pi)^+}$, so $\sigma(m) f(h) = f(h)$. Thus $\rho(\sigma) \leq \rho(\pi)$.

We now show that $\rho(\pi) \leq \rho(\sigma)$. There exists a $y \in \mathcal{B}(M) \subset \mathcal{B}(G)$ such that $W^{M_{y,\rho(\sigma)^+}}$ is nontrivial. Frobenius reciprocity states that

$$\mathrm{Hom}_G(V, \mathrm{Ind}_P^G \sigma) = \mathrm{Hom}_M(V_N, \sigma).$$

Therefore, there exist M-submodules $W_i \subset V_N$ such that the sequence

$$0 \to W_2 \to W_1 \to \sigma \to 0$$

is exact. Since taking $M_{y,\rho(\sigma)^+}$-fixed vectors is exact, we have $\{0\} \neq W_1^{M_{y,\rho(\sigma)^+}} \subset V_N^{M_{y,\rho(\sigma)^+}}$. From work of Jacquet and Harish-Chandra (see [11, Thm. 3.3.3]) we have that $V^{G_{y,\rho(\sigma)^+}}$ maps onto $V_N^{M_{y,\rho(\sigma)^+}}$, which is nontrivial. Thus $\rho(\pi) \leq \rho(\sigma)$. $\square$

We now state a corollary that results from repeating the proof of Lemma 5.2.4 with appropriate changes. Recall that if C is an alcove in $\mathcal{B}(G)$, then G_C denotes the associated Iwahori subgroup.

COROLLARY 5.2.5. *Suppose P is a parabolic subgroup of G with a Levi decomposition $P = MN$. Suppose that (σ, W) is an irreducible smooth representation of M and that C is an alcove in $\mathcal{B}(M)$. Suppose (π, V) is an irreducible subquotient of* $\mathrm{Ind}_P^G \sigma$. *Then*

$$W^{M_C} \neq \{0\} \textit{ if and only if } V^{G_C} \neq \{0\}.$$

See [4] for a complete treatment of admissible representations with nontrivial Iwahori-fixed vectors.

LEMMA 5.2.6. *Fix $r \geq 0$. If (π, V) is a smooth representation of G such that every irreducible subquotient of (π, V) has depth r, then (π, V) has depth r.*

Proof. Since $\rho(\pi) \leq r$, we must show that $\rho(\pi) \geq r$.

Choose $s \in \mathbb{R}_{\geq 0}$ for which there exists an $x \in \mathcal{B}(G)$ such that $V^{G_{x,s^+}} \neq \{0\}$. Choose $v \in V^{G_{x,s^+}}$ and let $W = \langle G \cdot v \rangle$. Since W is finitely generated, there exists a subrepresentation $W_1 \subsetneq W$ such that

$$0 \to W_1 \to W \to W/W_1 \to 0$$

is an exact sequence of G-modules and the quotient W/W_1 is irreducible. Because taking G_{x,s^+} fixed vectors is exact, we have $(W/W_1)^{G_{x,s^+}} \neq \{0\}$ (otherwise, $W_1 = W$). Thus $s \geq r$, which implies that $\rho(\pi) \geq r$. □

Corollary 5.2.7. *Suppose P is a parabolic subgroup of G with a Levi decomposition $P = MN$. If σ is an irreducible smooth representation of M, then $\rho(\mathrm{Ind}(\sigma)) = \rho(\sigma)$.*

5.3. An Interesting Space of Functions

Fix $r \in \mathbb{R}_{\geq 0}$. Suppose that P is a parabolic subgroup of G with a Levi decomposition $P = MN$. Let $x_0 \in \mathcal{B}(M)$ be a special point for G.

Definition 5.3.1.

$$\mathcal{H}_r^G := \sum_{x \in \mathcal{B}(G)} C_c(G/G_{x,r}).$$

We interpret the sum on the right in the following way. If $f \in \mathcal{H}_r = \mathcal{H}_r^G$, then we can write f as a finite sum $f = \sum_i f_i$ with $f_i \in C_c(G/G_{y_i,r})$ and $y_i \in \mathcal{B}(G)$.

Definition 5.3.2. For $f \in \mathcal{H}^G$, we define $f_P \in \mathcal{H}^M$ by

$$f_P(m) = \delta_P^{1/2}(m) \int_N dn \int_{G_{x_0}} f({}^h(mn))\, dh$$

for $m \in M$.

Here dn is a Haar measure on N, dh is the normalized Haar measure on G_{x_0}, and δ_P is the modular function for P.

5.4. Functions and Representations

In the following subsections we wish to make a precise statement about the Plancherel support of functions in $\mathcal{H}_r$. We also want to show that the map from $\mathcal{H}$ to $\mathcal{H}^M$ defined in Definition 5.3.2 takes $\mathcal{H}_r$ into $\mathcal{H}_r^M$. These results were originally pursued because of their relevance to certain homogeneity problems. I thank Alan Roche for his extremely helpful comments on an earlier version of this subsection.

Fix $r \in \mathbb{R}_{\geq 0}$. Let $\mathfrak{R}(G)$ denote the category of smooth complex representations of G. We recall the basic facts about the Bernstein decomposition of $\mathfrak{R}(G)$ (see [9] for a fuller recollection). The Bernstein decomposition allows us to write $\mathfrak{R}(G)$ as a direct product of full subcategories:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}} \mathfrak{R}^{\mathfrak{s}}.$$

The Bernstein spectrum $\mathfrak{B}$ consists of equivalence classes $[L, \sigma]$ where L is a Levi subgroup of G and σ is an irreducible, supercuspidal, smooth representation of L. (A pair (L', σ') belongs to the equivalence class $[L, \sigma]$ if and only if there exist $g \in G$ and an unramified character χ of L such that ${}^gL' = L$ and ${}^g\sigma' = \sigma \otimes \chi$.) If $\mathfrak{s} = [L, \sigma] \in \mathfrak{B}$, then $\mathfrak{R}^{\mathfrak{s}}$ consists of those smooth representations π of G for which each irreducible subquotient of π occurs as a subquotient of $\mathrm{Ind}_P^G(\sigma \otimes \chi)$ for some unramified character χ of L and some parabolic P with Levi L. It follows from Lemma 5.2.4 and Lemma 5.2.6 that every object of $\mathfrak{R}^{[L,\sigma]}$ has depth $\rho(\sigma)$.

With respect to the right regular representation, $\mathcal{H}$ is a smooth representation of G. Therefore, from the Bernstein decomposition of the category $\mathfrak{R}(G)$, we can write $\mathcal{H} = \bigoplus_{\mathfrak{s}} \mathcal{H}^{\mathfrak{s}}$. Each $\mathcal{H}^{\mathfrak{s}}$ is a G-stable subspace of $\mathcal{H}$.

Since irreducible depth-zero representations may or may not have nontrivial Iwahori-fixed vectors, we introduce some notation to distinguish these two cases.

DEFINITION 5.4.1. For $s \in \mathbb{R}_{>0}$, let $\Pi_s = \Pi_s^G$ denote the set of equivalence classes of irreducible smooth representations of G of depth strictly less than s. Let $\Pi_0 = \Pi_0^G$ denote the set of equivalence classes of irreducible smooth representations of G possessing nontrivial Iwahori-fixed vectors.

Thus, if $s \in \mathbb{R}_{\geq 0}$ and (π, V) is a representative for a class in Π_s, then V is generated (as a G-representation) by a $G_{x,s}$-fixed vector for some $x \in \mathcal{B}(G)$.

Let $\mathfrak{s}$ be a point in the Bernstein spectrum and let $\mathrm{Irr}(\mathfrak{R}^{\mathfrak{s}})$ denote the set of equivalence classes of irreducible objects in $\mathfrak{R}^{\mathfrak{s}}$. From Lemma 5.2.4 and Corollary 5.2.5, we have that either $\Pi_r \cap \mathrm{Irr}(\mathfrak{R}^{\mathfrak{s}})$ is trivial or $\mathrm{Irr}(\mathfrak{R}^{\mathfrak{s}}) \subset \Pi_r$. The following definitions therefore make sense.

DEFINITION 5.4.2.

$$\mathfrak{B}_r := \{\mathfrak{s} \in \mathfrak{B} \mid \mathrm{Irr}(\mathfrak{R}^{\mathfrak{s}}) \subset \Pi_r\};$$

$$\mathcal{H}'_r := \bigoplus_{\mathfrak{s} \in \mathfrak{B}_r} \mathcal{H}^{\mathfrak{s}}.$$

We will require one more definition before proving the main result of this subsection.

DEFINITION 5.4.3. Suppose $f \in \mathcal{H}$. We will say that $\mathrm{supp}(\hat{f}) \subset \Pi_0$ if $\hat{f}(\pi) = 0$ for all irreducible smooth representations π that do not possess a nontrivial Iwahori-fixed vector. For $s \in \mathbb{R}_{>0}$ we will say that $\mathrm{supp}(\hat{f}) \subset \Pi_s$ if $\hat{f}(\pi) = 0$ for all irreducible smooth representations π with $\rho(\pi) \geq s$.

In the following lemma, the second equality is valid for any subset of the Bernstein spectrum.

LEMMA 5.4.4. $\mathcal{H}_r = \{f \in \mathcal{H} \mid \mathrm{supp}(\hat{f}) \subset \Pi_r\} = \mathcal{H}'_r$.

Proof (P. Kutzko). From the definitions it follows that $\mathcal{H}_r \subset \{f \in \mathcal{H} \mid \operatorname{supp}(\hat{f}) \subset \Pi_r\}$.

We now show that $\{f \in \mathcal{H} \mid \operatorname{supp}(\hat{f}) \subset \Pi_r\} \subset \mathcal{H}'_r$. Suppose $f \in \mathcal{H}$ such that $\operatorname{supp}(\hat{f}) \subset \Pi_r$. Since $f \in \mathcal{H} = \bigoplus_{\mathfrak{s}} \mathcal{H}^{\mathfrak{s}}$, we can write $f = \sum_{\mathfrak{s}} f^{\mathfrak{s}}$ with $f^{\mathfrak{s}} \in \mathcal{H}^{\mathfrak{s}}$. Fix $\mathfrak{t} \in \mathfrak{B}$ such that $f^{\mathfrak{t}} \neq 0$. It will be enough to show that $\mathfrak{t} \in \mathfrak{B}_r$. Since $f^{\mathfrak{t}} \neq 0$, by [3, Prop. 2.12] there exists a smooth irreducible representation (π, V) such that $\pi(f^{\mathfrak{t}}) \neq 0$. Note that (π, V) is a nondegenerate $\mathcal{H}^{\mathfrak{t}}$-module, so it occurs as a quotient of $\mathcal{H}^{\mathfrak{t}}$; this implies that (π, V) is an object in $\mathfrak{R}^{\mathfrak{t}}$. Since for $\mathfrak{s} \neq \mathfrak{t}$ we have that (π, V) is not an object in $\mathfrak{R}^{\mathfrak{s}}$, we conclude that

$$\pi(f)w = \pi(f^{\mathfrak{t}})w$$

for all $w \in V$. Therefore, $\pi(f)$ is nonzero. It follows that (π, V) represents an equivalence class in Π_r and thus $\mathfrak{t} \in \mathfrak{B}_r$.

Finally, we show that $\mathcal{H}'_r \subset \mathcal{H}_r$. As a G-representation, the equivalence class of each irreducible subquotient of $\mathcal{H}'_r$ lies in Π_r, and each such representative is therefore generated (as a G-representation) by a $G_{x,r}$-fixed vector for some $x \in \mathcal{B}(G)$. We claim that $\mathcal{H}'_r$ is generated (as a G-representation) by a collection of such vectors. Indeed, suppose this is not the case. Then there exists a set of idempotents $E \subset \mathcal{H}$ such that

(1) $\sum_{e \in E} \mathcal{H}e\mathcal{H}'_r \neq \mathcal{H}'_r$ and
(2) for each irreducible subquotient X of $\mathcal{H}'_r$, there exists an $e' \in E$ such that $\mathcal{H}e'X = X$.

Since $\sum_{e \in E} \mathcal{H}e\mathcal{H}'_r \neq \mathcal{H}'_r$, we can produce G-modules W_1 and W_2 such that

$$\sum_{e \in E} \mathcal{H}e\mathcal{H}'_r \subset W_2 \subsetneqq W_1 \subset \mathcal{H}'_r$$

with W_1/W_2 irreducible. But there exists an $e' \in E$ such that $\mathcal{H}e'(W_1/W_2) = (W_1/W_2)$. Hence $\mathcal{H}e'W_1 + W_2 = W_1$, which implies that $W_1 = W_2$, a contradiction.

Because $\mathcal{H}'_r$ is generated by a collection of $G_{x,r}$-fixed vectors, we have $\mathcal{H}'_r \subset \mathcal{H}_r$. □

5.5. *Two Consequences for Harmonic Analysis*

Suppose $r \geq 0$. The function space $\mathcal{H}_r$ plays an important role in harmonic analysis. It is desirable to have an understanding of how the space $\mathcal{H}_r$ behaves under parabolic descent and the degree to which $\mathcal{H}_r$ depends on r.

The following lemma is the group analogue of [1, Rem. 4.2.10]. Suppose P is a parabolic subgroup of G with a Levi decomposition $P = MN$.

LEMMA 5.5.1. *Suppose that σ is an irreducible smooth representation of M. If $(\operatorname{Ind}_P^G(\sigma))(f) = 0$, then $\sigma(f_P) = 0$.*

Proof. This can be obtained from a minor modification of the computations found in [23, pp. 233–234]. □

LEMMA 5.5.2. *For $r \geq 0$, the map $f \mapsto f_P$ takes $\mathcal{H}_r^G$ into $\mathcal{H}_r^M$.*

Proof. Suppose $f \in \mathcal{H}_r^G$. By Lemma 5.4.4, it will be sufficient to show that $\operatorname{supp}(\widehat{f_P}) \subset \Pi_r^M$. Suppose σ is an irreducible smooth representation of M such that the equivalence class of σ is not an element of Π_r^M. From Lemma 5.2.4, Corollary 5.2.5, and Lemma 5.4.4 we have $(\operatorname{Ind}_P^G(\sigma))(f) = 0$. The lemma now follows from Lemma 5.5.1. □

To investigate the degree to which $\mathcal{H}_r$ depends on r, we introduce the space

$$\mathcal{H}_{r^+} := \sum_{x \in \mathcal{B}(G)} C_c(G/G_{x,r^+}).$$

We interpret this sum as in Definition 5.3.1.

Note that if π is a positive-depth irreducible smooth representation of G, then $\rho(\pi) = r$ if and only if $\operatorname{res}_{\mathcal{H}_r} \pi = 0$ and $\operatorname{res}_{\mathcal{H}_{r^+}} \pi \neq 0$. (Here $\operatorname{res}_{\mathcal{H}_r} \pi$ means the restriction of π to the space of functions $\mathcal{H}_r$.)

LEMMA 5.5.3. *Fix $s > 0$. There exists an $\varepsilon \in (0, s]$ such that, for all $r \in (s-\varepsilon, s)$, we have*

$$\mathcal{H}_{r^+} = \mathcal{H}_s.$$

Proof. Choose $\varepsilon \in (0, s]$ such that the set $(s - \varepsilon, s)$ does not intersect the set of optimal numbers. Fix $r, t \in (s - \varepsilon, s)$ with $t < r$. We have

$$\mathcal{H}_t \subset \mathcal{H}_{r^+} \subset \mathcal{H}_s.$$

Note that, if π is a smooth representation of G, then (by Remark 5.2.3) we have that $\rho(\pi) \geq s$ if and only if $\rho(\pi) \geq t$. Thus $\mathcal{H}_t = \mathcal{H}_s$ by Lemma 5.4.4. □

The following corollary follows from the proof of Lemma 5.5.3.

COROLLARY 5.5.4. *Fix $r \geq 0$. We have*

$$\mathcal{H}_{r^+} = \{f \in \mathcal{H} \mid \hat{f}(\pi) = 0 \text{ for all smooth irreducible representations } \pi \text{ of } G \text{ such that } \rho(\pi) > r\}.$$

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