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On Certain Loci of Smooth Degree $d \geq 4$ Plane Curves with d-Flexes

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0. Introduction and Notation

Vermeulen [Ve] studied the subvarieties $\mathcal{V}_\alpha \subseteq \mathfrak{M}_3$ (where $\mathfrak{M}_g$ is the moduli space of smooth, genus-g curves over the complex field $\mathbb{C}$) corresponding to plane, smooth quartics C having α hyperflexes—that is, pairs $h := (P, r) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2$ (with $\mathbb{P}^n$ the projective n-space over $\mathbb{C}$) such that $C \cdot r = 4P$.

Vermeulen proved that if $\alpha = 1, 2$ then $\mathcal{V}_\alpha$ is an irreducible subvariety of dimension $6 - \alpha$, and that $\mathcal{V}_3$ is the union of three irreducible components each of whose dimension is 3. He also studied $\mathcal{V}_\alpha$ for $\alpha \geq 4$. Since it follows from the results listed in [Ve] that each component of $\mathcal{V}_\alpha$ is unirational, we obtain that all such components are actually rational when $\alpha \geq 4$ (via Castelnuovo's and Lüroth's theorems, since their dimension is at most 2).

The aim of this paper is to generalize these kind of results by considering smooth, plane curves of degree d having d-flexes, that is, pairs $h := (P, r) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2$ such that $C \cdot r = dP$.

Let $d \geq 4$ and $g := \binom{d-1}{2}$, and denote by $\mathcal{V}_{d,\alpha} \subseteq \mathfrak{M}_g$ the locus of points representing isomorphism classes of smooth, plane curves having α d-flexes. In Section 2 we prove the following theorem.

Theorem A. *The loci $\mathcal{V}_{d,\alpha}$ $(\alpha = 1, 2)$ are irreducible, rational locally closed subvarieties of dimension $\binom{d+2-\alpha}{2} - 8 + 3\alpha$.*

The locus $\mathcal{V}_{4,1}$ has been considered also by Faber [Fa], who proved that the Chow ring $A(\mathfrak{M}_3)$ can be generated by it together with the hyperelliptic locus $\mathcal{H}_3$. Since $\mathcal{H}_3$ is known to be rational (see [BK] and [Ka]; see also [Do] and [PV]), it follows that $A(\mathfrak{M}_3)$ can be generated by rational subvarieties (see [CD] for a similar result about $A(\mathfrak{M}_4)$).

When $\alpha \geq 3$, the locus $\mathcal{V}_{d,\alpha}$ is no longer irreducible. Let $\{h_i\}_{i=1,\dots,\alpha}$ be the set of d-flexes of C. For any triple in this set, one can define a projective invariant $\Lambda_{i,j,k} := \lambda(h_i, h_j, h_k)$ that always satisfies $\Lambda^d_{i,j,k} = 1$ (see Section 1). In order to detect the irreducible components of $\mathcal{V}_{d,\alpha}$, one introduces the $\binom{\alpha}{3}$-tuple

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$\Lambda := (\Lambda_{i,j,k})_{i<j<k}$ and the related subloci $\mathcal{V}_{d,\alpha}^{\Lambda} \subseteq \mathcal{V}_{d,\alpha}$. This is what Vermeulen [Ve] did when dealing with the case $d = 4$.

As d increases, the loci $\mathcal{V}_{d,\alpha}^{\Lambda}$ become too numerous for all to be taken care of (though $\mathcal{V}_{d,\alpha}^{\Lambda} = \mathcal{V}_{d,\alpha}^{\Lambda'}$ for some $\Lambda \neq \Lambda'$, as shown in [Ve] when $\alpha = 3$, $\Lambda = (i)$, and $\Lambda' = (-i)$). Thus we examine here only the cases $\Lambda = (1), (-1), (\pm i)$ for $\alpha = 3$ and $\Lambda = (1, 1, 1, 1)$ for $\alpha = 4$.

In Section 3 we prove the following.

THEOREM B. *If $\Lambda = (1)$ and $d \geq 5$ or if $\Lambda = (-1)$ or $\Lambda = (\pm i)$, then the locus $\mathcal{V}_{d,3}^{\Lambda} \subseteq \mathfrak{M}_g$ is an irreducible, rational locally closed subvariety of dimension $\binom{d-1}{2}$.*

Observe that if $(P_1, r_1), \dots, (P_{d-1}, r_{d-1})$ are pairwise distinct d-flexes of C then the dth residual intersection P_d of C is a d-flex point; that is, there is an $r_d \in \check{\mathbb{P}}^2$ such that (P_d, r_d) is a d-flex (see Section 1). It follows that $\mathcal{V}_{d,3}^{(1)} = \mathcal{V}_{d,4}^{(1,1,1,1)}$.

In Section 4 we prove our final theorem.

THEOREM C. *If $d \neq 5$, then the locus $\mathcal{V}_{d,4}^{(1,1,1,1)} \subseteq \mathfrak{M}_g$ is an irreducible, rational locally closed subvariety of dimension $\binom{d-1}{2} - 2$.*

From Theorems B and C it follows, in particular, that all the components of $\mathcal{V}_{4,3}$ are rational.

To locate our results in a more general setting, let us consider another viewpoint. Let $\mathfrak{M}_{g,\alpha}$ be the moduli space of α-pointed smooth curves of genus $g \geq 3$, and fix $n^{(1)}, \dots, n^{(\alpha)} \in \mathbb{N}^g$.

Eisenbud and Harris [EH] define the locus $\mathcal{C}_{n^{(1)},\dots,n^{(\alpha)}} \subseteq \mathfrak{M}_{g,\alpha}$ of isomorphism classes $[C, P_1, \dots, P_\alpha]$ of α-pointed curves $(C, P_1, \dots, P_\alpha)$ such that the Weierstrass gap sequence of P_j is $n(P_j) = n^{(j)}$, and they then pose the problem of its description.

When $g = \binom{d-1}{2} \geq 3$ and the first two nongaps at each P_i are $d-1$ and d, it is known (see Lemma 1.5) that $[C, P_1, \dots, P_\alpha] \in \mathcal{C}_{n^{(1)},\dots,n^{(\alpha)}}$ if and only if C can be embedded in $\mathbb{P}^2$ and the images of the points P_i are d-flex points. In particular, one checks that $n(P_i)$ is the complement $n(d)$ of the semigroup generated by $d-1$ and d and that $\mathcal{V}_{d,\alpha}$ is the projection via the natural forgetful mapping $p \colon \mathfrak{M}_{g,\alpha} \to \mathfrak{M}_g$ of $\mathcal{C}_{n(d),\dots,n(d)}$.

We state our results in terms of d-flexes rather than in terms of Weierstrass points and gaps, because the former way is closer to our methods of proof.

NOTATION. $\mathbb{C}[x, y, z]$ is the ring of polynomials and $\mathbb{C}[x, y, z]_d$ is its vector subspace of degree-d forms. We use $\mathfrak{M}_g$ to denote the coarse moduli space of smooth, projective curves of genus g defined over the complex field $\mathbb{C}$ and use $[C]$ to denote the point in $\mathfrak{M}_g$ representing the isomorphism class of the smooth curve C of genus g. If $g_1, \dots, g_h$ are elements of a certain group G, then $\langle g_1, \dots, g_h \rangle$ denotes the subgroup of G generated by $g_1, \dots, g_h$. We denote by $\cong$ isomorphisms and by $\approx$ birational equivalences.

For all other definitions, results, and notation, we refer to [Ha].

1. Preliminary Results

For all results about d-flexes, we refer to [Ve]. Since this work has not (to our knowledge) been published in any journal, for the reader's benefit we summarize here the definitions and some of the statements that we shall need in the sequel.

Let $C \subseteq \mathbb{P}^2$ be a curve (i.e., a divisor in $\mathbb{P}^2$) of degree $d \geq 1$. A pair $(P, r) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2$ is a d-flex of C if $C \cdot r = dP$. We call P a d-flex point and r a d-flex tangent line.

Notice that, if (P_1, r_1) and (P_2, r_2) are d-flexes on a smooth curve C, then $P_1 \notin r_2$ and $P_2 \notin r_1$. We set

$$T := \{ (P, r) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 \mid P \in r \},$$

$$B_n := \{ (h_1, \ldots, h_n) \in T^n \mid P_i \notin r_j \text{ if } i \neq j \text{ and } P_i, P_j, P_k \text{ are collinear if } r_i, r_j, r_k \text{ are concurrent} \}.$$

If $h_1, \ldots, h_n \in T^n$ are distinct d-flexes on a curve C of degree d with $P_1, \ldots, P_n$ smooth, then $(h_1, \ldots, h_n) \in B_n$ [Ve, Lemma II.2.8].

We define a morphism $\lambda \colon B_3 \to \mathbb{C}^*$ as follows. If $h_i := (P_i, \{\ell_i(x, y, z) = 0\})$, $i = 1, 2, 3$, for a suitable linear form, then we set

$$\lambda(h_1, h_2, h_3) := -\frac{\ell_2(P_3)\ell_1(P_2)\ell_3(P_1)}{\ell_3(P_2)\ell_2(P_1)\ell_1(P_3)}$$

(see [Ve, formula 2.9]); $\lambda(h_1, h_2, h_3)$ is a projective invariant known as the *λ-invariant* of (h_1, h_2, h_3). Let

$$B_{n,d} := \{ (h_1, \ldots, h_n) \in B_n \mid \lambda(h_i, h_j, h_k)^d = 1 \text{ for all distinct } i, j, k \}.$$

PROPOSITION 1.1. *Let $(h_1, \ldots, h_n) \in B_n$ be given. There exists a curve of degree d for which $h_1, \ldots, h_n$ are d-flexes if and only if $(h_1, \ldots, h_n) \in B_{n,d}$.*

Proof. See [Ve, II.2.12]. □

COROLLARY 1.2. *Let C be a curve of degree d and let $P_1, \ldots, P_d \in C$ be d distinct collinear points. If $P_1, \ldots, P_{d-1}$ are d-flex points, then P_d is also a d-flex point.*

Proof. See [Ve, II.2.17]. □

For each $i, j, k \in \{1, \ldots, \alpha\}$, let $L_{i,j,k} := \lambda(P_i, P_j, P_k)$ (we're not concerned about the order of i, j, k). Then the point $L := (L_{i,j,k})_{i,j,k} \in (\mathbb{C}^*)^{\alpha(\alpha-1)(\alpha-2)}$ is a point whose coordinates satisfy the set of equations

$$\begin{aligned} L_{i,j,k} &= L^{\operatorname{sgn}(\sigma)}_{\sigma(i),\sigma(j),\sigma(k)}, \quad \sigma \in \mathfrak{S}_\alpha, \\ L_{i,j,k}L_{k,l,i} &= L_{j,k,l}L_{l,i,j}, \quad i, j, k, l \in \{1, \ldots, \alpha\} \end{aligned} \tag{1.3}$$

(see [Ve, II.3.2 and II.3.3]). In particular, the $\binom{\alpha}{2}$-tuple Λ defined in Section 0 must satisfy a set of equations arising from (1.3).

Let $(h_1,\dots,h_n)\in B_{n,d}$. We identify $\mathbb{P}(\mathbb{C}[x,y,z]_d)$ with the projective space $\mathbb{P}^{\binom{d+2}{2}-1}$. Under this identification, we denote by $\mathbb{P}(h_1,\dots,h_n)_d$ the subset consisting of forms representing integral curves of degree d carrying $h_1,\dots,h_n$ as d-flexes.

Notice that $\overline{\mathbb{P}(h_1,\dots,h_n)_d}$ is a projective space. In order to compute its dimension, let $h_i := (P_i,\{\ell_i(x,y,z)=0\})$ for $i=1,\dots,n$. Let $C := \{f(x,y,z)=0\}\in\mathbb{P}(h_1,\dots,h_n)_d$, $D := \{g(x,y,z)=0\}\in\overline{\mathbb{P}(h_1,\dots,h_n)_d}$, and $E := \{e(x,y,z) := \prod_{i=1}^n \ell_i(x,y,z)=0\}$. Notice that $C\cdot E=\sum_{i=1}^n dP_i$. We claim that $g\mathcal{O}_{\mathbb{P}^2,P_i}\subseteq (f,e)\mathcal{O}_{\mathbb{P}^2,P_i}$, $i=1,\dots,n$.

Indeed, in affine coordinates x,y we can assume that $P_1=(0,0)$ and $\ell_1=x$, so that $f(x,y,1)=xf_1(x,y)+y^d$ and $g(x,y,1)=xg_1(x,y)+g_2y^d$. Since $\ell_i(P_1)\neq 0$ when $i=2,\dots,n$, one has $(f,e)\mathcal{O}_{\mathbb{P}^2,P_i}=(y^d,x)\mathcal{O}_{\mathbb{P}^2,P_i}$.

It follows from the Max Noether theorem that $g\in(f,e)$; that is,

$$g=af+be=af+b\prod_{i=1}^n \ell_i, \tag{1.4}$$

where $a\in\mathbb{C}$ and $b\in\mathbb{C}[x,y,z]_{d-n}$. In particular, $\overline{\mathbb{P}(h_1,\dots,h_n)_d}$ is obtained by joining the point represented by f with the space $\mathbb{P}(e\mathbb{C}[x,y,z]_{d-n})$. Hence,

$$\dim(\mathbb{P}(h_1,\dots,h_n)_d)=\binom{d+2-n}{2}$$

(see Lemma II.2.15 of [Ve] and its proof).

We conclude this section with the following result, which was stated in Section 0.

LEMMA 1.5. *Fix $d\ge 4$ and $g:=\binom{d-1}{2}$. Let C be a smooth curve of genus g and let $P\in C$. Then the first nongaps of P are $d-1$ and d if and only if C can be embedded in $\mathbb{P}^2$ in such a way that P becomes a d-flex point.*

Proof. Let C be a plane smooth curve with a d-flex point P. In the proof of Lemma 1.1 of [CK] it is shown that $n(P)=n(d)$, where $n(d)$ is the complement in $\mathbb{N}^g$ of the semigroup generated by $d-1$ and d. For instance, $n(4)=(1,2,5)$, $n(5)=(1,2,3,6,7,11)$, and $n(6)=(1,2,3,4,7,8,9,13,14,19)$.

Conversely, let $D:=|dP|$ and $E:=|(d-1)P|$. Then it is easy to check that $\dim(E)=1$ and $\dim(D)=2$, hence D has no fixed points. The linear system D induces a finite map $\varphi\colon C\to C'\subseteq\mathbb{P}^2$ of degree n onto a curve of degree $\delta:=d/n\ge 2$. On the other hand, the linear system E induces a second finite map $\psi\colon C\to\mathbb{P}^1$ of degree $d-1$. Let $n\ge 2$; then n and $d-1$ are coprime and thus ψ cannot be composed with φ. Now we apply Castelnuovo's inequality (see [Ac, Thm. 3.5]) and obtain

$$\binom{d-1}{2}\le n\binom{\delta-1}{2}+(d-2)(n-1).$$

Taking into account that $n\delta=d$, simple computations yield $nd-2d+2\le 0$, which is absurd because $\delta\ge 2$. It follows that $n=1$ and that C is a nonsingular model of C' and hence the geometric genus of C is $g = \binom{d-1}{2}$. We conclude that C' is nonsingular; thus φ is an embedding. Moreover, dP is cut out on C by the lines in $\mathbb{P}^2$. □

2. $\alpha = 1, 2$ d-Flexes

2.1. $\alpha = 1$

Let C be a plane irreducible curve of degree d carrying one d-flex, that is, a nonsingular point P_1 whose tangent line r_1 intersects C only at the point P_1. We briefly write $h_1 := (P_1, r_1)$ in order to denote such a d-flex.

Up to a proper choice of the coordinates in $\mathbb{P}^2$, we can assume that $P_1 := [1, 0, 0]$ and $r_1 = \{z = 0\}$. Then the equation for the curve C must be of the form

$$f(x, y, z) = ay^d + z\psi(x, y, z), \tag{2.1.1}$$

where $a \in \mathbb{C}^*$ and $\psi \in \mathbb{C}[x, y, z]_{d-1}$. Let $V := \{ f \in \mathbb{C}[x, y, z]_d$ as in (2.1.1) $\}$. This is a vector space, and we now study the subgroup G of GL_3 sending V to itself.

LEMMA 2.1.2. *If $d \geq 4$ then the general $f \in V$ represents a smooth curve with exactly one d-flex.*

Proof. Computing dimensions yields $\mathbb{P}(V) = \overline{\mathbb{P}(h_1)_d}$. We must check that

$$\mathbb{P}(h_1)_d \setminus \bigcup_{h_2 \in T} \mathbb{P}(h_1, h_2)_d \neq \emptyset.$$

Indeed, the scheme $\bigcup_{h_2 \in T} \mathbb{P}(h_1, h_2)_d$ has a map φ onto T that is clearly surjective (since $B_2 = B_{2,d}$); hence, if $d \geq 4$ we have

$$\binom{d+1}{2} = \dim(\mathbb{P}(h_1)_d) > \dim\left(\bigcup_{h_2 \in T} \mathbb{P}(h_1, h_2)_d\right) = 3 + \binom{d}{2}.$$

Thus the subset $V_0 \subseteq V$ representing curves with exactly one d-flex is open and nonempty.

Finally, notice that $f(x, y, z) = y^d + xz(x - z)(x^{d-3} + z^{d-3})$ represents a smooth curve having the d-flex h_1; thus the subset $V_1 \subseteq V$ representing smooth curves is open and nonempty. Therefore, each $f \in V' := V_0 \cap V_1 \neq \emptyset$ satisfies the required conditions. □

REMARK 2.1.3. The condition $d \geq 4$ is sharp. Indeed, each conic contains infinitely many 2-flexes, whereas it is well known that each (possibly singular) cubic with two 3-flexes necessarily contains another flex.

If $g \in G$, then g must fix both P_1 and r_1. Thus

$$G := \left\{ \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} \\ 0 & \alpha_{1,1} & \alpha_{1,2} \\ 0 & 0 & \alpha_{2,2} \end{pmatrix} \right\}.$$

PROPOSITION 2.1.4. *Let $g := \binom{d-1}{2}$, $d \geq 4$. The locus $\mathcal{V}_{d,1} \subseteq \mathfrak{M}_g$ (whose general points represent plane curves of degree d with exactly one d-flex) is irreducible and rational of dimension $\binom{d+1}{2} - 5$.*

Proof. We have a natural rational map $\Psi: V \dashrightarrow \mathfrak{M}_g$ sending f to the class of the curve $C := \{f(x, y, z) = 0\}$ and defined on the open subset V' (see the preceding proof of Lemma 2.1.2). In particular, V' dominates $\mathcal{V}_{d,1}$, which then turns out to be irreducible. Clearly $Gf \subseteq \Psi^{-1}([C])$.

Conversely, let $C' := \{f'(x, y, z) = 0\} \in \operatorname{im}(\Psi)$ and assume the existence of $\phi: C \xrightarrow{\sim} C'$. Since C and C' are smooth, each of them carries exactly one very ample g_d^2 (see [ACGH])—say, $|D|$ and $|D'|$ (respectively). It follows that $\phi^*|D'| = |D|$; hence we have $\Phi \in \mathrm{GL}_3$ sending f to f' and thus $\Phi \in G$, that is, $f' \in Gf$. Then $\mathcal{V}_{d,1} := \operatorname{im}(\Psi) \approx V/G$ turns out to be rational, since G is triangular (see [Mi]; see also [Vi; Do; PV]).

Let $f(x, y, z) := y^d + xz(x - z)(x^{d-3} + z^{d-3})$. It represents a smooth curve C and so $\operatorname{Aut}(C)$ is finite. Let g be in the stabilizer G_f of f in G. If its restriction to C is the identity, then g fixes four points on C in general position and thus g is scalar. It follows that the restriction map $G_f \to \operatorname{Aut}(C)$ is injective, hence G_f must be finite. The statement then follows by a parameters computation. □

Since the general point of $\mathcal{V}_{d,1}$ represents a smooth plane curve of degree d with exactly one d-flex and no other, then the restriction to $\mathcal{C}_{n(d)}$ of the natural forgetful map $p: \mathfrak{M}_{g,1} \to \mathfrak{M}_g$ induces a birational equivalence $\mathcal{C}_{n(d)} \approx \mathcal{V}_{d,1}$. We thus conclude as follows.

COROLLARY 2.1.5. *Let $g := \binom{d-1}{2}$, $d \geq 4$. The locus $\mathcal{C}_{n(d)} \subseteq \mathfrak{M}_{g,1}$ is rational of dimension $\binom{d+1}{2} - 5$.*

2.2. $\alpha = 2$

Let C be a plane irreducible curve of degree d carrying two d-flexes, $h_1 := (P_1, r_1)$ and $h_2 := (P_2, r_2)$.

Up to a proper choice of the coordinates in $\mathbb{P}^2$, we can assume that $P_1 := [1, 0, 0]$, $P_2 := [0, 0, 1]$, $r_1 = \{z = 0\}$, and $r_2 := \{x = 0\}$. Then the equation for the curve C must be of the form

$$f(x, y, z) = ay^d + xz\psi(x, y, z), \tag{2.2.1}$$

where $a \in \mathbb{C}^*$ and $\psi \in \mathbb{C}[x, y, z]_{d-2}$. Let $V := \{ f \in \mathbb{C}[x, y, z]_d \text{ as in (2.2.1)} \}$. We have to deal with its stabilizer $G \subseteq \mathrm{GL}_3$.

LEMMA 2.2.2. *If $d \geq 4$, then the general $f \in V$ represents a smooth curve with exactly two d-flexes.*

Proof. Again one has $\mathbb{P}(V) = \overline{\mathbb{P}(h_1, h_2)_d}$. We check that

$$\mathbb{P}(h_1, h_2)_d \setminus \bigcup_{h_3 \in T_0} \mathbb{P}(h_1, h_2, h_3)_d \neq \emptyset,$$

where $T_0 := \{ h_3 \in T \mid (h_1, h_2, h_3) \in B_{3,d} \}$. Indeed, if $h_3 = ([x_0, y_0, z_0], \{ax + by + cz = 0\})$, then $x_0^d a^d - (-1)^d z_0^d c^d = 0$. Thus $\dim(T_0) \leq 2$ and hence, if $d \geq 4$, we have

$$\binom{d}{2} = \dim(\mathbb{P}(h_1, h_2)_d) > \dim\left(\bigcup_{h_3 \in T_0} \mathbb{P}(h_1, h_2, h_3)_d \right) = 2 + \binom{d-1}{2}.$$

Thus the subset $V_0 \subseteq V$ representing curves with exactly two d-flexes is open and nonempty.

Finally, observe that $f(x, y, z) = y^d + xz(x - z)(x^{d-3} + z^{d-3})$ represents a smooth curve having the d-flexes h_1 and h_2. Thus we conclude as in the proof of Lemma 2.1.2. □

REMARK 2.2.3. Again, the condition $d \geq 4$ is sharp.

If $g \in G$, then g must fix the set $\{P_1, P_2\}$ as well as the set $\{r_1, r_2\}$. Thus $G := G_0 \rtimes \langle i \rangle$, where

$$G_0 := \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \right\} \quad \text{and} \quad i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

PROPOSITION 2.2.4. *Let $g := \binom{d-1}{2}$, $d \geq 4$. The locus $\mathcal{V}_{d,2} \subseteq \mathfrak{M}_g$ (whose general points represent plane curves of degree d with exactly two d-flexes) is irreducible and rational of dimension $\binom{d}{2} - 2$.*

Proof. As in the proof of Proposition 2.1.4, $\mathcal{V}_{d,2}$ is irreducible and $V/G \approx \mathcal{V}_{d,2}$.

Let

$$V' := \{ f \in V \mid f(x, y, z) = xz(ax^{d-2} + bx^{d-3}y + cz^{d-3}y + dz^{d-2}) \}.$$

It is easy to check that $V \cong V' \oplus V''$ as a representation of the group G. We will prove the rationality of V/G by proving that V' is an almost free representation of G; because the quotient V'/G has dimension 1 and is unirational, the rationality of V/G will follow from the method of reducible representation (see [Do, Cor. 1] or [PV, Thm. 2.13]).

Let

$$g := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

stabilize $f(x, y, z) := xz(ax^{d-2} + bx^{d-3}y + cz^{d-3}y + dz^{d-2})$. Then

$$\begin{aligned} \alpha^{d-1}\gamma &= 1, \\ \alpha^{d-2}\beta\gamma &= 1, \\ \alpha\gamma^{d-1} &= 1, \\ \alpha\beta\gamma^{d-2} &= 1 \end{aligned}$$

and hence $\alpha/\beta = \gamma/\beta = 1$; that is, g is a scalar matrix. By substitution we also get that $\beta^d = 1$. Thus the intersection of G_f (the stabilizer of f) with the component G_0 acts trivially on V (it is also contained in the center of G and thus is a normal subgroup).

Now let $g \in G_f \cap G_0 i$. Then

$$\begin{aligned} a\alpha^{d-1}\gamma &= d, \\ b\alpha^{d-2}\beta\gamma &= c, \\ d\alpha\gamma^{d-1} &= a, \\ c\alpha\beta\gamma^{d-2} &= b; \end{aligned}$$

hence

$$\frac{\alpha}{\beta} = \frac{bd}{ac} = \frac{\beta}{\gamma} \quad \text{and} \quad \left(\frac{\alpha}{\gamma}\right)^{d-2} = \frac{d^2}{a^2}.$$

We then conclude that

$$\frac{d^2}{a^2} = \left(\frac{bd}{ac}\right)^{2(d-2)},$$

which is a nontrivial relation among the coefficients of f. Thus, for f general enough, we have $G_f = G_f \cap G_0$.

For the statement on the dimension we repeat the argument of Lemma 2.1.2. □

We conclude this section with the following.

Proof of Theorem A. Propositions 2.1.4 and 2.2.4 yield Theorem A. □

3. $\alpha = 3$ d-Flexes

3.1. $\lambda = 1$ *and* $d \geq 5$

Let C be a plane irreducible curve of degree d carrying three collinear d-flexes: $h_1 := (P_1, r_1)$, $h_2 := (P_2, r_2)$, and $h_3 := (P_3, r_3)$.

Since these d-flexes lie on a curve of degree d, their λ-invariant must satisfy $\lambda(h_1, h_2, h_3)^d = 1$ (see Proposition 1.1). Thus we have the particular case $\lambda(h_1, h_2, h_3) = 1$, which (by the same proposition) means that P_1, P_2, P_3 are collinear.

Hence, up to a proper choice of the coordinates in $\mathbb{P}^2$, we can assume that $P_1 := [1, 0, 0]$, $P_2 := [0, 0, 1]$, and $P_3 := [1, 0, 1]$ as well as $r_1 = \{z = 0\}$, $r_2 := \{x = 0\}$, and $r_3 := \{ax + by + cz = 0\}$. We have

$$1 = \lambda(h_1, h_2, h_3) = -c/a$$

and $a + c = 0$, so $r_3 = \{a(x - z) + by = 0\}$ where $a \neq 0$. Since $b = 0$ if and only if the lines r_1, r_2, r_3 are concurrent, we will assume from now on that $b \neq 0$.

The projectivity associated to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a/b & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

allows us to assume that $r_3 = \{x + y - z = 0\}$. Thus, the equation for C must be of the form

$$f(x, y, z) = ay^d + xz(x + y - z)\psi(x, y, z), \tag{3.1.1}$$

where $a \in \mathbb{C}^*$ and $\psi \in \mathbb{C}[x, y, z]_{d-3}$. Let $V^{(1)} := \{ f \in \mathbb{C}[x, y, z]_d$ as in (3.1.1) $\}$. It is a vector space, and we study its stabilizer $G \subseteq \mathrm{GL}_3$.

LEMMA 3.1.2. *If $d \geq 5$, then the general $f \in V^{(1)}$ represents a smooth curve with exactly three collinear d-flexes.*

Proof. As in the proof of Lemma 2.2.2, we define

$$T_0 := \{ h_4 \in T \mid (h_1, h_2, h_3, h_4) \in B_{4,d} \}.$$

We have $\dim(T_0) \leq 2$ and hence, if $d \geq 5$,

$$\begin{aligned} \binom{d-1}{2} &= \dim(\mathbb{P}(h_1, h_2, h_3)_d) > \dim\Bigg(\bigcup_{h_4 \in T_0} \mathbb{P}(h_1, h_2, h_3, h_4)_d \Bigg) \\ &= 2 + \binom{d-2}{2}, \end{aligned}$$

whence $\mathbb{P}(h_1, h_2, h_3)_d \setminus \bigcup_{h_4 \in T_0} \mathbb{P}(h_1, h_2, h_3, h_4)_d \neq \emptyset$.

Consider the pencil of curves $C_{[\alpha,\beta]} := \{\alpha(y^d + xz(x - z)(x^{d-3} + z^{d-3})) + \beta xyz(x^{d-3} + z^{d-3}) = 0\}$. Since the curve $C_{[1,0]}$ is smooth, it follows that, for general $[1, \beta] \in \mathbb{P}^1$, the same is true for $C_{[\alpha,\beta]}$. Because each such curve is projectively isomorphic in $\mathbb{P}^2$ to a curve whose equation lies in $V^{(1)}$, the statement can be proved using the same argument of Lemmas 2.1.2 and 2.2.2. □

REMARK 3.1.3. Corollary 1.2 implies $d \geq 5$, so Lemma 3.1.2 is sharp.

If $g \in G$, then g must fix the line $r_0 := \{y = 0\}$ and the set $\{r_1, r_2, r_3\}$. In particular, the restriction of its action to r_0 coincides with the action of $\mathfrak{S}_3$ on the set $\{P_1, P_2, P_3\}$. With some easy computation one checks that $G = G_0 \times \mathfrak{S}_3$, where $G_0 \subseteq \mathrm{GL}_3$ is the torus of scalar matrices and

$$\mathfrak{S}_3 = \left\langle \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle.$$

LEMMA 3.1.4. *Let $\mathfrak{S}_3$ act linearly on the vector space $V^{(1)}$. Then $V^{(1)}/\mathfrak{S}_3$ is rational.*

Proof. Let $\varrho\colon \mathfrak{S}_3 \to \mathrm{GL}(V^{(1)})$ be any representation. If ϱ is not faithful, then it is a linear representation of $\mathbb{Z}_2 \trianglelefteq \mathfrak{S}_3$.

It is known (see [Fi]) that each linear representation $\tau\colon K \to \mathrm{GL}(U)$ of a finite abelian group K has rational quotient U/K. Thus $V^{(1)}/\mathfrak{S}_3$ is rational if ϱ is not faithful.

If ϱ is faithful then $V^{(1)} = V' \oplus V''$, where V' is the irreducible representation of degree 2. Since $V'/\mathfrak{S}_3$ is unirational of dimension 2, a theorem of Castelnuovo yields its rationality, whence we get the rationality of $V^{(1)}/\mathfrak{S}_3$ by the method of reducible representation (see [Do, Cor. 1] or [PV, Thm. 2.13]). □

PROPOSITION 3.1.5. *Let $g := \binom{d-1}{2}$, $d \ge 5$. The locus $\mathcal{V}_{d,3}^{(1)} \subseteq \mathfrak{M}_g$ (whose points represent plane curves of degree d with exactly three collinear d-flexes) is irreducible, and it is rational of dimension $\binom{d-1}{2}$.*

Proof. As in the proof of Proposition 2.1.4, $\mathcal{V}_{d,3}^{(1)}$ is irreducible and $V^{(1)}/G \approx \mathcal{V}_{d,3}^{(1)}$.

We have $\mathbb{C}(V^{(1)})^G = (\mathbb{C}(\mathbb{P}(V^{(1)})))^{\mathfrak{S}_3} = \mathbb{C}(\mathcal{U})^{\mathfrak{S}_3}$, where

$$\mathcal{U} := \{\text{ classes of } f \text{ as in (3.1.1) with } a \neq 0\,\}.$$

Via the isomorphism

$$\Psi\colon \mathcal{U} \to \mathbb{C}[x,y,z]_{d-3},$$
$$\text{class of } [y^d + xz(x+y-z)\psi(x,y,z)] \to \psi(x,y,z),$$

we obtain a linear action of $\mathfrak{S}_3$ onto $\mathbb{C}[x,y,z]_{d-3}$ and $V^{(1)}/G \approx \mathcal{U}/\mathfrak{S}_3 \approx \mathbb{C}[x,y,z]_{d-3}/\mathfrak{S}_3$, whose rationality follows from Lemma 3.1.4.

For the dimension, we repeat the argument of Lemma 2.1.2 with $f(x,y,z) := y^d + xz(x+2y-z)(x^{d-3}+z^{d-3})$. □

3.2. $\lambda = -1$

We now consider the other easy case, when the three d-flexes $h_1 := (P_1, r_1)$, $h_2 := (P_2, r_2)$, and $h_3 := (P_3, r_3)$ are such that $\lambda(h_1,h_2,h_3) = -1$ (hence d must be even). By Proposition 1.1, it follows that they must lie on an integral conic D.

Up to a proper choice of the coordinates in $\mathbb{P}^2$, we can assume that $P_1 := [1,0,0]$, $P_2 := [0,0,1]$, and $P_3 := [1,1,1]$ and that $r_1 = \{z = 0\}$, $r_2 := \{x = 0\}$, and $r_3 := \{ax + by + cz = 0\}$. We have

$$-1 = \lambda(h_1,h_2,h_3) = -c/a$$

and $a+b+c = 0$, thus $r_3 = \{x - 2y + z = 0\}$. The unique conic having h_1, h_2, h_3 as 2-flexes is $D := \{y^2 - xz = 0\}$. Hence, the equation for C must be of the form

$$f(x,y,z) = a(y^2 - xz)^{d/2} + xz(x-2y+z)\psi(x,y,z) \tag{3.2.1}$$

(see (1.4)), where $a \in \mathbb{C}^*$ and $\psi \in \mathbb{C}[x,y,z]_{d-3}$. Let $V^{(-1)} := \{\, f \in \mathbb{C}[x,y,z]_d$ as in (3.2.1) $\}$. This is a vector space, and we can consider its stabilizer $G \subseteq \mathrm{GL}_3$.

LEMMA 3.2.2. *If $d \ge 4$ is even, then the general $f \in V^{(-1)}$ represents a smooth curve with exactly three d-flexes on a conic.*

Proof. If $d \ge 6$ then we use the argument of Lemmas 2.1.2, 2.2.2, and 3.1.2. In any case, we have a natural rational map $\Psi\colon V^{(-1)} \dashrightarrow \mathfrak{M}_g$ and $\mathrm{im}(\Psi) = \mathcal{V}_{d,3}^{(-1)}$ (see the proof of Proposition 2.1.4). If $d = 4$, Vermeulen showed that $\dim(\mathcal{V}_{d,3}^{(-1)}) = 3$, whereas the sublocus of points representing curves with at least four 4-flexes has dimension 2 (see [Ve, Props. II.10.11, II.12.5, II.12.9). Then, for each even $d \geq 4$, the general $f \in V^{(-1)}$ represents a curve with exactly three d-flexes.

Now consider the pencil of curves

$$C_{[\alpha,\beta]} := \{\alpha((y^2 - xz)^{d/2} + xz(x+z)(x^{d-3} - z^{d-3})) \\ + 2\beta xyz(x^{d-3} - z^{d-3}) = 0\},$$

and imitate the argument of Lemma 3.1.2. □

If $g \in G$ then g must fix D. Since D is an irreducible conic, it is isomorphic to $\mathbb{P}^1$; thus $G = G_0 \times \mathfrak{S}_3$, where G_0 is the torus of scalar matrices in GL_3. Explicitly, since the set $\{h_1, h_2, h_3\}$ must remain fixed, we have

$$\mathfrak{S}_3 = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \right\rangle.$$

PROPOSITION 3.2.3. *Let $g := \binom{d-1}{2}$, $d \geq 4$ even. The locus $\mathcal{V}_{d,3}^{(-1)} \subseteq \mathfrak{M}_g$ (whose points represent plane curves of degree d with exactly three d-flexes on a smooth conic) is irreducible and rational of dimension $\binom{d-1}{2}$.*

Proof. The argument is the same as that for the proof of Proposition 3.1.5. □

3.3. $\lambda = \pm i$

Finally, consider the case when the three d-flexes $h_1 := (P_1, r_1)$, $h_2 := (P_2, r_2)$, and $h_3 := (P_3, r_3)$ satisfy $\lambda(h_1, h_2, h_3) = \pm i$ (hence d is divisible by 4). If this is the case then they must lie on a quartic D but not on a conic (see Proposition 1.1).

Notice that $\lambda(h_1, h_2, h_3) = \lambda(h_1, h_3, h_2)^{-1}$, so the two cases give rise to the same curves. If $\lambda(h_1, h_2, h_3) = i$ then, up to projective isomorphisms, we can take $P_1 := [1, 0, 0]$, $P_2 := [0, 0, 1]$, and $P_3 := [1, 1, 1]$ as well as $r_1 := \{z = 0\}$, $r_2 := \{x = 0\}$, and $r_3 := \{(1+i)x - 2y + (1-i)z = 0\}$.

The curve $D := \{y^4 - xz(2y - x)(2y - z) = 0\} \in \mathbb{P}(h_1, h_2, h_3)_4$ and hence the equation for $C \in \mathbb{P}(h_1, h_2, h_3)_d$ must be of the form

$$f(x, y, z) = a(y^4 - xz(2y - x)(2y - z))^{d/4} \\ + xz((1+i)x - 2y + (1-i)z)\psi(x, y, z) \qquad (3.3.1)$$

(see (1.4)), where $a \in \mathbb{C}^*$ and $\psi \in \mathbb{C}[x, y, z]_{d-3}$. Let $V^{(\pm i)} := \{ f \in \mathbb{C}[x, y, z]_d$ as in (3.3.1) $\}$ and let $G \subseteq \mathrm{GL}_3$ be its stabilizer.

LEMMA 3.3.2. *If $d \geq 4$ is divisible by 4, then the general $f \in V^{(\pm i)}$ represents a smooth curve with exactly three d-flexes on a quartic (but not on any conic).*

Proof. If $d \geq 6$ then we use the argument of Lemmas 2.1.2, 2.2.2, and 3.1.2. We have a natural rational map $\Psi \colon V^{(\pm i)} \dashrightarrow \mathfrak{M}_g$ and $\mathrm{im}(\Psi) = \mathcal{V}_{d,3}^{(\pm i)}$. Again, if $d = 4$ then $\dim(\mathcal{V}_{d,3}^{(\pm i)}) = 3$, whereas the sublocus of points representing curves with at least four 4-flexes has dimension 2 (see [Ve, Props. II.10.16, II.11.4, II.12.9, II.13.4).

Consider the pencil of curves $C_{[\alpha,\beta]} := \{\alpha((y^4 - xz(2y-x)(2y-z))^{d/4} + xz(x-z)(x^{d-3}+z^{d-3})) + 2\beta x(ix-2y+iz)(x^{d-3}+z^{d-3}) = 0\}$ and imitate the argument of Lemma 3.1.2. □

Every $g \in G$ must fix the set $\{h_1, h_2, h_3\}$. Hence $G \subseteq G_0 \times \mathfrak{S}_3$, where $G_0 \subseteq \mathrm{GL}_3$ is the torus of scalar matrices. Explicitly,

$$\mathfrak{S}_3 = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1+i & 0 & 0 \\ 1+i & -1-i & 0 \\ 1+i & -2 & 1-i \end{pmatrix} \right\rangle.$$

PROPOSITION 3.3.3. *Let $g := \binom{d-1}{2}$ with $d \geq 4$ divisible by 4. The locus $\mathcal{V}_{d,3}^{(\pm i)} \subseteq \mathfrak{M}_g$ (whose points represent plane curves of degree d with exactly three d-flexes on a quartic, but not on any conic) is irreducible and rational of dimension $\binom{d-1}{2}$.*

Proof. Again, the argument is the same as that for the proof of Proposition 3.1.5. □

Proof of Theorem B. Theorem B now follows from Propositions 3.1.5, 3.2.3, and 3.3.3. □

4. Four Collinear *d*-Flexes

Let C be a plane irreducible curve of degree d carrying four collinear 4-flexes: $h_1 := (P_1, r_1)$, $h_2 := (P_2, r_2)$, $h_3 := (P_3, r_3)$, and $h_4 := (P_4, r_4)$. Since these d-flexes lie on a line $r := \{\ell = 0\}$, the equation for C must be of the form

$$f(x,y,z) = \ell^d + \varphi(x,y,z)\psi(x,y,z), \tag{4.1}$$

where $\varphi, \psi \in \mathbb{C}[x,y,z]$ are forms of respective degrees 4 and $d-4$ and where φ can be factorized in linear forms.

Consider the Segre map $s\colon H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))^{\otimes 4} \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-4)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ and let $X := \mathrm{im}(s)$. Since the projectivization of s corresponds to the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{\binom{d-2}{2}-1} \hookrightarrow \mathbb{P}^{\binom{d+2}{2}-1}$, it follows that $\dim(X) = \binom{d-2}{2} + 8$. We must therefore consider

$$V := \{\, f \in \mathbb{C}[x,y,z]_d \text{ as in (4.1)} \,\} \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \times X,$$

and the action of GL_3 on V is induced by the natural action of GL_3 on

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \times H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)).$$

The subvariety $W \subseteq V$ defined by

$$\begin{aligned} W &:= \{\, f \in V \mid f(x,y,z) = \ell^d + xyz(x+y+z)\psi(x,y,z) \,\} \\ &\cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \times H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-4)) \end{aligned}$$

is GL_3-dense in V. We now consider the subgroup G of GL_3 sending W to itself.

LEMMA 4.2. *If $d \geq 4$ and $d \neq 5$, then the general $f \in V$ represents a smooth curve with exactly three d-flexes.*

Proof. If $d \geq 6$ then we use the argument of Lemmas 2.1.2, 2.2.2, or 3.1.2. The case $d = 4$ follows from Propositions II.10.4 and II.11.4 of [Ve] (imitate the proofs of Propositions 3.2.2 and 3.3.2). $\square$

REMARK 4.3. Lemma 4.2 is sharp, again by Corollary 1.2.

If $d \geq 4$ and $d \neq 5$, then we have two actions on the variables x, y, z in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ that leave $\mathbb{C}xyz(x + y + z)$ fixed. The first one is the canonical action of the symmetric group $\mathfrak{S}_4 \subseteq \mathrm{GL}_3$ generated by the transposition matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}.$$

The second one is the action of the torus of scalar matrices $G_0 \subseteq \mathrm{GL}_3$. Since the elements of G_0 and $\mathfrak{S}_4$ commute each other, it follows that we have a natural action of $H := G_0 \times \mathfrak{S}_4$ on W.

PROPOSITION 4.4. *If $d \geq 4$ and $d \neq 5$, then $W/H \approx V/G$.*

Proof. It suffices to prove that W is a (G, H)-section of V. To this purpose let $W' \subseteq W$ be the open subset consisting of forms with irreducible ψ. Let $g \in \mathrm{GL}_3$ satisfy $g(W') \subseteq W$. Then g fixes $xyz(x+y+z)$ and so $g = \mu\sigma$, where $\sigma \in \mathfrak{S}_4 \subseteq \mathrm{GL}_3$ and μ is a scalar matrix. $\square$

Proof of Theorem C. We study W/H. Notice that $\mathbb{C}(W)^H = (\mathbb{C}(W)^{\mathfrak{S}_4})^{G_0}$; since $\mathfrak{S}_4$ is finite, $\mathbb{C}(W)^{\mathfrak{S}_4}$ is the field of fractions of the algebra $\mathbb{C}[W]^{\mathfrak{S}_4}$. On the other hand, $\mathfrak{S}_4$ is generated by pseudoreflections and so $\mathbb{C}[W]^{\mathfrak{S}_4} = \mathbb{C}[a, b, c]$ for three suitable homogeneous $\mathfrak{S}_4$-invariants $a, b, c \in \mathbb{C}[W]$ (see [Sp, Thm. 4.2.5]). This implies that $\mathbb{C}(W)^H = \mathbb{C}(a, b, c)^{G_0}$, where the action of G_0 is diagonal; thus, we have $\mathbb{C}(a, b, c)^{G_0} = \mathbb{C}(A, B, C)$ for three suitable G_0-invariants $A, B, C \in \mathbb{C}(a, b, c)$. $\square$

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