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Automorphism Groups and Derivation Algebras of Finitely Generated Vertex Operator Algebras

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1. Introduction

In this paper we investigate the general structure of the automorphism group and the Lie algebra of derivations of a finitely generated vertex operator algebra. We prove two main results. The automorphism group is isomorphic to an algebraic group. Under natural assumptions, the derivation algebra has an invariant bilinear form and the ideal of inner derivations is nonsingular.

Definition 1.1. Let V be a vertex operator algebra. We say that $a \in \mathrm{GL}(V)$ is an *automorphism of V* if and only if it leaves the vacuum element and the principal Virasoro element fixed ($a\mathbf{1} = \mathbf{1}$ and $a\omega = \omega$) and preserves all V-compositions; that is, for all $m \in \mathbb{Z}$ and $u, v \in V$, we have $a(u_m v) = a(u)_m a(v)$. It follows that an automorphism fixes all the V_i since they are eigenspaces for an operator in the series for the principal Virasoro element.

The set of all automorphisms is a group, denoted $\mathrm{Aut}(V)$.

In the definition, it suffices to restrict u and v to homogeneous elements. Note that, in some definitions of VOA automorphism, there is no requirement that the principal Virasoro element be fixed.

So far, we know the automorphism groups explicitly for relatively few vertex operator algebras, such as $V^\natural$ [FLM], vertex operator algebra V_L for a positive definite even lattice L [DN], certain vertex operator algebras with central charge 1 [DG; DGR], vertex operator algebras associated to highest weight representations for affine algebras (cf. [DLY]), vertex operator algebras associated to codes [M], and a few special cases (see e.g. [G]).

The determination of each of these automorphism groups has its own story and depends heavily on the specifics of the auxiliary object used to construct the VOA, such as a lattice, Lie algebra, or code. Nevertheless, one can observe that all these automorphism groups have similarities.

We denote by (V, k^{th}) the algebra with underlying vector space V and product $a_k b$ for $a, b \in V$, where a_k is the coefficient at z^{-k-1} in the vertex operator for a.

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The linear subspace V_m is closed under this product if $m-k=1$. This algebra is denoted $(V_m, (m-1)^{th})$. The cases $m=1$ and 2 are especially interesting.

For simplicity of exposition, let us assume that V is a simple vertex operator algebra of CFT type (see [DLMM], or Definition 2.8 of this paper). Then $(V_1, 0^{th})$ is a Lie algebra with bracket $[u,v]=u_0v$ for $u, v \in V_1$, where $Y(u,z)=\sum_{n\in\mathbb{Z}} u_n z^{-n-1}$. The endomorphism u_0 is a derivation of V in the sense that $u_0\mathbf{1}=0$, $u_0\omega=0$, and

$$u_0(Y(v,z)w)=Y(u_0v,z)w+Y(u,z)u_0w$$

for any $v, w\in V$ (cf. [DN]). Moreover, the exponential e^{u_0} is an automorphism of V (e^{u_0} is a well-defined operator on V because each V_n is finite dimensional). Denote by $\mathrm{Aut}_1(V)$ the subgroup of the automorphism group $\mathrm{Aut}(V)$ of V generated by e^{u_0} for $u\in V_1$. Then $\mathrm{Aut}_1(V)$ is a finite dimensional connected algebraic normal subgroup of $\mathrm{Aut}(V)$. In all the examples mentioned previously, $\mathrm{Aut}(V)/\mathrm{Aut}_1(V)$ is a finite group. We think that this is probably a general phenomenon for rational vertex operator algebras. There are counterexamples when V is not rational (cf. [DM1] and Examples 2.6 and 4.1 in this paper).

For some time it has been a feeling that any rational vertex operator algebra of CFT type is finitely generated. If V is regular in the sense that any weak module is a direct sum of ordinary modules (see [DLM2]), then it is proved in [KL] and [L3] (see also [GN]) that V is finitely generated. It is also felt that rational vertex operator algebras must be regular. Interest in the category of modules is motivation to study automorphism groups of finitely generated vertex operator algebras.

The main result of this paper is that the automorphism group $\mathrm{Aut}(V)$ of a finitely generated vertex operator algebra is isomorphic to a finite dimensional algebraic group. It is well known that a finite dimensional algebraic group G has only finitely many connected components and so G/G^0 is a finite group, where G^0 is the connected component of G containing the identity. We expect that the normal subgroup $\mathrm{Aut}_1(V)$ of $\mathrm{Aut}(V)$ is exactly $\mathrm{Aut}(V)^0$ for all rational vertex operator algebras V of CFT type. This property holds for all examples discussed so far.

There is a close relation between the automorphism group and the Lie algebra of derivations of a vertex operator algebra. If d is a derivation of a vertex operator algebra V, then e^d is an automorphism of V (see Section 3). If V is a finitely generated rational vertex operator algebra of CFT type, then equality of $\mathrm{Aut}_1(V)$ and $\mathrm{Aut}(V)^0$ is equivalent to all the derivations of V being given by u_0 for $u\in V_1$.

The paper is organized as follows. In Section 2 we prove that the automorphism group of a finitely generated vertex operator algebra is a finite dimensional algebraic group. We also give an example of a non–finitely generated vertex operator algebra whose automorphism group is not isomorphic to an algebraic group. In Section 3 we study derivations of vertex operator algebras. For $v\in V$ we define a linear operator $o(v)$ by the conditions that $o(v)=v_{\mathrm{wt}v-1}$ if v is homogeneous. We show in Section 3 that $o(v)$ is a derivation of V if and only if $v\in V_1$. We also show that the Lie algebra V_1 is an ideal of the Lie algebra of the derivations and has an orthogonal complement with respect to a suitable invariant symmetric bilinear form. In Section 4 we discuss an example of a nonsimple finitely generated vertex operator algebra and its automorphism group.

2. Automorphism Groups

We suppose that the VOA V is finitely generated (cf. [FHL]). This is equivalent to assuming the existence of an $n \in \mathbb{Z}$ such that $U = \bigoplus_{m \leq n} V_m$ generates V in the sense that

$$V = \mathrm{span}\langle u^1_{i_1} \cdots u^s_{i_s} u \mid u^j, u \in U,\ s \in \{0, 1, 2, \ldots\},\ i_j \in \mathbb{Z}\rangle.$$

For a subset A of V, set $A^{r+1} := A \times \cdots \times A$ ($r+1$ times) and $A^\infty := \bigcup_{r \geq 0} A^{r+1}$. An element of A^∞ is a finite length vector $\vec{x} = (x_0, \ldots, x_r)$, and we call $r+1$ the *length* of $\vec{x}$. For every nonempty finite sequence $\vec{m} := (m_1, \ldots, m_r)$ of integers, we define the function (called *$\vec{m}$-composition*) $\mu := \mu_{\vec{m}} \colon V^{r+1} \to V$ by $\mu(x_0, \ldots, x_r) := (x_0)_{m_1}(x_2)_{m_2} \cdots (x_{r-1})_{m_r} x_r$. Call $r+1$ the *length* of μ. Thus, $\mu(\vec{x})$ is defined if and only if μ and $\vec{x}$ have the same lengths, in which case we say that $(\mu, \vec{x})$ is an *admissible pair*. If the entries of such $\vec{x}$ are restricted to a subset A of V, then we call the pair an *A-admissible pair*. If the coordinates of $\vec{x} = (x_0, \ldots, x_r)$ are homogeneous, then $\mu(\vec{x})$ is homogeneous and we define the *weight* of $(\mu, \vec{x})$ to be $\sum_{i=0}^r \mathrm{wt}(x_i) - \sum_{i=1}^r (m_i + 1)$. This is just the weight of $\mu(\vec{x})$ if $\mu(\vec{x})$ is nonzero.

Such a function, for some $\vec{m}$, is called a *V-composition*. For a subset A of V, the restriction of μ to tuples of elements in A is denoted μ_A.

REMARK 2.1. The property that U generates V means that, for each integer m, there exists a finite set S of V-compositions such that $V_m = \sum_{\mu \in S} \mathrm{Im}(\mu_U) \cap V_m$.

We choose a basis $\mathcal{B}$ of U consisting of homogeneous elements, including $\mathbf{1}$. Let $\mathcal{Q}$ be the set of $\mathcal{B}$-admissible pairs. Define $\mathcal{Q}_m$ to be the set of pairs in $\mathcal{Q}$ of weight m. Then V_m is spanned by a finite set of "monomials" in $\mathcal{B}$ of weight m, that is, elements of certain $\mathrm{Im}(\mu_{\mathcal{B}})$.

There is a finite set $\mathcal{R}_m$ of pairs $(\mu, \vec{x}) \in \mathcal{Q}_m$ such that the set $\mathcal{B}_m := \{\mu(\vec{x}) \mid (\mu, \vec{x}) \in \mathcal{R}_m\}$ forms a basis for V_m. Choose $\mathcal{R}_0 = \{(\mu_0, \mathbf{1})\}$ (where μ_0 is the trivial length-1 composition) and set $\mathcal{R} := \bigcup_{m \in \mathbb{Z}} \mathcal{R}_m$.

We write "res" for the restriction homomorphism $\mathrm{Aut}(V) \to \mathrm{GL}(U)$. Since U generates V, it follows that res is injective. We consider the question of when $g \in \mathrm{GL}(U)$ is in the image of res.

We shall define a set function $e \colon \mathrm{GL}(U) \to \mathrm{End}(V)$ as follows. For $g \in \mathrm{GL}(U)$, define $e(g) \in \mathrm{End}(V)$ by its action on the basis elements $\mu(\vec{x})$, $(\mu, \vec{x}) \in \mathcal{R}$:

$$e(g)(\mu(\vec{x})) := \mu(g(\vec{x})). \tag{2.1}$$

This endomorphism will turn out to be invertible in cases of interest to us.

Now consider the following set of conditions on $e(g) \in \mathrm{End}(V)$:

$$e(g)(\mu(\vec{u})) = \mu(g(\vec{u})), \tag{2.2}$$

$$e(g)e(g^{-1})(\mu(\vec{u})) = \mu(\vec{u}) = e(g^{-1})e(g)(\mu(\vec{u})) \tag{2.3}$$

for all U-admissible pairs $(\mu, \vec{u})$.

We may assume that the components of $\vec{x}$ are homogeneous elements and even that $(\mu, \vec{u}) \in \mathcal{Q}_m$. Both sides of (2.2) are expanded in the basis $\mathcal{B}_m$. Equating the coefficients of both sides gives *polynomial conditions* on the entries of (g_{ij}), the matrix representing g with respect to $\mathcal{B}$. A similar discussion applies to (2.3).

There is an ideal $I_{(\mu,\vec{x})}$ in the ring $\mathbb{C}[x_{ij}, \det^{-1} \mid i, j = 1, \dots, \dim(U)]$ of polynomial functions on $\mathrm{GL}(U)$ associated to conditions (2.2) and (2.3).

Finally, for $u \in U$, define the ideal I_u by the condition $gu = u$. Set $I := \sum_{(\mu,\vec{x})\in\mathcal{Q}} I_{(\mu,\vec{x})} + I_{\mathbf{1}} + I_\omega$ and set

$$G_U := \{g \in \mathrm{GL}(U) \mid p(g) = 0 \text{ for all } p \in I\}.$$

Then G_U is a variety contained in $\mathrm{GL}(U)$. Clearly, $\mathrm{res}(\mathrm{Aut}(V)) \le G_U$.

LEMMA 2.2. *G_U is a subgroup of $\mathrm{GL}(U)$; that is, G_U is an algebraic group. Also, e is a homomorphism.*

Proof. First, $1 \in G_U$. Observe that, for $g \in G_U$, $e(g)$ and $e(g^{-1})$ are invertible because their restrictions to each V_i are invertible. Moreover, they form an inverse pair, whence $e(g^{-1}) = e(g)^{-1}$. We now show that g^{-1} satisfies (2.2). Let $(\mu, \vec{y})$ be a U-admissible pair of length $r+1$. Since $g \in G$ we have, for all $\vec{y}$, $e(g)\mu(g^{-1}(\vec{y})) = \mu(\vec{y})$ and so $\mu(g^{-1}(\vec{y})) = e(g)^{-1}\mu(\vec{y}) = e(g^{-1})\mu(\vec{y})$. Since g^{-1} satisfies (2.2), we have $G_U = G_U^{-1}$.

To prove closure under products, we let $(\mu, \vec{u})$ be a U-admissible pair and $g, h \in G_U$. We must show that $e(gh)\mu(\vec{u}) = \mu(gh(\vec{u}))$. Write

$$\mu(\vec{u}) = \sum_{(\nu,\vec{y})\in\mathcal{R}} a_{(\nu,\vec{y})}\nu(\vec{y})$$

for unique scalars $a_{(\nu,\vec{y})}$ almost all zero. Then,

$$e(gh)\mu(\vec{u}) = \sum_{(\nu,\vec{y})\in\mathcal{R}} a_{(\nu,\vec{y})}e(gh)\nu(\vec{y}) = \sum_{(\nu,\vec{y})\in\mathcal{R}} a_{(\nu,\vec{y})}\nu(gh(\vec{y})),$$

by definition of $e(gh)$. Since $g \in G_U$, this equals

$$\sum_{(\nu,\vec{y})\in\mathcal{R}} a_{(\nu,\vec{y})}e(g)\nu(h(\vec{y})) = \sum_{(\nu,\vec{y})\in\mathcal{R}} a_{(\nu,\vec{y})}e(g)e(h)\nu(\vec{y}) = e(g)e(h)\mu(\vec{u}).$$

Also $\mu(gh(\vec{u})) = e(g)\mu(h(\vec{u})) = e(g)e(h)\mu(\vec{u})$ because $g, h \in G_U$. We conclude that $gh \in G_U$ and so G_U is a group.

Since the $\mu(\vec{u})$ span V, we also deduce that $e(gh) = e(g)e(h)$, whence e is a homomorphism. □

LEMMA 2.3. *For all $u, v \in V$ and $n \in \mathbb{Z}$, we have*

$$e(g)(u_n v) = (e(g)u)_n(e(g)v).$$

That is, $\mathrm{Im}(e) \subseteq \mathrm{Aut}(V)$.

Proof. We may assume that u is "monomial" (i.e., has the form $\mu(\vec{x})$) for a U-admissible pair $(\mu, \vec{x})$. We argue by induction on the length of $(\mu, \vec{x})$. First, we assume that the length is 1. We may also assume that v is monomial, so $v = \nu(\vec{y})$ for a U-admissible pair $(\nu, \vec{y})$. Say ν is an $\vec{m}$-composition, $\vec{m} = (p, \dots, q)$ and $\vec{y} = (y_1, \dots, y_t)$. Then $u_n v = u_n(y_1)_p \cdots {}_q(y_t)$ and $e(g)(u_n v) = (gu)_n(gy_1)_p \cdots {}_q(gy_t)$, by Lemma 2.2. By Lemma 2.2 applied to $(\nu, \vec{y})$, we deduce $e(g)(u_n v) = (gu)_n(e(g)\nu(\vec{y})) = (gu)_n(e(g)v)$. Finally, since $e(g)x = gx$ for $x \in U$, this is $(e(g)u)_n(e(g)v)$.

Suppose next that the length is $r \geq 2$ and that μ is an $\vec{m}$-composition, $\vec{m} = (m_1, \dots, m_r)$. Set $k = m_1$, $b = x_1$, and $a = \nu(\vec{y})$, where $y = (x_2, \dots, x_r)$ and ν is the V-composition associated to the $(r-1)$-tuple $(m_2, \dots, m_r)$. Then $u = b_k a$.

We now perform a residue calculation to verify that

$$e(g)(Y(u, z)v) = Y(e(g)u, z)(e(g)v).$$

Extracting the coefficient at z^{-n-1} will give the lemma.

Since $u = b_k a$, we have from the Jacobi identity for vertex operators (see the formula before (3.3) of [D]) that

$$Y(u, z)v = \mathrm{Res}_w\{(w - z)^k Y(b, w)Y(a, z)v - (-z + w)^k Y(a, w)Y(b, z)v\}.$$

Write h for $e(g)$. Then

$$\begin{aligned} &h[Y(u, z)v] \\ &\quad = \mathrm{Res}_w\{(w - z)^k h[Y(b, w)Y(a, z)v] - (-z + w)^k h[Y(a, w)Y(b, z)v]\}. \end{aligned}$$

Using repeated induction on length (applied to b and a) together with the foregoing consequence of the Jacobi identity, we deduce that this equals

$$\begin{aligned} \mathrm{Res}_w&\left\{(w - z)^k Y(hb, w)h[Y(a, z)v] - (-z + w)^k Y(ha, w)h[Y(b, z)v]\right\} \\ &= \mathrm{Res}_w\{(w - z)^k [Y(hb, w)Y(ha, z)](hv) \\ &\qquad - (-z + w)^k [Y(ha, w)Y(hb, z)(hv)]\} \\ &= Y((hb)_k(ha), z)(hv) \\ &= Y(h(b_k a), z)(hv) \\ &= Y(hu, z)(hv), \end{aligned}$$

as desired. □

THEOREM 2.4. *The two maps*

$$\mathrm{res}\colon \mathrm{Aut}(V) \to G_U \quad \textit{and} \quad e\colon G_U \to \mathrm{Aut}(V)$$

form a pair of inverse isomorphisms. Therefore, $\mathrm{Aut}(V)$ *is isomorphic to the algebraic group* G_U.

Proof. Since U generates V, res is a monomorphism. Because $\mathrm{Im}(e)$ is contained in $\mathrm{Aut}(V)$ and $\mathrm{res} \circ e = \mathrm{Id}_{G_U}$, it follows that res is an epimorphism and hence an isomorphism. Since the set map e is a one-sided inverse of an isomorphism (hence a two-sided inverse), it is an isomorphism of groups. (We proved before that e is a homomorphism, but we do not need to quote that result here.) □

REMARK 2.5. The most well-known vertex operator algebras are finitely generated. For examples, Heisenberg vertex operator algebras [FLM] and affine vertex operator algebras (cf. [DL; FZ; L2]) are generated by their weight-1 subspaces; Virasoro vertex operator algebras (cf. [FZ; L2]) and the moonshine vertex operator algebra (see [B; FLM]) are generated by weight-2 subspaces. The lattice vertex operator algebra V_L (see [B; FLM]) is generated by $\bigoplus_{m\leq n}(V_L)_m$, where n is any positive integer such that L has a direct sum decomposition $L=\bigoplus_{i=1}^{n}\mathbb{Z}\alpha_i$ satisfying $\langle\alpha_i,\alpha_i\rangle/2\leq n$. In fact, V_L is generated by $e^{\pm\alpha_i}$ for $i=1,\dots,n$.

EXAMPLE 2.6. If G is not finitely generated then, in general, $\mathrm{Aut}(V)$ is not an algebraic group. Here is an example. Let $(U,Y,\mathbf{1},\omega)$ be a vertex operator algebra with infinitely many irreducible modules $U^i=(U^i,Y^i)$ $(i=1,2,\dots)$ not isomorphic to U such that $U^i=\bigoplus_{n\geq 0}U^i_{\lambda_i+n}$ with $U^i_{\lambda_i}\neq 0$ and $\lambda_1<\lambda_2<\cdots$. Set $V=U\oplus\bigoplus_{i>0}U_i$. Then V has a vertex operator algebra structure with vertex operator Y' defined in the following way (see [L1]). Since V is a U-module, $Y'(u,z)v$ (for $u\in U$ and $v\in V$) is defined in an obvious way. Using the idea of skew symmetry, for $v\in U^i$ and $u\in U$ we define $Y'(v,z)u:=e^{zL(-1)}Y'(u,-z)v$. Finally, we define $Y'(v,z)w:=0$ for all $v,w\in\bigoplus_i U^i$. We refer the reader to [L1] for the proof that $(V,Y',\mathbf{1},\omega)$ is indeed a vertex operator algebra.

For $k,\lambda\in\mathbb{C}$, let $L(k,\lambda)$ be the irreducible highest weight module for the Virasoro algebra with central charge k and highest weight λ. Then $L(1,0)$ is a vertex operator algebra and $L(1,\lambda)$ is an irreducible $L(1,0)$-module for any λ (cf. [FZ]). Now we take $U=L(1,0)$ and $U^i=L(1,i)$ for $i=1,2,\dots$. Let u^i be a nonzero highest weight vector of U^i (which is unique up to a scalar). Then V is generated by ω and u^i for $i>0$. Clearly, V is not finitely generated since a finite set of generators would lie in the sum of U and finitely many U^i. Note also that the sum of any set of the U^i is an ideal.

PROPOSITION 2.7. *The automorphism group of the VOA $\bigoplus_{n=0}^{\infty}L(1,n)$ is isomorphic to the infinite direct product $\prod_{i=1}^{\infty}\mathbb{C}_i^{\times}$, where $\mathbb{C}_i^{\times}$ is a copy of multiplicative group $\mathbb{C}^{\times}$ acting faithfully on U^i, trivially on U^j for $j\neq i$, and trivially on U. In particular,* $\mathrm{Aut}(V)$ *is not an algebraic group.*

Proof. Let $\lambda=(\lambda_1,\lambda_2,\dots)\in\prod_{i=1}^{\infty}\mathbb{C}^{\times}$. We define a U-module homomorphism $g_\lambda\in\prod_{i=1}^{\infty}\mathbb{C}_i^{\times}$ on $V:=\bigoplus_{n=0}^{\infty}L(1,n)$ such that $g_\lambda\mathbf{1}=\mathbf{1}$ and $g_\lambda u^i=\lambda_i u^i$. It is easy to see from the definition of Y that g_λ is an automorphism of V. On the other hand, any automorphism g is the identity on U because U is generated by the Virasoro element. So g preserves the space of highest weight vectors that is spanned by $\mathbf{1}$ and u^i for $i>0$. Since the weights of any two highest weight vectors are different, we immediately have that $gu^i=\lambda_i u^i$ for some nonzero constant λ_i for all i. Set $\lambda=(\lambda_1,\lambda_2,\dots)\in\prod_{i=1}^{\infty}\mathbb{C}^{\times}$. Then $g=g_\lambda$. Clearly, $\lambda\mapsto g_\lambda$ is an isomorphism. □

Next we discuss the automorphism group of V for a "nice" vertex operator algebra. We need more definitions.

DEFINITION 2.8. A vertex operator algebra V has *CFT type* if $V_n = 0$ for $n < 0$ and $\dim(V_0) = 1$ (so $V_0 = \mathbb{C}\mathbf{1}$).

In the following definition we use the notion of admissible modules as introduced in [Z] and [DLM2]. We refer the reader to [DLM2] for details.

DEFINITION 2.9. A vertex operator algebra V is *rational* if any admissible module is a direct sum of irreducible admissible modules.

DEFINITION 2.10. A vertex operator algebra V is C_k*-cofinite* if $\dim(V/C_k(V))$ is finite, where $C_k(V)$ is the subspace of V spanned by $u_{-k}v$ for $u, v \in V$.

The C_2-cofinite condition has been called, in the literature, the C_2-finite condition or C_2-condition (as in [Z]). In the case of vertex operator algebras associated to highest weight modules for affine Lie algebras and the Virasoro algebras, $V/C_k(V)$ are the spaces of coinvariants (cf. [FF; FKLMM]). This should explain why we are changing the terminology. In this paper only the C_2-cofinite condition is used.

As we have already mentioned, if V is of CFT type then $(V_1, 0^{th})$ is a Lie algebra under $[u, v] = u_0 v$. Part (1) of the following theorem can be found in [DM2]; the rest follows from the general structure of algebraic groups.

THEOREM 2.11. *Let V be a simple, C_2-cofinite rational vertex operator algebra of CFT type with $L(1)V_1 = 0$. Then the following statements hold.*

(1) *V_1 is a reductive Lie algebra; write $V_1 = \mathfrak{s} \oplus \mathfrak{t}$, where the first summand is semisimple and the second is toral.*

(2) *$G := \mathrm{Aut}(V)$ contains the connected component G^0 of the identity with finite index and satisfies $G^0 = G_1 C_1$ (central product), where $G_1 := \langle \exp(x_0) \mid x \in V_1 \rangle$ and $C_1 := C_G(V_1)^0$. We have $G_1 = S_1 T_1$, where $S_1 := \langle \exp(x_0) \mid x \in \mathfrak{s} \rangle$ and $T_1 := \langle \exp(x_0) \mid x \in \mathfrak{t} \rangle$. Also, $T_1 = (C_1 \cap G_1)^0$, and there is a connected group K_1 that is normal in G and has the following properties: $C_1 = T_1 K_1$, $[T_1, K_1] = 1$, and $T_1 \cap K_1$ is finite.*

We remark that the condition $L(1)V_1 = 0$ is not a strong assumption. It seems that all rational vertex operator algebras of CFT type satisfy this condition. For example, it is satisfied by well-known rational vertex operator algebras associated to highest weight integral modules for affine algebras (cf. [L2]), to minimal series for the Virasoro algebras (the weight-1 space is zero in this case), and to positive definite even lattices (cf. [FLM]). It is proved in [L1] that, for a simple vertex operator algebra of CFT type, the condition $L(1)V_1 = 0$ is equivalent to there being a nondegenerate symmetric invariant bilinear form on V in the sense of [FHL].

3. Derivations

There is a close relation between automorphisms and derivations for a vertex operator algebra. In this section we discuss the Lie algebra of the derivations of a vertex operator algebra.

Define a linear map o on V by setting $o(v) = v_{\mathrm{wt}(v)-1}$ for homogeneous elements v. Then $o(v)V_n \subset V_n$ for all n.

A *derivation* of the vertex operator algebra V is an endomorphism d of V that satisfies $d\mathbf{1} = 0$, $d\omega = 0$, and $[d, Y(u, z)] = Y(du, z)$. Since $d\omega = 0$, it follows that d preserves all the V_n (which are the eigenspaces of an operator in $Y(\omega, z)$), whence d is locally finite. The derivation d is an *inner derivation* if there is a $v \in V$ such that $o(v) = d$ (see Lemma 3.1). Since the V_k are finite dimensional, any endomorphism preserving the graded pieces is locally finite.

Since d is a locally finite derivation of V, the exponential e^d is an automorphism of V. On the other hand, $\mathrm{Aut}(V)^0$ (when V is finitely generated) is a connected Lie group, and its Lie algebra acts on V as derivations.

Set $\mathrm{IDer}(V) := o(V) \cap \mathrm{Der}(V)$, the space of *inner derivations.*

Let V be of CFT type such that $L(1)V_1 = 0$. Then V is a direct sum of irreducible modules for $\mathrm{span}\{L(-1), L(0), L(1)\} \cong \mathrm{sl}(2, \mathbb{C})$, the *principal* sl_2 [DLinM]. For homogeneous v, since $o(L(-1)v) = -(\mathrm{wt}(v) - 1)v_{\mathrm{wt}(v)-1}$, we have equality of $\{o(v) \mid v \in V\}$ and $\{o(v) \mid v \in \mathrm{Ker}(L(1))\}$.

Let $QV := \mathrm{Ker}(L(1))$, the space of *quasi-primary vectors.*

LEMMA 3.1. *We have $o(v) \in \mathrm{Der}(V)$ for $v \in V_1$.*

Proof. Since $[o(v), Y(u, z)] = [v_0, Y(u, z)] = Y(v_0 u, z) = Y(o(v)u, z)$ for $v \in V_1$ and $u \in V$, the result is clear. □

LEMMA 3.2. *Assume that V has CFT type. If $v = \sum_{i\geq 2} v^i$ with $v^i \in V_i \cap QV$ and $o(v) \in \mathrm{Der}(V)$, then $v = 0$.*

Proof. Since $o(v) = \sum_{i\geq 2} v^i_{i-1}$, we have

$$\begin{aligned}[o(v), Y(u, z)] &= \sum_{i\geq 2} [v^i_{i-1}, Y(u, z)] \\ &= \sum_{i\geq 2} \sum_{j\geq 0} \binom{i-1}{j} Y(v^i_j u, z) z^{i-1-j} \\ &= \sum_{i\geq 2} Y(v^i_{i-1} u, z).\end{aligned}$$

It follows that

$$\sum_{i\geq 2} \binom{i-1}{j} \sum_{j=0}^{i-2} Y(v^i_j u, z) z^{i-1-j} = 0$$

and

$$\lim_{z\to 0} \left\{ \sum_{i\geq 2} \binom{i-1}{j} \sum_{j=0}^{i-2} Y(v^i_j u, z) z^{i-1-j-1}\mathbf{1} \right\} = 0.$$

This implies

$$\sum_{i\geq 2} \binom{i-1}{i-2} v^i_{i-2} u = 0$$

for all u. Thus

$$\sum_{i\geq 2}\binom{i-1}{i-2}v^{i}_{i-2}=0$$

on V. Since the v^i are quasi-primary vectors, $[L(1), v^i_{i-2}] = iv^i_{i-1}$. As a result,

$$\sum_{i\geq 2}(i-1)iv^{i}_{i-1}=0.$$

By Theorem 2.2 of [DLMM], we now have $\sum_{i\geq 2} i(i-1)v^i \in V_1$, whence $v^i = 0$ for all i and so $v = 0$. □

The following corollary is immediate.

COROLLARY 3.3. *For V of CFT type,* $\mathrm{IDer}(V) = o(V_1) = \{o(v) \mid v \in V_1\}$.

Recall from [DLMM] that the radical $J(V)$ of V consists of those vectors $v \in V$ such that $o(v) = 0$. We shall need a result from [DM2].

LEMMA 3.4. *Let V be a C_2-cofinite rational vertex operator algebra of CFT type. Then $J(V) = (L(-1) + L(0))V$.*

From now on we assume that V is a C_2-cofinite rational vertex operator algebra of CFT type. Then $\mathfrak{g} = (V_1, 0^{th})$ is a reductive Lie algebra, and each V_n is a finite dimensional $\mathfrak{g}$-module via $v \mapsto o(v)$. Define the invariant symmetric bilinear form $(\cdot,\cdot)_M$ on $\mathfrak{g}$ for any $\mathfrak{g}$-module $(u, v)_M = \mathrm{tr}_M(uv)$ for $u, v \in \mathfrak{g}$.

Recall that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{t}$ is abelian. Then each finite dimensional module for $\mathfrak{g}$ is a direct sum of indecomposable modules, which are tensor products of irreducible modules for $\mathfrak{s}$ and indecomposable modules for $\mathfrak{t}$.

LEMMA 3.5. *Let M be a finite dimensional $\mathfrak{g}$-module such that M contains $\mathfrak{s}$ as an $\mathfrak{s}$-module. Let $\mathfrak{s} = \mathfrak{s}^1 \oplus\cdots\oplus \mathfrak{s}^p$ be the decomposition into simple ideals. Write $\mathfrak{s}^0 := \mathfrak{t}$. Then $(\mathfrak{s}^i, \mathfrak{s}^j)_M = 0$ if $i \neq j$ and, if $i > 0$, the restriction of the form to each $\mathfrak{s}^i$ is nondegenerate.*

Proof. First we prove that the restriction of the form to each $\mathfrak{s}^i$ is nondegenerate. Note that, as an $\mathfrak{s}^i$-module, M is completely reducible and $\mathfrak{s}^i$ is an irreducible submodule.

Let $i > 0$. It is well known that, for each irreducible $\mathfrak{s}^i$-module W, the corresponding invariant symmetric bilinear form $(\cdot,\cdot)_W$ is a nonnegative multiple of $(\cdot,\cdot)_{\mathfrak{s}^i}$, the Killing form on $\mathfrak{s}^i$, which is nondegenerate (cf. [H]). As a result, $(\cdot,\cdot)_M$ is nondegenerate when restricted to $\mathfrak{s}^i$.

In order to prove that $(\mathfrak{s}^i, \mathfrak{s}^j)_M = 0$ if $i \neq j$, we may assume that M is irreducible. Then $\mathfrak{t}$ acts as scalars on M, and $M|_{\mathfrak{s}} = M^0 \otimes\cdots\otimes M^p$ is a tensor product of irreducible modules M^i for $\mathfrak{s}^i$.

Let $i \geq 0$ be any index and let $x \in \mathfrak{s}^i$. Then M^i is a direct sum of generalized eigenspaces under x. We can therefore choose a basis B_k for M^k consisting of generalized eigenvectors for the action of x on M^i. Let $m^k \in B_k$, associated to the generalized eigenvalue λ_k. Let $\lambda := \sum_{k\neq j}\lambda_k$. Now fix $j > 0$ and assume $j \neq i$. Let $y \in \mathfrak{s}^j$. Think of a matrix for the action of y that is written in a basis taken from the subspaces of the form $m^0 \otimes m^1 \otimes \cdots \otimes m^{j-1} \otimes M^j \otimes m^{j+1} \otimes \cdots \otimes m^p$. Observe that M can be written as a direct sum of such subspaces. These spaces are not invariant by x, but the action of xy has block-triangular form with respect to this direct sum.

The contribution to the trace $\mathrm{tr}_M(xy)$ from the subspace $m^0 \otimes m^1 \otimes \cdots \otimes m^{j-1} \otimes M^j \otimes m^{j+1} \otimes \cdots \otimes m^p$ is equal to $\lambda\,\mathrm{tr}_{M^j}(y)$ for $y \in \mathfrak{s}^j$. Since $\mathfrak{s}^j$ is simple and $[\mathfrak{s}^j, \mathfrak{s}^j] = \mathfrak{s}^j$, we see that $\mathrm{tr}_{M^j}(y) = 0$. Thus $(\mathfrak{s}^i, \mathfrak{s}^j)_M = 0$, since M is a direct sum as described previously. □

For convenience we denote the bilinear form $(\cdot,\cdot)_{V_n}$ on $\mathfrak{g}$ by $(\cdot,\cdot)_n$ for $n \geq 0$.

We need the following result from [DM2].

PROPOSITION 3.6. *Let V be a rational, C_2-cofinite vertex operator algebra of CFT type, and let $L(1)V_1 = 0$. Then, for any $u \in V_1$, there exist $n \geq 0$ and $v \in V_1$ such that $(u, v)_n \neq 0$.*

The next result sharpens Proposition 3.6.

THEOREM 3.7. *Let V be as in Proposition 3.6. Then there exists an n such that $(\cdot,\cdot)_n$ is nondegenerate.*

Proof. Take n large enough so that $\sum_{m=0}^n V_m$ generates V. We claim that $(\cdot,\cdot)_n$ is nondegenerate.

Recall that $\mathbb{C}L(-1) + \mathbb{C}L(0) + \mathbb{C}L(1)$ is isomorphic to the Lie algebra $\mathrm{sl}(2,\mathbb{C})$. Let $M(\lambda)$ be the irreducible highest weight module for $\mathrm{sl}(2,\mathbb{C})$ with highest weight λ. Then

$$M(\lambda) = \bigoplus_{m\geq 0} M(\lambda)_{\lambda+m},$$

so that $M(\lambda)_\lambda$ is spanned by a highest weight vector v_λ and $M(\lambda)_{\lambda+m}$ is spanned by $L(-1)^m v_\lambda$. If $\lambda = 0$ then $M(\lambda)$ is trivial. If $\lambda > 0$, each $M(\lambda)$ is a Verma module and $M(\lambda)_{\lambda+m} \neq 0$ for all m.

First we prove that the representation of $\mathfrak{g}$ on V_n is faithful. Assume that $u_0 = 0$ on V_n. Since V is of CFT type and $L(1)V_1 = 0$, it follows from Corollary 3.2 of [DLinM] that (i) V is a direct sum of copies of $M(\lambda)$ for $m \geq 0$ and (ii) the multiplicity of $M(0)$ in the decomposition is 1. Note that $[L(i), u_0] = 0$ for $i = -1, 0, 1$ and $u \in \mathfrak{g}$. Let $M(\lambda)$ occur in the decomposition of V such that $0 \neq m \leq n$; then $u_0 = 0$ on $M(\lambda)$. Also note that $u_0 V_0 = 0$. Thus, $u_0 = 0$ on $\bigoplus_{m=0}^n V_m$. Since u_0 is a derivation on V and since V is generated by $\bigoplus_{m=0}^n V_m$, we immediately see that $u_0 = 0$ on V. This contradicts Proposition 3.6.

We can therefore identify $\mathfrak{g}$ with its image $o(\mathfrak{g})_n = \{u_0|_{V_n}\}$. If the form $(\cdot,\cdot)_n$ is degenerate then, by Lemma 3.5, there exists an $x \in \mathfrak{t}$ such that $(x, y)_n = 0$ for all $y \in \mathfrak{g}$. By Lemma 4.3 of [H], x_0 is nilpotent on V_n. In particular, all the eigenvalues of x_0 on V_n are zero. A similar argument as before then shows that x_0 has only zero eigenvalues on $\bigoplus_{m=0}^n V_m$. Because $\bigoplus_{m=0}^n V_m$ generates V, we see immediately that x_0 has only zero eigenvalues on V_m for all m, since

$$u_0 v^1_{m_1} \cdots v^k_{m_k} v = \sum_{j=1}^{k} v^1_{m_1} \cdots (u_0 v^j)_{m_j} \cdots v^k_{m_k} v + v^1_{m_1} \cdots v^k_{m_k} u_0 v$$

for $v^j, v \in V$ and $m_j \in \mathbb{Z}$.

Note that $\mathfrak{t}$ is an abelian Lie algebra and that all irreducible modules are 1-dimensional. Hence $\mathrm{tr}_{V_m}(x_0 y_0) = 0$ for all $y \in \mathfrak{g}$ and $m \in \mathbb{Z}$. But again, by Proposition 3.6, this is impossible. □

THEOREM 3.8. *Let V be a C_2-cofinite rational vertex operator algebra of CFT type such that $L(1)V_1 = 0$. Let $n \geq 0$ be such that $\sum_{m=0}^{n} V_m$ generates V. Then* $\mathrm{Der}(V)$ *is a direct sum of ideals $o(\mathfrak{g})$ and $\mathfrak{g}^{\perp}$, where $\mathfrak{g}^{\perp}$ consists of $d \in D$ such that* $\mathrm{tr}_{V_n}\, o(u)d = 0$ *for all $u \in V_1$.*

Proof. Let V be as in Lemma 3.5 and Theorem 3.7, and let $n > 0$ be as in the proof of Theorem 3.7. Then the action of $D := \mathrm{Der}(V)$ is also faithful on V_n, and $(d, d')_n = \mathrm{tr}_{V_n}(dd')$ defines a symmetric invariant bilinear form on D. Hence, by Theorem 3.7, the restriction of the form to $o(\mathfrak{g})$ is nondegenerate. Let $\mathfrak{g}^{\perp}$ be the orthogonal complement of $o(\mathfrak{g})$. Then the intersection of $\mathfrak{g}^{\perp}$ and $o(\mathfrak{g})$ must be zero. On the other hand, $[d, u_0] = (du)_0$ tells us that $o(\mathfrak{g})$ is an ideal of D and so is $\mathfrak{g}^{\perp}$. Thus $[d, u_0] = (du)_0 = 0$ for $d \in \mathfrak{g}^{\perp}$ and $u \in V_1$. Since the action of D is faithful on V_n, we have $du = 0$. □

4. Example

In this section we show by example that a finitely generated VOA with an infinite descending chain of ideals can still have a reductive group of automorphisms. Our example is V_L^G, for which we find all ideals and find that they form a countable descending chain.

EXAMPLE 4.1. We consider the vertex operator algebra $V = V_L = L(\Lambda_0)$, where $L = \mathbb{Z}\alpha$ such that $(\alpha, \alpha) = 2$ and where $L(\Lambda_0)$ is the fundamental representation for the affine algebra $A_1^{(1)}$. Then the automorphism group of V_L is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$ (see [DLY] and [DN]).

Let $L(c, h)$ be the highest weight module for the Virasoro algebra with central charge c and highest weight h. Let W_m for $m \geq 1$ be the irreducible module for $\mathrm{sl}(2, \mathbb{C})$ of dimension m. Then

$$V_L = \bigoplus_{m \geq 0} L(1, m^2) \otimes W_{2m+1}$$

(cf. [DG]) and $\mathrm{SL}(2, \mathbb{C})$ acts on V_L by acting on W_{2m+1}. Moreover, W_{2m+1} regarded as a $\mathrm{SL}(2, \mathbb{C})$-submodule of V_L is generated by the highest weight vector $e^{m\alpha}$.

Consider the subgroup $G = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{C} \right\}$ of $\mathrm{SL}(2, \mathbb{C})$. Clearly G is not compact. Hence the space of G-invariants W_{2m+1}^G is spanned by $e^{m\alpha}$. As a result, we have a direct sum decomposition of the fixed point set for G into irreducible modules for the Virasoro algebra:

$$V_L^G = \bigoplus_{m \geq 0} L(1, m^2),$$

where the highest weight module $L(1, m^2)$ for the Virasoro algebra is generated by $e^{m\alpha}$. For distinct m, these modules are pairwise nonisomorphic. It is not hard to see that, for any $n \geq 1$, $\sum_{m \geq n} L(1, m^2)$ is an ideal of V_L^G, because $u_k v \in L(1, (s+t)^2)$ for any $u \in L(1, s^2)$, $v \in L(1, t^2)$, and $k \in \mathbb{Z}$.

We now prove that all the ideals of V_L^G are given in this way. Let I be a nonzero ideal of V_L^G. Then I is a module for the Virasoro algebra and thus is a sum of a family of the $L(1, n^2)$. Let $n \geq 1$ be the smallest positive integer such that $L(1, n^2)$ is a subspace of I. Then $e^{\alpha}_{-n(\alpha,\alpha)-1} e^{n\alpha} = e^{(n+1)\alpha} \in I$ (see [FLM]) and $L(1, (n+1)^2)$ is contained in I. Continuing in this way, we see that $I = \sum_{m \geq n} L(1, m^2)$. It was proved in [DLM1] that, if H is a compact Lie group acting continuously on a simple vertex operator algebra V, then V^H is also a simple vertex operator algebra. Since our G is not compact, V_L^G is permitted to be nonsimple, and it is.

We now show that the automorphism group of V_L^G is isomorphic to $\mathbb{C}^{\times}$. Note that V_L^G is generated by ω and e^{α}. In fact, we have already proved that $e^{(n+1)\alpha}$ can be generated from e^{α} and $e^{n\alpha}$. Hence all the highest weight vectors can be generated from e^{α}; using ω then generates the whole space. For any $\lambda \in \mathbb{C}^{\times}$, we define a linear isomorphism σ_{λ} of V_L^G such that σ_{λ} acts on $L(1, m^2)$ as λ^m. From the previous discussion it is clear that σ_{λ} is an automorphism of V_L^G. On the other hand, any automorphism σ of V_L^G maps e^{α} to λe^{α} for some nonzero constant λ, since e^{α} is the only highest weight vector with highest weight 1 (up to a constant) for the Virasoro algebra. As V_L^G is generated by ω and e^{α}, we immediately see that $\sigma = \sigma_{\lambda}$.

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