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# Acylindrical Surfaces and Branched Surfaces I

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## 1. Introduction

In this paper we give a finiteness property of some embedded surfaces in a 3-manifold $M$. These embedded surfaces will be called "acylindrical" or "pseudo-acylindrical" in Section 2. Acylindrical surfaces are important for hyperbolic 3-manifolds; for example, if $M$ is a hyperbolic 3-manifold and if $S$ is an embedded totally geodesic surface, then $S$ is acylindrical. Furthermore, if $S$ is acylindrical or pseudo-acylindrical, then $S$ is a quasi-Fuchsian surface; that is, the limit set of the image $\pi_1(S) \to \pi_1(M) \to \mathrm{Isom}^+ \mathbb{H}^3$ is a simple closed curve in $S^2_\infty$ ([10]).

In Section 3, we prove the following finiteness result.

Theorem 1.1. *There exist only finitely many pseudo-acylindrical surfaces, up to isotopy, in a compact orientable 3-manifold.*

A result similar to Theorem 1.1 was obtained by Hass [3] and Sela [9]. Since Gabai suggested that [3, Thm. 10] can be obtained using techniques of branched surface theory, we give a proof of Theorem 1.1 using some results about branched surfaces obtained by Floyd and Oertel [1].

In Section 2, we give definitions for acylindrical and pseudo-acylindrical surfaces and for branched surfaces.

In Section 4, we consider the finite cyclic covering spaces of a 3-manifold. In fact, for a Haken 3-manifold $M$ with positive Betti number, some finite cyclic covering space $M'$ contains pseudo-acylindrical surfaces, or $M'$ is a surface bundle over the circle or constructed by books of $I$-bundles.

In Section 5, we will give some examples of 3-manifolds that are related to our results. As an application of Theorem 1.1, one can construct infinitely simple 3-manifolds (cf. Proposition 5.1).

## 2. Preliminaries

Unless stated otherwise, we let $M$ be a compact orientable 3-manifold, and we let $F$ be a 2-manifold that (a) is not homeomorphic to $S^2$ or $P^2$ and (b) is properly embedded in $M$. We denote a regular neighborhood of a subset $X \subset M$ by $N(X; M)$ and denote its interior by $\mathring{X}$. For a topological space $X$, we denote the

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number of components by $|X|$. For a set $S$, we denote the cardinality of $S$ by $\sharp S$. By a *surface* we mean a compact 2-manifold. For a surface $F$ properly embedded in a 3-manifold $M$, the frontier of $N(F; M)$ is denoted by $\partial N(F; M)$.

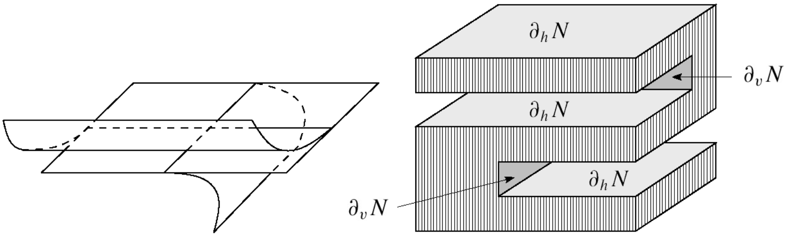
A surface $F$ properly embedded in a 3-manifold $M$ is said to be *injective* ($\pi_1$-*injective*) in $M$ if the map $i_*\colon \pi_1(F) \to \pi_1(M)$ is injective. A *compressing disk* for a surface $F$ embedded in a 3-manifold $M$ is an embedded 2-disk $D$ such that $F \cap D = \partial D$ and $\partial D$ does not bound a disk in $F$. A surface $F$ embedded in a 3-manifold $M$ is *incompressible* in $M$ if there exists no compressing disk for $F$. A surface $F$ is $\partial$-*incompressible* if, for each disk $D \subset M$ with $\partial D = \alpha \cup \beta$ (where $D \cap F = \alpha$ is a properly embedded arc in $F$ and $\partial M \cap D = \beta$), there is a disk $D' \subset F$ with $\partial D' = \alpha \cup \beta'$ and $\beta' = D' \cap \partial F$.

An embedded surface $F$ is *two-sided* in $M$ if $N(F)$ is homeomorphic to $F \times I$; otherwise, $F$ is said to be *one-sided*. It is known that an injective surface is incompressible and that a two-sided incompressible surface is injective (see [4, Chap. 6]). A 3-manifold $M$ is said to be *irreducible* if each embedded 2-sphere bounds a 3-ball in $M$, and $M$ is $P^2$-*irreducible* if $M$ is irreducible and contains no two-sided projective plane $P^2$. A manifold $M$ is said to be: *sufficiently large* if $M$ contains some two-sided incompressible surface; *Haken* if $M$ is compact, $P^2$-irreducible, and sufficiently large; and $\partial$-*irreducible* if each component of $\partial M$ is incompressible in $M$. A surface $F$ properly embedded in $M$ is said to be $\partial$-*parallel* if there exists an embedding $f\colon F \times [0,1] \to M$ such that $f(F \times \{0\}) = F$ and $f(\partial F \times [0,1] \cup F \times \{1\}) \subset \partial M$. If $M$ contains no incompressible torus that is not $\partial$-parallel, then $M$ is said to be *atoroidal*. An annulus $A$ properly embedded in a 3-manifold $M$ is said to be *essential/* if $A$ is incompressible and not $\partial$-parallel in $M$. A 3-manifold $M$ is said to be *anannular* if $M$ contains no properly embedded essential annuli. A 3-manifold that is irreducible, $\partial$-irreducible, atoroidal, and anannular is called a *simple* 3-manifold.

A closed two-sided incompressible surface $F$ in $M$ is said to be *pseudo-acylindrical* if, for some $i = 0, 1$, each essential annulus in $M - \mathring{N}(F)$ has boundaries in $F \times \{i\}$, where $N(F)$ is identified with the product $F \times [0,1]$. An incompressible closed surface $F$ in a 3-manifold $M$ is *acylindrical* if each component of $M - \mathring{N}(F)$ contains no essential annuli with boundaries contained in $\partial N(F)$.

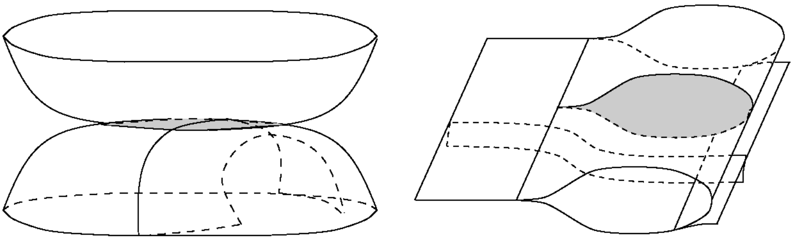
A compact 2-polyhedron $B \subset M$ is called a *branched surface* if the local structure is modeled on the space in Figure 1(A) (see [1] for details). The *branch locus* of $B$ is the set of points in $B$ each of which has no neighborhood homeomorphic to $\mathbb{R}^2$. A neighborhood $N$ of $B$ in the 3-manifold $M$ is naturally constructed as indicated in Figure 1(B). Such a neighborhood is called a *fibered neighborhood*. Observe that $\partial N$ is the union of three compact subsurfaces ($\partial_h N$, $\partial_v N$, and $N \cap \partial M$) that meet only in their common boundary points. A fiber of $N$ meets $\partial_h N$ transversely at its endpoints, while a fiber of $N$ intersects $\partial_v N$ in a closed interval in the interior of the fiber. A surface $F$ is *carried by $B$* if $F$ can be isotoped into $\mathring{N}$ so that $F$ intersects the fibers transversely. A surface $F$ is carried by $B$ *with positive weights* if $F$ can be isotoped into $\mathring{N}$ so that $F$ intersects all fibers transversely.

A branched surface $B$ properly embedded in a Haken 3-manifold $M$ is said to be *incompressible* if the following conditions are satisfied (see Figure 2).




(A) (B)

**Figure 1**



(A) disk of contact (B) monogon

**Figure 2**

(I.1) There exists no disk $D \subset N$ such that $D$ is transverse to the fibers of $N$ and $\partial D \subset \mathring{\partial}_v N$ (such a disk is called a *disk of contact*); and there is no disk $D \subset N$ such that $D$ is transverse to the fibers of $N$ with $\partial D = \alpha \cup \beta$, where $\mathring{\alpha} \subset \mathring{\partial}_v N$ and $\beta \subset \partial M$ are arcs and $\alpha \cap \beta = \partial\alpha = \partial\beta$.

(I.2) Each component of $\partial_h N$ is incompressible and $\partial$-incompressible in $M - \mathring{N}$.

(I.3) There exists no disk $D \subset M - \mathring{N}$ with $\partial D = D \cap N = \alpha \cup \beta$, where $\alpha \subset \partial_v N$ is a fiber and $\beta \subset \partial_h N$ (such a disk is called a *monogon*).

## 3. Proofs

Throughout this section, we let $B$ be a branched surface that carries some connected surface with positive weights in an orientable irreducible 3-manifold $M$. We assume that each surface $F$ carried by $B$ with positive weights is isotoped into a fibered neighborhood $N'$ of $B$ so that $F$ intersects all fibers transversely. Since $S = \partial N(F)$ intersects each fiber of $N'$ at least twice, we may assume that $N(F)$ is isotoped so that $\partial_h N' \subset S$ and $S \cap \partial N' = \partial_h N'$ (see Figure 3). If we let $L'$ be the closure of $N' - N(F)$, then each component of $L'$ is an $I$-bundle over a surface. Unless stated otherwise, the base spaces of all $I$-bundles are homeomorphic to neither $S^2$ nor $P^2$.

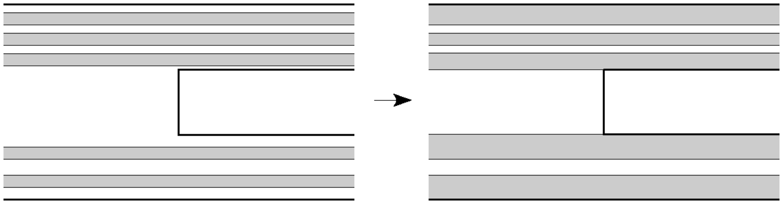


**Figure 3**

If some component $A$ of $\partial_v N'$ is compressible in $M - \mathring{N}'$ then, by condition (I.2), both components of $\partial A$ bound disks $D_0$ and $D_1$ in $\partial_h N'$. By the irreducibility of $M$, the sphere $D_0 \cup A \cup D_1$ bounds a 3-ball $C$ on the side containing the compressing disk $D$ for $A$. The 3-ball $C$ can be identified with the product $D \times I$ so that $D \times \{0\} = D_0$ and $D \times \{1\} = D_1$. Hence, the $I$-bundle $L'$ is naturally extended to the $I$-bundle $L = L' \cup_A C$, and we put $N = N' \cup C$. It can be seen that conditions (I.1)–(I.3) hold for $N$, by an argument similar to [1, Thm. 2]. We call the 3-manifold $N$ constructed as described here an *extended fibered neighborhood of $B$*. Note that the extended neighborhood $N$ may not be a regular neighborhood of a branched surface. In fact, the manifold $N$ could be the whole manifold $M$, or some of the induced fibers of $N$ may be $S^1$ or noncompact if $B$ has "Reeb components".

A branched surface $B$ is said to be *reduced* if $B$ is an incompressible branched surface such that no component of $\partial_h N$ is a closed surface. The following theorem is a consequence of the main result of Floyd and Oertel in [1].

THEOREM 3.1. *Let $M$ be a Haken 3-manifold. There exist a finite number of reduced branched surfaces and incompressible surfaces $B_1, \dots, B_n$ such that each two-sided closed incompressible surface in $M$ is carried with positive weights by some $B_i$.*

*Proof.* If $M$ is $\partial$-reducible, then we let $D_1, \dots, D_m$ be disks properly embedded in $M$ such that each component of $M - \mathring{N}\left(\bigcup_{i=1}^m D_i\right)$ is irreducible and $\partial$-irreducible (see [5, Lemma III.21]). We put $M_0 = M - \mathring{N}\left(\bigcup_{i=1}^m D_i\right)$. If $M$ is $\partial$-irreducible then we put $M_0 = M$. Let $B_1, \dots, B_n$ be branched surfaces for $M_0$ given in [1, Thm. 1] without boundaries. Then each two-sided closed incompressible surface in $M_0$ is carried with positive weights by some $B_i$. By the construction in [1], these branched surfaces are incompressible in $M_0$. If some component $S$ of $\partial_h N_i$ is closed, where $N_i$ is an extended fibered neighborhood of $B_i$, then each connected two-sided surface carried by $B_i$ with positive weights is isotopic to $S$. If some $B_i$ is a one-sided surface, then we replace $B_i$ by $\partial N(B_i)$. Now it follows that each branched surface $B_i$ is either a reduced branched surface or a two-sided surface. Since $M$ is irreducible, each closed incompressible surface in $M$ is isotopic to an incompressible surface in $M_0$. Furthermore, each $B_i$ is also incompressible in $M$. Therefore, each closed incompressible surface in $M$ is carried by some $B_i$ with positive weights. $\square$

We call the branched surfaces $B_1, \ldots, B_n$ given in Theorem 3.1 *basic branched surfaces for M.*

LEMMA 3.2. *Let $M$ be an orientable Haken 3-manifold. For each closed connected incompressible surface $F$ carried by a reduced branched surface $B$ with positive weights, each component of $\partial_v N$ is an essential annulus in $M - \mathring{N}(F)$, where $N$ is an extended fibered neighborhood of $B$.*

*Proof.* Let $A$ be a component of $\partial_v N$. Since $M$ is orientable, $A$ is an annulus. First, we show that $A$ is incompressible in $M - \mathring{N}(F)$. If it is not, then—for a compressing disk $D$ for $A$ with the number $|D \cap \partial_v N|$ minimal among all compressing disks—we let $D_0$ be an innermost disk $D$ with respect to $D \cap \partial_v N$. We claim that $D_0$ is isotopic to a disk of contact for $N$. Notice that $D_0$ is properly embedded in $N$ because each component of $\partial_v N$ is incompressible in $M - \mathring{N}$. Since $D$ is contained in $M - \mathring{N}(F)$, it follows that $D_0$ is contained in $L$, where $L$ is the closure of $N - N(F)$, which is an $I$-bundle. Since $\partial D_0$ is contained in $\partial_v N$, the component of $L$ containing $D_0$ is homeomorphic to a product $D_0 \times I$. Thus, there exists a disk of contact that is isotopic to $D_0$. This contradicts condition (I.1).

Next, we show that $A$ is $\partial$-incompressible in $M - \mathring{N}(F)$. Let $D$ be a $\partial$-compressing disk for $A$ in $M - \mathring{N}(F)$. By the incompressibility of $\partial_v N$, we may assume that $D \cap \partial_v N$ consists of properly embedded arcs in $D$. Let $D_0$ be an outermost disk of $D$ with $\partial D_0 = \alpha \cup \beta$, where $\partial D_0 \cap \partial_v N = \alpha$. We let $A_0$ be the component of $\partial_v N$ such that $\alpha \subset A_0$. By the minimality of $|D \cap \partial_v N|$, it follows that the two points $\partial\alpha$ are contained in mutually distinct component of $\partial A_0$. Since $L$ is an $I$-bundle, it follows that $D$ is contained in $M - \mathring{N}$. This contradicts condition (I.3). □

LEMMA 3.3. *Let $M$, $F$, and $B$ be as in Lemma 3.2. Then $F$ is not pseudo-acylindrical in $M$.*

*Proof.* Let $N$ be an extended fibered neighborhood of $B$. If $\partial N$ is empty, then each component of $M - \mathring{N}(F)$ is an $I$-bundle. Hence $F$ is not pseudo-acylindrical in $M$.

We assume that $\partial N$ is not empty and that $F$ is pseudo-acylindrical in $M$. By Lemma 3.2, for each component $A$ of $\partial_v N$ it follows that $\partial A \subset F \times \{0\}$, since $F$ is pseudo-acylindrical; here, $N(F)$ is identified with the product $F \times [0, 1]$. Since $\partial_h N$ has no closed component, $F \times \{1\}$ is contained in $\mathring{N}$. Hence, some component $L_1$ of $L$ with $\partial L_1 \cap F \times \{1\} \neq \emptyset$ is an $I$-bundle over a closed surface. By the hypothesis that $\partial_v N$ is not empty, $L_1$ must be a twisted $I$-bundle over a closed surface. This shows that there exists an essential annulus $A' \subset L_1$ with $\partial A' \subset F \times \{1\}$. Hence $F$ is not pseudo-acylindrical. □

Now let us prove Theorem 1.1. First, we prove the following lemma about reducible 3-manifolds.

LEMMA 3.4. *Let $M$ be a reducible 3-manifold. Then $M$ does not contain acylindrical surfaces.*

*Proof*. Let $F$ be any two-sided closed incompressible surface embedded in $M$. Our plan is to show that $F$ cannot be pseudo-acylindrical. Let $l$ be a nontrivial simple closed curve in $F$ and let $E$ be an essential sphere embedded in $M$. By the incompressibility of $F$, we can choose the sphere $E$ such that $E$ does not meet $F$. We join points $x \in l$ and $y \in E$ with an arc $\alpha$ such that $\alpha \cap F = x$ and $\alpha \cap E = y$, and we put $A' = \partial N(l \cup \alpha \cup E; M)$. Clearly, $A'$ consists of three components: one of the three is a sphere parallel to $E$, and the other two are annuli. For an annular component $A$ of $A' - F$ "surrounding $\alpha \cup E$", we see that $A \cap (M - \mathring{N}(F))$ is incompressible and not $\partial$-parallel in the $M - \mathring{N}(F)$. Therefore, $A$ is essential in $M - \mathring{N}(F)$ and hence $F$ is not acylindrical. □

*Proof of Theorem 1.1.* First, we prove the theorem for irreducible 3-manifolds. By Theorem 3.1, each closed two-sided incompressible surface is carried with positive weights by some basic branched surface $B$ that is a reduced branched surface or has no branch loci. By Lemma 3.3, $F$ is pseudo-acylindrical if and only if $B$ has no branch loci and the surface $B$ is pseudo-acylindrical. Hence, each pseudo-acylindrical surface in $M$ is isotopic to one of the basic surfaces that are pseudo-acylindrical surfaces. Since the number of basic branched surfaces $B_1, \dots, B_n$ is finite, the conclusion follows.

Next, we consider the case where $M$ is reducible. By the same argument as Lemma 3.4, if $M$ is reducible and $S$ is a pseudo-acylindrical surface in $M$, then $S$ is isotoped off the reducing spheres and $S$ is a separating surface in $M$. Let $M_0$ be the cutting result of $M$ along the a union of reducing spheres of $M$, and let $\hat{M}_0$ be the manifold obtained from $M_0$ by capping off spherical boundary components with 3-balls $C$. Then $\hat{M}_0$ is an irreducible 3-manifold. Suppose $M$ contains infinitely many pseudo-acylindrical surfaces $S_1, S_2, \dots$, up to isotopy. Then each $S_i$ is separating and does not meet the reducing spheres, and $S_i$ is contained in $\hat{M}_0$. Since $\hat{M}_0$ is irreducible, some $S_i$ and $S_j$ is isotopic in $\hat{M}_0$. The isotopy can be chosen so that $C$ is preserved. Thus, if $S_i$ is not isotopic to $S_j$ in $M$, then (a) $S_i$ can be isotoped to $S_i'$ in $M$ so that $S_i' \cap S_j = \emptyset$ and (b) in $\hat{M}_0$, $S_i' \cup S_j$ bounds a product $W = S_i \times I$ such that $C$ is contained in $W$. Hence, the number of isotopy classes of pseudo-acylindrical surfaces in $M$ is at most twice of that of $\hat{M}_0$. The proof is complete. □

In [9], Sela obtained a stronger result than Theorem 1.1 and [3, Thm. 10], a result concerning a "$k$-acylindrical surface" for simple 3-manifolds (see [9] for the definition of *$k$-acylindrical surface*). In [12], the author gave a proof of a "$k$-acylindrical finiteness property" for irreducible 3-manifolds using branched surfaces. Furthermore, it was shown in [12] that, if $M$ is hyperbolic and if each component of $M - \mathring{N}(F)$ is not an $I$-bundle, then the incompressible surface $F$ is $k$-acylindrical for some $k$.

## 4. Finite Covering Spaces and Books of $I$-Bundles

In this section, we will search for pseudo-acylindrical surfaces in finite-fold cyclic covering spaces of an atoroidal 3-manifold. In fact, if we fail to find pseudo-acylindrical surfaces, then the 3-manifold would be finitely covered by some 3-manifold that can be decomposed into "books of $I$-bundles".

Let $M$ be a 3-manifold. We say a surface $S$ in $\partial M$ is *$\partial$-incompressible* if there exists no properly embedded disk $D$ in $M$ such that $D \cap S$ is a single essential arc in $S$. A 3-manifold pair $(M, R)$ is called an *incompressible pseudo-sutured manifold pair* if each component of $R$ is an incompressible torus or an incompressible, $\partial$-incompressible annulus in $\partial M$. An essential loop in a component of $R$ is called a *suture.* Notice that, if $B$ is an incompressible branched surface in $M$, then (by Lemma 3.2) the pair $(M - \mathring{N}, \partial_v N)$ is a pseudo-sutured manifold pair, where $N$ is the extended neighborhood of $B$.

Let $(V_1, R_1), \ldots, (V_m, R_m)$ be incompressible pseudo-sutured solid torus pairs with $|R_i| \geq 2$. Each $R_i$ is a union of mutually parallel disjoint annuli in $\partial V_i$ with nonmeridional slopes. Let $F_1, \ldots, F_n$ be compact surfaces with boundaries. If we glue $\bigcup_{i=1}^{n}(F_i \times I)$ to $\bigcup_{i=1}^{m} V_i$ with a homeomorphism $\bigcup_{i=1}^{n}(\partial F_i \times I) \to \bigcup_{i=1}^{m} R_i$, then we obtain a compact 3-manifold. Such a 3-manifold is called a *book of $I$-bundles*; cores of the $V_i$ (or the solid tori $V_i$) are called *binders* and the $F_i$ are called *pages.*

Let $\mathcal{BI}$ be the set of 3-manifolds $M$ in which there exists a finite union of two-sided incompressible surfaces $S_1, \ldots, S_m$ such that each component of $M - \mathring{N}\left(\bigcup_{i=1}^{m} S_i\right)$ is a book of $I$-bundles.

We describe some properties of books of $I$-bundles. Hereafter, we shall use the following notation for books of $I$-bundles: $(V_i, R_i)$ denotes a binder pseudo-sutured solid torus pair; $F_i$ denotes a page; and we put $V = \bigcup V_i$, $R = \bigcup R_i$, and $F = \bigcup F_i$.

Lemma 4.1. *Let $M$ be a book of $I$-bundles. If each page $F_i$ has negative Euler characteristic, then $M$ is irreducible, $\partial$-irreducible, and atoroidal.*

*Proof.* By the hypothesis that $\chi(F_i) < 0$, each annulus $R_i$ is incompressible and $\partial$-incompressible in $M$. Thus, if $M$ is reducible then there exists an essential sphere in some $V_i$ or $F_i \times I$, which is impossible because there are handlebodies. Hence we can conclude $M$ is irreducible. If $M$ is $\partial$-reducible then we let $D$ be a compressing disk for $\partial M$. By the incompressibility and $\partial$-incompressibility of $R_i$, we may assume that $D \cap R = \emptyset$, since $R_i$ is incompressible and $\partial$-incompressible. The disk $D$ is therefore contained in $V$ or $F \times I$. If $D$ is contained in $V$ then, by the incompressibility of $R$, $\partial D$ must bound a disk in $\partial V - R_i$. If $\partial D$ is contained in $F \times \{0\}$ then, since $F \times \{0\}$ is incompressible in the product $F \times I$, it follows that $\partial D$ bounds a disk in $F \times \{0\}$. These statements contradict the assumption that $D$ is a compressing disk for $\partial M$. Hence, $M$ is irreducible and $\partial$-irreducible. Let $T$ be an incompressible torus in $M$. By the incompressibility of $R$, we may assume that the closure of each component of $T - R$ is an essential annulus in $V$ or $F \times I$ and that $V \cap T \neq \emptyset$ and $T \cap (F \times I) \neq \emptyset$. However, since $\chi(F_i) < 0$, each component of $T \cap (F \times I)$ cannot be essential in $F \times I$. The proof is complete. $\square$

Lemma 4.2. *Let $M$ be a book of $I$-bundles with each page $F_i$ of negative Euler characteristic. Let $\partial_0 M$ be a component of $\partial M$. Suppose that* (1) *$\partial_0 M$ contains at most one component of $F_i \times \{0, 1\}$ for any $i$,* (2) *each component $R_i$ of $R$ has integral slope on $\partial V_i$, and* (3) *$\partial_0 M$ contains at most one component of $\partial V_i - R_i$ for any $i$. Then $\partial_0 M$ is acylindrical in $M$.*

*Proof.* Let $A$ be an essential annulus in $M$ with $\partial A \subset \partial_0 M$. Let $F'$ be the maximal union of components of $F$ such that $(F' \times I) \cap \partial_0 M = \emptyset$. Since each component of $F$ has negative Euler characteristic, we may assume that $A \cap (\partial F' \times I) = \emptyset$. Because the slope of each suture is integral on the boundary of the binder solid torus, we may assume (by a suitable choice of a union $A'$ of essential annuli in $M$ with $A' \cap (\partial_0 M \cup A) = \emptyset$) that the component $M_0$ of $M - \mathring{N}(A')$ that contains $\partial_0 M$ is homeomorphic to the product $\partial_0 M \times I$. Thus, the annulus $A$ is $\partial$-parallel to $\partial_0 M$. This is a contradiction to the essentiality of $A$. $\square$

The main result in this section is the following theorem. A surface $S$ in a 3-manifold $M$ is said to be *taut* if $\sum_i(|\chi(S_i)|)$ is minimal among surfaces in the homology class of $S$, where the sum is over components $S_i$ of $S$ with $\chi(S_i) \leq 0$.

THEOREM 4.3. *Let $M$ be a closed atoroidal* 3*-manifold with $\beta_1(M) \geq 1$. Then, for any primitive element $e \in H_2(M; \mathbb{Z})$, $M$ has a finite-fold cyclic cover $p\colon M' \to M$ that is dual to $e$ with at least one of the following properties*:

(a) $M' \in \mathcal{BI}$;

(b) *$M'$ contains a pseudo-acylindrical surface.*

In order to prove Theorem 4.3, we need some lemmas. Let $S$ be a two-sided surface in $M$; namely, $N(S)$ is identified with the product $S \times [0, 1]$. An essential annulus (or a Seifert pair) $A$ properly embedded in $M - \mathring{N}(S)$ is said to be of *type* $\mathcal{A}_0$ ($\mathcal{A}_1$, $\mathcal{A}_{01}$, resp.) if $\partial A \subset S \times \{0\}$ ($\partial A \subset S \times \{1\}$, one component of $\partial A$ is contained in $S \times \{0\}$ and the other in $S \times \{1\}$, resp.).

LEMMA 4.4. *Let $M$ be an atoroidal closed* 3*-manifold and let $S$ be a two-sided nonseparating incompressible surface embedded in $M$ such that $M - \mathring{N}(S)$ is not an $I$-bundle. Then there exists a finite-fold cyclic cover $p\colon M' \to M$ such that, for some lift $S'$ of $S$, the manifold $M' - \mathring{N}(S')$ contains no essential annulus of type $\mathcal{A}_{01}$. Furthermore, if $S$ is taut in $M$, then $p\colon M' \to M$ can be chosen so that $S'$ is taut in $M'$.*

*Proof.* Let $(\Sigma, \Phi)$ be the characteristic Seifert submanifold $\Sigma$ of $M$. Suppose that some $\mathcal{A}_{01}$-type component of $\Sigma$ is not an $S^1$-pair. Let $q : \tilde{M} \to M$ be the infinite cyclic cover of $M$ that is dual to $S$, and let $\tau : \tilde{M} \to \tilde{M}$ be a generator of the covering translation. Let $M_1$ be a fundamental domain in $\tilde{M}$ such that the pair $(M_1, \partial M_1)$ is homeomorphic to the pair $(M - \mathring{N}(S), \partial N(S))$, so that: $q|_{\mathring{M}_1}$ is a homeomorphism; $\partial M_1$ is a union of two copies of $S$; and $q|_{\partial M_1}$ is a 2-fold cover of $S$. We put $M_j = \bigcup_{i=1}^{j} \tau^{i-1}(M_1)$. Then each $M_j$ has the common boundary component $\partial_0 M_j = \partial_0 M_1$. We put $\partial_1 M_j = \partial M_j - \partial_0 M_j$. Let $(\Sigma_j, \Phi_j)$ be the characteristic Seifert pair of $(M_j, \partial M_j)$, and let $\Sigma_j^{01}$ be the subset that consists of components of $\Sigma_j$ of type $\mathcal{A}_{01}$. We put $\Phi_j^0 = \Sigma_j^{01} \cap \partial_0 M_j$ and $\Phi_j^1 = \Sigma_j^{01} \cap \partial_1 M_j$.

We claim that $\Sigma_m^{01} = \emptyset$ for some positive integer $m$. By a suitable isotopy, each component of $\Sigma_{i+1} \cap M_i$ that meets $\partial_0 M_i$ is contained in $\Sigma_i^{01}$; this follows because, after eliminating trivial circle components of $\partial\Sigma_{i+1} \cap \partial_1 M_i$ via their incompressibility, the intersection $\Sigma_{i+1} \cap M_i$ forms an essential Seifert pair of $M_i$.

So, the equation $\chi(\Phi^0_{i+1}) = \chi(\Phi^0_i) - \chi(\Phi^0_i - \Phi^0_{i+1})$ holds. Put $a(i) = -\chi(\Phi^0_i)$ and $b(i) = -|\partial\Phi^0_i|$. Set $c(i) = (a(i), b(i))$, a complexity that is ordered lexicographically. As we have seen before, $a(i+1) = a(i) + \chi(\Phi^0_i - \Phi^0_{i+1})$. Thus, we have $a(i+1) \leq a(i)$.

CLAIM 4.5. $c(i) > c(2i)$.

*Proof*. We may assume that each component of $\Sigma^{01}_{2i} \cap \partial_1 M_i$ is incompressible and $\partial$-incompressible or parallel to $\Phi^1_{2i}$ in $\Sigma^{01}_{2i}$. Let $\Sigma'$ be the union of the closures of components of $\Sigma^{01}_{2i} - \partial_1 M_i$ each of which meets $\partial_1 M_{2i}$. If $\Sigma' = \emptyset$, we are done. It follows that $\tau^{-i}(\Sigma')$ can be isotoped into $\Sigma^{01}_i$. If $c(i) = c(2i)$ then, for each component $l$ of $\partial(\Sigma' \cap \partial_1 M_i)$, the loop $\tau^{-1}(l)$ is parallel to a component of $\partial\Phi^0_i$ in $\partial_0 M_i$. Furthermore, $\Sigma^{01}_i$ is isotopic to $\tau^{-i}(\Sigma')$. Hence we can find an incompressible torus (possibly immersed) in $\tilde{M}/\tau^i$. Now the torus theorem [5] yields a contradiction to the condition that $M$ is atoroidal and contains a nonseparating surface such that the exterior is not an $I$-bundle. □

It is true (see [11]) that the number of mutually nonparallel disjoint essential annuli properly embedded in an atoroidal 3-manifold $M$ is bounded by a number that is dependent only on $\chi(\partial M)$. As a result, the inequality $|b(i)| < n$ holds for some $n$, and thus we have $\Sigma^{01}_m = \emptyset$ for some $m$.

Now we let $M'$ be an $m$-fold cyclic cover of $M$ that is dual to $S$. Let $S'$ be a lift of $S$. If there exists an essential annulus in $M' - \mathring{N}(S')$ of type $\mathcal{A}_{01}$, then some component of the characteristic Seifert submanifold $\Sigma'$ of $M' - \mathring{N}(S')$ is an $S^1$-pair. Let $k$ be the number of $S^1$-pairs in $\Sigma'$. By the atoroidality of $M$, for the $(k+1)$-fold cyclic cover $M''$ of $M'$ dual to $S'$, the exterior $M'' - \mathring{N}(S'')$ contains no essential annuli of type $\mathcal{A}_{01}$, where $S''$ is a lift of $S'$. So, by taking a finite cyclic cover of $M$, we can eliminate the essential annulus of type $\mathcal{A}_{01}$.

Now we prove the latter part of this lemma. Let $p\colon M' \to M$ be the resulting cyclic cover. Suppose $S$ is taut in $M$ and there is an incompressible surface $F'$ that is homologous to $S'$ in $M'$ with $\chi(F') > \chi(S')$. By an argument similar to [2, Lemma 3.6], we can find a surface $F''$ that is homologous to $S'$ in $M'$ with $\chi(F'') > \chi(S')$ such that $F'' \cap p^{-1}(S) = \emptyset$. Since $F''$ and $S'$ are homologous in $M'$, there exists a compact 3-manifold $B'$ embedded in $M'$ with $\partial B' = F'' \cup S'$ and $B' \cap p^{-1}(S) = S'$. Hence, $p(F'')$ is an embedded surface homologous to $S$ in $M$, since $p|_{B'}$ is an embedding. This contradicts the assumption that $S$ is taut in $M$. □

*Proof of Theorem 4.3.* Let $S$ be a taut, incompressible, nonseparating surface such that $[S] = e \in H_2(M; \mathbb{Z})$. Let $\Sigma$ be the characteristic submanifold of $M - \mathring{N}(S)$. If $\Sigma = M - \mathring{N}(S)$, then $M$ is a surface bundle over $S^1$ with a fiber $S$ (property (a)). If $\Sigma = \emptyset$, then $S$ is acylindrical (property (b)). By Lemma 4.4, we may assume that $M - \mathring{N}(S)$ contains no essential annuli of type $\mathcal{A}_{01}$. Furthermore, since the cyclic covering space dual to $S$ is unique up to homology class of $S$, by taking a cyclic cover of $M$ we may assume that, for each incompressible surface $F$ that is homologous to $S$ in $M$ with $\chi(F) = \chi(S)$, the manifold $M - \mathring{N}(F)$ contains no essential annulus of type $\mathcal{A}_{01}$, since there exists only finitely many incompressible surfaces in $M$ (up to isotopy) with any fixed Euler characteristic [6, Cor. 2.3].

Let $q\colon \tilde{M} \to M$ be an infinite cyclic covering space of $M$ that is dual to $S$. Let $\tilde{S}_0$ be a lifting of $S$ into $\tilde{M}$, and let $A_0$ be an annulus in $\tilde{M}$ such that $A_0 \cap \tilde{S}_0 = \partial A_0$ and $(\tilde{M} - \mathring{N}(\tilde{S}_0)) \cap A_0$ is an essential annulus of type $\mathcal{A}_1$.

We construct surfaces $\{\tilde{S}_i\}$ in $\tilde{M}$ successively as follows. We are given an incompressible surface $\tilde{S}_i$ in $\tilde{M}$ such that $S_i = q(\tilde{S}_i)$ is an embedded incompressible surface in $M$, so that $M - \mathring{N}(S_i)$ contains no essential annulus of type $\mathcal{A}_{01}$. We identify the regular neighborhood $N(\tilde{S}_i)$ with the product $\tilde{S}_i \times [0,1]$, so that $\tilde{S}_i \times \{1/2\} = \tilde{S}_i$ and $\tilde{S}_i \times \{1\}$ is contained in the *front-side* of $\tilde{M} - \tilde{S}_i$, that is, the component of $\tilde{M} - \tilde{S}_i$ that contains $\tau(\tilde{S}_i)$, where $\tau\colon \tilde{M} \to \tilde{M}$ is a generator of the covering translation group. See Figure 4.

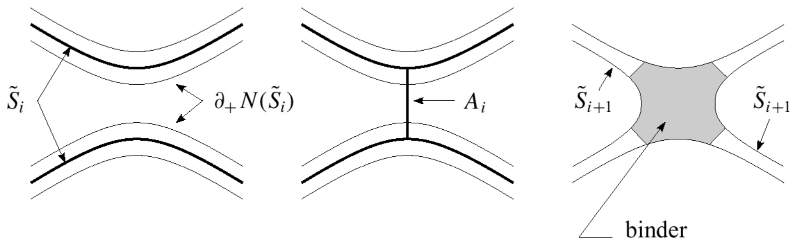



**Figure 4**

If there exists an annulus $A_i$ in $\tilde{M}$ such that $A_i \cap \tilde{S}_i = \partial A_i$ and $(\tilde{M} - \mathring{N}(\tilde{S}_i)) \cap A_i$ is an essential annulus of type $\mathcal{A}_1$, then we set $\tilde{S}_{i+1} = \partial_+ N(\tilde{S}_i \cup A_i)$, where $\partial N(\tilde{S}_i \cup A_i) = \partial_- N(\tilde{S}_i \cup A_i) \cup \partial N_+(\tilde{S}_i \cup A_i)$ and $\partial_- N(\tilde{S}_i \cup A_i)$ is parallel to $\tilde{S}_i$. Notice that the manifold $B_i$ cobounded by $\tilde{S}_i \cup \tilde{S}_{i+1}$ is a book of $I$-bundles and that $\chi(\tilde{S}_{i+1}) = \chi(\tilde{S}_i)$. Furthermore, $S_{i+1} = q(\tilde{S}_{i+1})$ is embedded in $M$; otherwise, for the annulus $A_i$ we would have $q(\mathring{A}_i) \cap q(\tilde{S}_i) \neq \emptyset$. In this case, some component of $q(A_i) \cap (M - \mathring{N}(q(\tilde{S}_i)))$ is an essential annulus of type $\mathcal{A}_{01}$, which contradicts the absence of essential annuli of types $\mathcal{A}_{01}$ for all surfaces in $M$ of Euler characteristic equal to $\chi(S)$. Furthermore, since $(\tilde{S}_i \cup \tau(\tilde{S}_i)) \cap \tilde{S}_{i+1} \neq \emptyset$, the surface $S_{i+1}$ is embedded in $M - \mathring{N}(S_i)$. Thus, the book of $I$-bundles $q(B_i)$ is embedded in $M$ and hence the surface $S_i$ is homologous to $S_{i+1}$ in $M$. Here we can prove that the surface $\tilde{S}_{i+1}$ is incompressible in $\tilde{M}$ directly as follows. For otherwise, let $\tilde{D}$ be a compressing disk for $\tilde{S}_{i+1}$. Recall that $S_{i+1} = q(\tilde{S}_{i+1})$ is embedded in $M - \mathring{N}(S_i)$. Since $\tilde{S}_i$ is incompressible, we may assume that $\tilde{D} \cap \bigcup_{j=-\infty}^{\infty} \tau^j(\tilde{S}_i) = \emptyset$. Therefore, $D = q(\tilde{D})$ is a compressing disk for $S_{i+1}$. This contradicts the assumption that $S = S_0$ is taut in $M$ and so proves our claim.

If there exists no such annulus in $\tilde{M}$ then $\tilde{S}_i$ is pseudo-acylindrical in $\tilde{M}$, and $q(\tilde{S}_i)$ is also pseudo-acylindrical in $M$ (property (b)).

Because $\chi(\tilde{S}_i) = \chi(S)$, the manifold $M$ contains only finitely many incompressible surfaces $q(\tilde{S}_i)$, up to isotopy [6, Cor. 2.3]. As a result, the surface $S_i = q(\tilde{S}_i)$

is isotopic to some $S_j = q(\tilde{S}_j)$ for some $i > j$. The isotopy between $S_i$ and $S_j$ is lifted to an isotopy between $\tilde{S}_i$ and $\tau^n(\tilde{S}_j)$ for some $n$. Hence there exists a map $g\colon \tilde{S}_i \times [0,1] \to \tilde{M}$ such that $g(\tilde{S}_i \times \{0\}) = \tilde{S}_i$ and $g(\tilde{S}_i \times \{1\}) = \tau^n(\tilde{S}_j)$. Since $\tilde{S}_i \times [0,1]$ is compact, we may assume that $g(\tilde{S}_i \times [0,1]) \cap \tilde{S}_j = \emptyset$. By an argument similar to [13, Cor. 5.5], we can construct an isotopy $f_t\colon \tilde{M} \to \tilde{M}$ such that $f_0$ is the identity, each $f_t|_{\tilde{S}_j}$ is the identity, and $f_1(\tilde{S}_i) = \tau^n(\tilde{S}_j)$. The compact submanifold $M''$ of $\tilde{M}$ cobounded by $\tilde{S}_j \cup \tau^n(\tilde{S}_j)$ is thus homeomorphic to the union of books of $I$-bundles $B'' = B_j \cup \cdots \cup B_{i-1}$, and so the manifold $M$ is finitely covered by a union of books of $I$-bundles $M''/\tau^n$ (property (b)).

## 5. Examples

As an application of Theorem 1.1, we give a method to show the existence of infinitely many certain Haken 3-manifolds, up to homeomorphism.

To state our result, we use the following notation. Recall that $\mathcal{BI}$ is defined to be the set of 3-manifolds (up to homeomorphism) such that, for each $M$ in $\mathcal{BI}$, there exists a union of two-sided incompressible surfaces $S_1, \ldots, S_m$ such that each component of $M - \mathring{N}\big(\bigcup_{i=1}^m S_i\big)$ is a book of $I$-bundles. Let $\vec{n} = (n_1, \ldots, n_k)$ be a $k$-tuple of positive integers $n_i$ (possibly $n_i = n_j$ for $i \neq j$). Put $\mathcal{ABI}_\phi = \{M \in \mathcal{BI} \mid M$ is closed orientable, irreducible and atoroidal$\}$. Let $\mathcal{ABI}_{\vec{n}}$ be the set of compact, orientable, irreducible, $\partial$-irreducible, atoroidal and anannular 3-manifolds in $\mathcal{BI}$ such that $\partial M$ consists of $k$ components $\partial_1 M, \ldots, \partial_k M$ with $\text{genus}(\partial_i M) = n_i$ for $\vec{n} = (n_1, \ldots, n_k)$.

It is easy to prove the following proposition using a result of Myers [7].

Proposition 5.1 [11]. *$\sharp\mathcal{ABI}_{\vec{n}} = \infty$ for any $\vec{n}$ (possibly $\vec{n} = \emptyset$).*

The proposition is applied when one constructs 3-manifolds that contain acylindrical surfaces arbitrarily; a proof based on the finiteness result on acylindrical surfaces is given in [11].

Here we give a sketch proof for the “infinitely many” part.

*Proof.* If $\mathcal{ABI}_{\vec{n}}$ is a finite set then there exists a number $g$ such that, for any 3-manifold $M \in \mathcal{ABI}_{\vec{n}}$, $M$ does not contain acylindrical surface of genus greater than $g$, since each manifold in $\mathcal{ABI}_{\vec{n}}$ contains only finitely many acylindrical surfaces (up to isotopy) by Theorem 1.1. However, this contradicts the following argument. Let $W_{g+1}$ be a 3-manifold in $\mathcal{ABI}_{\vec{n'}}$, where $\vec{n'} = (n_1, \ldots, n_k, g+1)$. If we identify $\partial M_{g+1}$ with the component $\partial_+ W_{g+1}$ of $\partial W_{g+1}$ whose genus is $g+1$, then the result is in $\mathcal{ABI}_{\vec{n}}$ and contains the acylindrical surface $\partial M_{g+1} = \partial_+ W_{g+1}$ with genus $g+1$. Hence, $\sharp\mathcal{ABI}_{\vec{n}}$ is infinite. □

Recall that an annulus properly embedded in $M$ is defined to be *essential* if it is incompressible and not $\partial$-parallel in $M$. An annulus properly embedded in $M$ is said to be *strictly essential* if it is incompressible and $\partial$-incompressible in $M$. Notice that, if $M$ is irreducible and $\partial$-irreducible, then these definitions are equivalent. However, for a reducible 3-manifold we can prove the following proposition. Here we say that a surface $S$ embedded in $M$ is *weakly acylindrical* if $M - \mathring{N}(S)$ does not contain properly embedded strictly essential annuli.

PROPOSITION 5.2. *There exists a reducible closed 3-manifold $M$ such that $M$ contains infinitely many weakly acylindrical surfaces, up to isotopy.*

*Proof.* Let $M_0$ be a closed irreducible 3-manifold that contains a nonseparating acylindrical surface $S_0$. Let $V$ be a solid torus in $M$ that meets $S_0$ with a single meridian disk of $V$. Let $x_1$ be a point in $\mathring{V} - S$ and let $x_2$ be a point in $M_0 - (V \cup S_0)$. We attach the product $S^2 \times [0,1]$ to $M_0 - \mathring{N}(x_1 \cup x_2)$ and obtain a reducible 3-manifold $M$ with the nonseparating sphere $E = S^2 \times \{1/2\}$, where we take $N(x_1 \cup x_2)$ sufficiently small so that it does not meet $\partial V$. Then there exists an embedding $g\colon [-1,1] \times S^1 \times S^1 \to M$ such that $g(\{0\} \times S^1 \times S^1) = \partial V$, $g([-1,1] \times \{0\} \times S^1) \subset S_0$, and $x_1, x_2$ are not contained in the image of $g$, where $S^1 = \mathbb{R}/(\bmod 2)$. Thus, the map $g$ can be thought to be local coordinates of $N(\partial V; M) = \mathrm{Im}(g)$.

There exists a homeomorphism $f\colon M \to M$ that agrees with the identity on $M - \mathring{N}(\partial V; M)$ and such that the map $f \circ g\colon [-1,1] \times S^1 \times S^1 \to M$ is given with $f \circ g(t, \theta_1, \theta_2) = g(t, \theta_1 + t + 1, \theta_2)$. Such a homeomorphism $f$ is sometimes called a *Dehn-twist along* $\partial V$. Put $S_i = f^i(S_0)$. It is easy to see that each $S_i$ is incompressible in $M$. We show that $S_i$ is weakly acylindrical in $M$. Since each $S_i$ can be identified with $S_0$ by the homeomorphism $f\colon M \to M$, it suffices to show that $S_0$ is weakly acylindrical in $M$. Suppose $S_0$ is weakly acylindrical in $M$; then $M - \mathring{N}(S_0)$ contains a strictly essential annulus $A$. By the incompressibility of $A$, we may assume that $A \cap E = \emptyset$. Therefore, $A$ is a properly embedded annulus in $M_0 - \mathring{N}(S_0)$. Since $S_0$ is acylindrical in $M_0$, the annulus $A$ is compressible or $\partial$-compressible in $M_0 - \mathring{N}(S_0)$. However, if $D$ is a compressing or a $\partial$-compressing disk for $A$ in $M_0 - \mathring{N}(S_0)$, then $D$ is still a compressing or a $\partial$-compressing disk for $A$ in $M - \mathring{N}(S_0)$. Hence, $A$ is not strictly essential in $M - \mathring{N}(S_0)$, so $S_0$ is weakly acylindrical in $M$. Because there exists a loop $l$ dual to $E$ such that $l$ intersects $S_i$ with the algebraic intersection number $i$, it follows that the weakly acylindrical surfaces $S_0, \ldots$ are mutually nonisotopic in $M$. □

It is not true that incompressible surfaces in a reducible 3-manifold $M$ are isotoped off the reducing spheres; if we choose a separating incompressible surface $F$ in a 3-manifold and remove one point on each side of $F$, then the surface $F$ is still incompressible and cannot be isotoped to be disjoint from a sphere bounding a twice-punctured ball. It is true that, for the 3-manifold constructed in Proposition 5.2, any incompressible surfaces can be isotoped off the essential sphere; however, it contains infinitely many weakly acylindrical surfaces up to isotopy. Therefore, some result on irreducible 3-manifolds is not inherited by reducible 3-manifolds. Before Lemma 3.4, the condition that $M$ be irreducible is necessary for the result of Floyd and Oertel [1] and for constructing an extended fibered neighborhood $N$ of the incompressible branched surface $B$ in Section 3.

PROPOSITION 5.3. *Each pseudo-acylindrical surface in a Seifert manifold is $\partial$-parallel.*

*Proof.* It is known that any closed incompressible surface in a Seifert manifold is isotopic, either vertical to the dual-Seifert fibration or horizontal [5, Thm. VI.34]. Let $F$ be a closed incompressible surface in a Seifert manifold. If $F$ is horizontal then, by [5, Thm. VI.34], each component of $M - \mathring{N}(F)$ is an $I$-bundle. If $F$ is vertical, then $F$ is $\partial$-parallel or each component of $M - \mathring{N}(F)$ contains vertical essential annuli. □

By Proposition 5.3, our interest in studying acylindrical surfaces is directed to atoroidal 3-manifolds.

PROPOSITION 5.4. *There exists an atoroidal* 3*-manifold $M$ containing infinitely many two-sided surfaces $F$, up to isotopy, with one of the following properties*:

(A) *each essential annulus $A$ in $M - \mathring{N}(F)$ is type $\mathcal{A}_{01}$; or*
(B) *each essential annulus $A$ in $M - \mathring{N}(F)$ is type $\mathcal{A}_0$ or $\mathcal{A}_1$.*

*Proof.* (A) For example, let $M$ be a surface bundle over $S^1$ that is atoroidal and let $\beta_1(M) \geq 2$. Neumann [8] showed that such a 3-manifold contains a nonseparating fiber surface of arbitrarily high genus. If $F$ is a nonseparating fiber surface in $M$, we have $M - \mathring{N}(F) = F \times I$. Consequently, there exists an essential annulus of type $\mathcal{A}_{01}$ and there does not exist an essential annulus of type $\mathcal{A}_0$ and $\mathcal{A}_1$. Hence the conclusion follows.

(B) Let $\tau$ be the train-track indicated on the left-hand side in Figure 5, and let $B_0$ be the 2-complex $\tau \times S^1$. The 2-complex $B_0$ is naturally embedded in $S^3$ and forms a branched surface with the branch loci union of four circles and six sectors, each of which is an annulus. Let $B$ be a branched surface obtained from $B_0$ by attaching a handle to each sector of $B_0$. The branched surface $B$ is still embedded in $S^3$ and has six sectors, each of which is a torus with two disks removed. Let $N$ be a fibered neighborhood of $B$ in $S^3$. Notice that $N$ is a book of $I$-bundles, since $B$ has no "triple points". We cap off $N$ with a 3-manifold in $\mathcal{ABI}_{(3,3,5,5)}$ so that the resulting manifold $M$ is orientable.

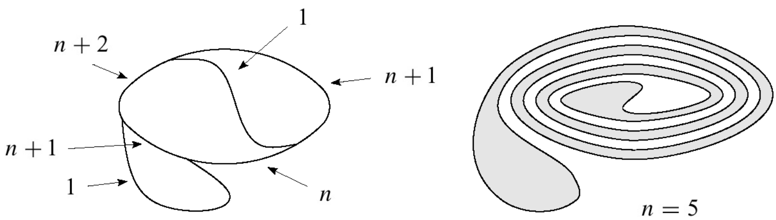



**Figure 5**

**Table 1**

| | $\mathcal{A}_1$ | $\mathcal{A}_0$ | $\mathcal{A}_{01}$ | finiteness | |
|---|---|---|---|---|---|
| pseudo-acylindrical | $\emptyset$ | | $\emptyset$ | Yes | Theorem 1.1 |
| | | $\emptyset$ | $\emptyset$ | Yes | |
| | $\emptyset$ | $\emptyset$ | | No | Proposition 5.4(A) |
| | | | $\emptyset$ | No | Proposition 5.4(B) |

For the book of $I$-bundles $N$, each page has negative Euler characteristic. Thus, by Lemma 4.1, $N$ is irreducible, $\partial$-irreducible, and atoroidal. Hence $M$ is irreducible and atoroidal. It can be seen that condition (I.1) follows because $N$ is irreducible and $\partial$-irreducible by Lemma 4.1. Furthermore conditions (I.2) and (I.3) hold for $N$ in $M$ because $M - \mathring{N}$ is irreducible, $\partial$-irreducible, atoroidal, and anannular. Therefore, $B$ is an incompressible branched surface in $M$. Let $F_n$ be a surface carried by $B$ with the positive weights indicated in Figure 5; an abstract diagram for $n = 5$ is shown on the right-hand side of the figure. Notice that $F_n$ is a connected two-sided surface with genus$(F_n) = 4n + 7$. By [1, Thm. 2], $F_n$ is incompressible in $M$. Furthermore, there exists no essential annulus of type $\mathcal{A}_{01}$, since $M - \mathring{N}$ is anannular and has essential annuli of type $\mathcal{A}_0$ and $\mathcal{A}_1$ in the fibered neighborhood $N$. □

We have Table 1 as our conclusion.

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