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# Valuative Arf Characteristic of Singularities

ANTONIO CAMPILLO & JULIO CASTELLANOS

## 1. Introduction

The proof by Hironaka [5] of resolution of singularities of algebraic varieties over fields of characteristic zero raised the problem of classifying singularities by looking at the resolution process. Thus, equisingularity of plane curve singularities was introduced and developed by Zariski in [15] by showing that the combinatorics of the resolution processes is equivalent data to Puiseux invariants, that is, the numerical data consisting of the Puiseux exponents of the branches and the intersection multiplicities among pairs of branches. For space curves, it is known that the combinatorics of the resolution processes is equivalent data to the Arf characteristic (see [5]). Arf closure lies between the singularity and its normalization, and its definition can be given in terms of the set of valuations centered at the singularity. Notice that, since those valuations correspond one-to-one to branches of the curve, the aforesaid set is finite and is canonically associated to the singularity.

In higher dimension the situation becomes much more complicated, as the resolution of singularities is not unique at all. In this paper we define the Arf characteristic for schemes of arbitrary dimensions. We define Arf closure relative to a finite set of valuations centered at a singularity and study its algebraic-geometric properties. Using appropiate canonically defined sets of divisorial valuations, we define the *valuative Arf characteristic of singularity* (note that it is not related at all to the well-known *Arf invariants* of the $\mathbb{Z}_2$ quadratic form associated topologically with the link of plane curve singularities). These invariants can be viewed as a generalization of Puiseux characteristic to higher dimensions. Arf closure relative to a single divisorial valuation was introduced in [2], showing that the corresponding invariants describe the geometry of certain arcs on the singularity.

There are two natural sets of valuations canonically associated to a singularity. First, one can associate the so-called essential valuations: those valuations that appear explicitly at every resolution. For surface singularities one has a minimal resolution, so that the essential valuations are nothing but the divisorial valuations centered at the components of the exceptional divisor of the minimal resolution. For dimension higher than two, essential valuations are not determined except in a few cases (see e.g. [1]). A result by Nash of 1964 (recently published in [10]) shows that the set of essential valuations contains the set of valuations coming

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from the components of the space of arcs. The Nash conjecture states that both sets are equal, but it is only established for a certain subclass of rational surface singularities containing the minimal ones (see [11]). Thus, in practice, what one has associated to a singularity is the set of valuations given by the components of the space of arcs. For recent work on space of arcs, see [4] and [6].

Second, one can consider the set of Rees valuations of canonically associated ideals to the singularity, such as, for instance, the Jacobian ideal. Rees valuations of an ideal are nothing but the divisorial valuations given by the components of the exceptional divisor of the normalized Nash blow-up of the ideal (the Nash blow-up in the case of the Jacobian ideal). When the singularity is embedded in a smooth space, the Jacobian ideal can be alternatively considered on that smooth space and then the corresponding Arf characteristic can be better interpreted in geometric terms. Thus, for isolated hypersurface singularities we will show how these invariants are finer than the polar invariants of the singularity introduced in [14]. For general singularities one has a good theory of equisingularity due to Teissier [13], which applies to families and which uses as invariants appropriate extensions of the polar invariants.

This paper presents the algebraic-geometric theory of the Arf characteristic relative to finite sets of divisorial valuations. We first introduce the algebraic notion of Arf closure and use it to define the Arf characteristic. Then we show how the invariants can be interpreted in terms of the geometry of sets of arcs transversal to corresponding divisors. For the case of valuations on smooth varieties, Arf invariants can be related to the proximity relations among points obtained by blowing up centers of the valuations under consideration. Finally, for Rees valuations associated to a primary ideal on a smooth variety, we show that the Arf characteristic is related to the geometry of the curves given as a complete intersection of hypersurfaces defined by general elements of the ideal.

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## 2. Arf Closure with Respect to Discrete Valuations

Let $K$ be a commutative field and $\mathbf{v} = \{v_1, \ldots, v_d\}$ a set of pairwise different discrete rank-1 valuations on $K$; let $R_{v_i} = \{x \in K : v_i(x) \geq 0\}$ and $\mathfrak{m}_{v_i} = \{x \in K : v_i(x) > 0\}$ be the corresponding valuation rings and their respective maximal ideals.

Consider $R_{\mathbf{v}} = \bigcap_{i=1}^{d} R_{v_i}$ and a subring $B \subset R_{\mathbf{v}}$ with quotient field $K$. For each $i$ $(1 \leq i \leq d)$, the center of $v_i$ at $\mathrm{Spec}_{\mathbf{v}}(B)$ is defined to be the prime ideal $\mathfrak{p}_i = \mathfrak{m}_{v_i} \cap B \in \mathrm{Spec}(B)$. Thus, associated with the pair $(B, \mathbf{v})$ we have the set $\mathrm{Spec}_{\mathbf{v}} B = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_d\}$, that is, the image of the natural mapping $\mathrm{SpecMax}(R_{\mathbf{v}}) \to \mathrm{Spec}(B)$, which we will called *the valuation spectrum* of $B$. Note that $\#\mathrm{Spec}(B) \leq d$ and that equality holds if and only if the centers in $B$ are different. Also note that $\mathfrak{p}_i$ is not the zero ideal for every $i$, since otherwise one would have $Q(B) = K \subseteq R_{\mathbf{v}}$.

A subring $C \subset R_{\mathbf{v}}$ will be said to be *Arf with respect to* $\mathbf{v}$ if it has the following property:

(A) If $z, z_1, z_2 \in C$ with $z \neq 0$ and if $v_j(z_h) \geq v_j(z)$ for $h = 1, 2$ and $j = 1, 2, \ldots, d$, then $z_1 z_2 z^{-1} \in C$.

For a subring $B \subset R_{\mathbf{v}}$ with quotient field $K$, the *Arf closure of $B$ with respect to* $\mathbf{v}$ is defined to be the smallest Arf subring of $R_{\mathbf{v}}$ containing $B$. Let us denote by $R'_{\mathbf{v}}B$ (or simply $B'$, if the reference to the valuations is obvious) the Arf closure of $B$ with respect to $\mathbf{v}$.

The following result gives us the first properties of $B'$.

THEOREM 2.1. *Given $B$, $B'$ and $\mathbf{v}$ as just described, the following statements hold.*

(i) *The mapping* $\operatorname{Spec}_{\mathbf{v}} B' \to \operatorname{Spec}_{\mathbf{v}} B$ *is bijective. Moreover, if $\mathfrak{p}$ and $\mathfrak{p}'$ are the corresponding elements in* $\operatorname{Spec}_{\mathbf{v}} B$ *and* $\operatorname{Spec}_{\mathbf{v}} B'$, *then* $k(\mathfrak{p}) = k(\mathfrak{p}')$ (*here* $k(-)$) *stands for the residue field*).

(ii) $\operatorname{Spec}_{\mathbf{v}} B'$ *is the maximal spectrum for* $B'$ (*in particular, the topology on* $\operatorname{Spec}_{\mathbf{v}} B'$ *is the discrete one*).

*Proof.* (i) Obviously, the map is onto, so we have to check the injectivity. Take $\mathfrak{p} \in \operatorname{Spec}_{\mathbf{v}} B$ and consider the set $H = \{i : \mathfrak{m}_{v_i} \cap B = \mathfrak{p}\}$. We build $B' = \bigcup_{h=1}^{\infty} B_h$, where $B_0 = B$ and $B_{h+1}$ is the subring of $R_{\mathbf{v}}$ consisting of the sum of elements of type $w = z_1 z_2 z^{-1}$, with $z, z_1, z_2$ as in (A) and belonging to $B_h$. It suffices to show that, for every $h \in \mathbb{Z}_+$, we have $\mathfrak{m}_{v_j} \cap B_h = \mathfrak{m}_{v_i} \cap B_h$ if $j, i \in H$. We prove this by induction on $h$. The statement is obvious for $h = 0$. For the induction step, take $w = w_1 + \cdots + w_r$ (with $w_t$ of the above type) and set $w = w' + w''$, where $w'$ is the sum of those $z_1 z_2 z^{-1}$ such that $v_j(z) = v_j(z_1) = v_j(z_2) = 0$ for some $j \in H$. Now, if $j \in H$ and $v_j(w) > 0$ then $v_j(w') > 0$. By removing denominators in $w'$ we obtain an expression in $B_h$ with $v_j$-value greater than 0, so by the induction hypothesis its $v_i$-value is greater than 0, too. Hence we can insert the denominators and obtain $v_i(w') > 0$. This shows that $\mathfrak{m}_{v_j} \cap B'_{h+1} = \mathfrak{m}_{v_i} \cap B'_{h+1}$ as required.

Using the same kind of induction, it is easy to show that, for corresponding $\mathfrak{p}, \mathfrak{p}'$, both of the domains are related by the natural mapping $B/\mathfrak{p} \to B'/\mathfrak{p}'$ and have the same residue field, so $k(\mathfrak{p}) = k(\mathfrak{p}')$.

(ii) First we consider the multiplicative set $S = B - \bigcap_{i=1}^{d} \mathfrak{p}_i$; then we have $B' = (S^{-1}B)'$. This follows from the fact that if $s \in S$ then $v_i(s) = 0$ for $i = 1, \ldots, d$, so $s^{-1} = 1 \cdot 1 \cdot s^{-1} \in B'$. From the same argument, the units in $B'$ are exactly the elements in $B' - \bigcap_{i=1}^{d} \mathfrak{p}'_i$ with $\mathfrak{p}'_i = \mathfrak{m}_{v_i} \cap B'$, so $B'$ has finitely many maximal ideals—exactly those maximal ideals in the set $\operatorname{Spec}_{\mathbf{v}} B'$. Moreover, let us show that, in fact, each $\mathfrak{p}'_i$ is a maximal ideal. Consider the set $L = \{j : \mathfrak{p}_j \subseteq \mathfrak{p}_i\}$, and take $s \in \left(\bigcap_{j \notin L} \mathfrak{p}_j\right) - \mathfrak{p}_i$ and $y \in \left(\bigcap_{j \in L} \mathfrak{p}_j\right)$ with $y \neq 0$. Note that, by replacing $s$ by some power of $s$, we can assume $v_j(y) \leq v_j(s)$ for $j \notin L$. Thus, for every $j$ we have $v_j(s^2 + y) \leq v_j(s)$, so from (A) we get $y(s^2 + y)^{-1} = 1 - s^2(s^2 + y)^{-1} \in B'$. Now, since $v_i(y(s^2 + y)^{-1}) > 0$ and $v_j(y(s^2 + y)^{-1}) = 0$ if $j \notin L$, we conclude

that $\mathfrak{p}_i' \not\subseteq \mathfrak{p}_j'$ for $j \notin L$; for $j \in L$, on the other hand, $\mathfrak{p}_i' \subseteq \mathfrak{p}_j'$ implies $\mathfrak{p}_i = \mathfrak{p}_j$. Thus, $\mathfrak{p}_i'$ is maximal in $\operatorname{Spec}_{\mathbf{v}} B'$ and hence is a maximal ideal in $B'$. □

REMARK 2.1. The theorem shows that, if we set $\mathfrak{p}_i' = \mathfrak{m}_{v_i} \cap B'$ for $i = 1, \ldots, d$, then $\mathfrak{p}_i = \mathfrak{p}_j$ if and only if $\mathfrak{p}_i' = \mathfrak{p}_j'$, since the residue fields for $\mathfrak{p}_i$ and $\mathfrak{p}_i'$ are the same. On the other hand, if $\mathfrak{p}_i \neq \mathfrak{p}_j$ then $\mathfrak{p}_i' \not\subseteq \mathfrak{p}_j'$ and $\mathfrak{p}_j' \not\subseteq \mathfrak{p}_i'$, although we may have $\mathfrak{p}_i \subset \mathfrak{p}_j$. Thus, one concludes that Arf closure eliminates the specialization relations between the centers.

The preceding remark will allow to us to separate subsets of valuations having the same center. Later, we will see how—after sucessive blowing up of centers—one derives a complete separation of the valuations of the considered set.

Take $\mathfrak{p} \in \operatorname{Spec}_{\mathbf{v}} B$ and consider the set of valuations

$$\mathbf{v}(\mathfrak{p}) = \{v_i : \mathfrak{m}_{v_i} \cap B = \mathfrak{p}\}$$

and the corresponding $R_{\mathbf{v}(\mathfrak{p})}$.

First, we have $B \subset B' \subset R'_{\mathbf{v}(\mathfrak{p})} B$ and $B'_{\mathfrak{p}'} \subset R_{\mathbf{v}(\mathfrak{p})}$. We claim that $B'_{\mathfrak{p}'}$ is Arf relative to $\mathbf{v}(\mathfrak{p})$. From the claim one may conclude that $R'_{\mathbf{v}(\mathfrak{p})} B = R'_{\mathbf{v}(\mathfrak{p})} B_{\mathfrak{p}} \subset B'_{\mathfrak{p}'}$. In fact, since $\mathbf{v}(\mathfrak{p}) = \{v_i : \mathfrak{m}_{v_i} \cap B' = \mathfrak{p}'\}$ for the corresponding $\mathfrak{p}$, if $H = \{i : v_i \in \mathbf{v}(\mathfrak{p})\}$ then we can take $s^* \in \left(\bigcap_{j \notin H} \mathfrak{p}_j'\right) - \mathfrak{p}'$. Now, if $z_1 s_1^{-1}, z_2 s_2^{-1}, z s^{-1} \in B'_{\mathfrak{p}'}$ with $v_i(z_h) \geq v_j(z)$ for $i \in H$ and $h = 1, 2$, then for some $q \in \mathbb{Z}_+$ we have $v_j(z_1 (s^*)^q), v_j(z_2 (s^*)^q) \geq v_j(z)$ for $j = 1, \ldots, d$. Thus, $z_1 (s^*)^q z_2 (s^*)^q z^{-1} \in B'$ and hence $(z_1 s_1^{-1})(z_2 s_2^{-1})(z s^{-1})^{-1} \in B'_{\mathfrak{p}'}$ as required.

Second, we have $B'_{\mathfrak{p}'} \subset R'_{\mathbf{v}(\mathfrak{p})} B$ because $R'_{\mathbf{v}(\mathfrak{p})} B$ is Arf with respect to $\mathbf{v}$, so $B' \subset R'_{\mathbf{v}(\mathfrak{p})} B$ and hence $B'_{\mathfrak{p}'} \subset R'_{\mathbf{v}(\mathfrak{p})} B$.

One concludes that $R'_{\mathbf{v}(\mathfrak{p})} B = B'_{\mathfrak{p}'}$. Moreover, the data $B' = R'_{\mathbf{v}} B$ are equivalent data to $R'_{\mathbf{v}(\mathfrak{p})} B$ for all $\mathfrak{p} \in \operatorname{Spec}_{\mathbf{v}} B$. This follows from the obvious fact that $B' = \bigcap B'_{\mathfrak{p}'}$.

This leads us to analyze the case in which $\operatorname{Spec}_{\mathbf{v}} B$ consists of only one ideal $\mathfrak{p}$, that is, where all the valuations have the same center in $B$. We also will assume that $B$ is Noetherian and that $B/\mathfrak{p}$ is infinite; in particular, we can pick an element $x \in \mathfrak{p}$ such that $v_j(x) = v_j(\mathfrak{p})$, $j = 1, \ldots, d$. Blow up $\mathfrak{p}$ to obtain a scheme $X_1 = \operatorname{Bl}_{\mathfrak{p}} \operatorname{Spec}(B)$ and a proper mapping $X_1 \to X = \operatorname{Spec}(B)$; thus, the center of each $v_j$ on $X_1$ lies in the affine open set $\operatorname{Spec}(B[\mathfrak{p}/x])$ of $X_1$. We now consider the Arf closure $B_1'$ of the subring $B_1 = B[\mathfrak{p}/x]$ of $R_{\mathbf{v}}$. By property (A), it follows that $B_1'$ does not depend on the choice of $x$; therefore, it will be called the Arf closure of $X_1$ with respect to $R_{\mathbf{v}}$. In fact, it is nothing but the Arf closure of the localization of $B_1$ at the complement of the union of the prime ideals in $\operatorname{Spec}_{\mathbf{v}} B_1$, and this localization is nothing but the intersection of the local rings of $X_1$ at the centers of the valuations.

Next, we will compare Arf closure for $X$ and $X_1$. First, let us analyze the structure of an Arf ring $C$ relative to $\mathbf{v}$.

We consider the semigroup of values

$$S(C) = \{(v_1(z), \ldots, v_d(z)) : z \in C,\ z \neq 0\} \subset \mathbb{N}^d.$$

Now, for each $\underline{m} \in S(C)$ we have a new ring $C(\underline{m})$ defined as follows: Take $z \in C$ such that $(v_1(z), \ldots, v_d(z)) = \underline{m}$; then

$$C(\underline{m}) = \{w \in R_{\mathbf{v}} : wz \in C\}.$$

Since $C$ is Arf, it is easy to check that $C(\underline{m})$ is a subring of $R_{\mathbf{v}}$ that is Arf and does not depend on the choice of $z$.

Now, coming back to the preceding situation, for the Arf closure $B'$ of $B$ we have that, if $\underline{n} = (v_1(x), \ldots, v_d(x))$ is the minimum element in $S(B') - \{0\}$, then considering $B'(\underline{n})$ leads to the following.

PROPOSITION 2.1. *Arf closure and blowing up at the centers of the valuations commute. That is,*

$$B'(\underline{n}) = B'_1.$$

*Proof.* First, $B'(\underline{n})$ is Arf and contains $B'_1$. Now consider the set

$$C = [(B - \mathfrak{p}) + xB'_1 \cup xB'_1]_T,$$

where $T$ is the multiplicative closed set given by $T = (B - \mathfrak{p}) + xB'_1$. One has that $C$ is an Arf ring, and $B \subset C$ so $B' \subset C$. By construction, $C(\underline{n}) = B'_1$, so $B'(\underline{n}) \subset B'_1$. □

## 3. Valuative Arf Characteristic

We now assume that $B$ is a Noetherian domain and that $B/\mathfrak{p}$ is infinite for every $\mathfrak{p} \in \operatorname{Spec}_{\mathbf{v}} B$. Consider the Arf closure $B'$ of $B$ relative to $\mathbf{v}$.

The first invariant obtained from the Arf characteristic of $B'$ is the *semigroup of values* $S(B') \subset \mathbb{N}^d$. To analyze the properties of this semigroup, consider the set $I = \{1, 2, \ldots, d\}$ and the partition $I = \bigcup H(\mathfrak{p})$, where $H(\mathfrak{p}) = \{i : v_i \in \mathbf{v}(\mathfrak{p})\}$ and $\mathfrak{p} \in \operatorname{Spec}_{\mathbf{v}} B$. The partition can be deduced from the semigroup, since it is the partition associated to the equivalence relation

$$i \sim j \iff \forall \underline{a} \in S(B'),\ pr_i(\underline{a}) = 0 \text{ iff } pr_j(\underline{a}) = 0.$$

Now, if $d(\mathfrak{p}) = \#H(\mathfrak{p})$, then the semigroup of values of $R'_{\mathbf{v}(\mathfrak{p})}B = B'_{\mathfrak{p}'}$ is contained in $\mathbb{N}^{d(\mathfrak{p})}$ and the relation just displayed has only one equivalence class (we will call these semigroups *local*). Moreover, after relabeling the indices, we have

$$S(B') = \prod_{\mathfrak{p} \in \operatorname{Spec}_{\mathbf{v}} B} S(B'_{\mathfrak{p}'}).$$

In fact, in order to check the nontrivial part of this equality, take $a_{\mathfrak{p}} \in \mathbb{N}^{d(\mathfrak{p})}$ and $x \in B'_{\mathfrak{p}'}$ such that $a_{\mathfrak{p}}$ is the $\mathbf{v}(\mathfrak{p})$-value of $x$. Note that $x$ can be chosen to be in $B'$ because the values of elements in $B' - \mathfrak{p}'$ are zero. Now, pick $y \in B$ with $y \in \mathfrak{p}'$ and such that, for any $\mathfrak{q}' \in \operatorname{Spec}_{\mathbf{v}} B'$, we have $y \in \mathfrak{q}'$ iff $x \notin \mathfrak{q}'$ (this is possible since the elements of $\operatorname{Spec}_{\mathbf{v}} B'$ are maximal ideals). Consider $x' = x + y^h$ for $h$ large enough; then $a_{\mathfrak{p}}$ is the $\mathbf{v}(\mathfrak{p})$-value of $x'$, and the $\mathbf{v}(\mathfrak{q})$-values of $x'$ are zero

for $\mathfrak{q} \neq \mathfrak{p}$. Thus, every element in the product semigroup is the sum of $\mathbf{v}$-values of elements of this form and hence belongs to $S(B')$.

Note that the decomposition of $S(B')$ as a product of local semigroups is unique, so the datum $S(B')$ is equivalent to the set of local semigroups $S(B'_{\mathfrak{p}'})$.

Observe also that the semigroups and local semigroups considered have the *Arf property*: for any $a, b, c$ in the semigroup with $a \geq c$ and $b \geq c$, we have that $a+b-c$ is in the semigroup. To each one of these local semigroups (or the classes $H(\mathfrak{p})$) we associate the *residue field* $k(\mathfrak{p}') = B'/\mathfrak{p}'$.

Next, suppose that $\operatorname{Spec}_{\mathbf{v}} B$ consists of only one element $\mathfrak{p}$. We can define the value $\underline{n} = \min(S(B' - \{0\}))$ and the Arf ring $B'(\underline{n})$. Now everything can be repeated for $B'(n)$ instead of $B$, and thus we have a partition of $H(\mathfrak{p})$ into classes; the semigroup of $B'(\underline{n})$ is a product of local semigroups, and associated to each factor we have a residue field. We remark that (i) the new residue fields are extensions of $k(\mathfrak{p})$ and (ii) the semigroup of $B'(\underline{n})$ is easily deduced from $S(B')$, since in general for $\underline{a} \in S(B')$ we have

$$S(B'(\underline{a})) = \{\underline{m} \in \mathbb{N}^d : \underline{m} + \underline{a} \in S(B')\}.$$

Going on recursively, we obtain a set of invariants consisting of the preceding *residue fields* ( *field extensions of* $k$), each of which is associated with any value $\underline{n}$ of the semigroup $S(B')$.

We will refer the semigroup $S(B')$ and those residue fields as the *valuative Arf characteristic of* $B$ *(Arf characteristic) with respect to* $\mathbf{v}$ or simply as the *valuative Arf characteristic (Arf characteristic) of* $B$.

## 4. Arf Characteristic and Transversal Curves to Divisors

Next we will relate the Arf characteristic for $B$ with respect to $\mathbf{v}$ with geometric properties of successive blow-ups of $\operatorname{Spec}(B) = X$. Consider the valuations of $\mathbf{v}$, and blow up sucessively the centers of any $v_i$. That is, consider $\xi_\alpha$ ($\alpha \in \Sigma_0$), the centers of all the $v_i$ at $X$, and let $X_{\alpha 1}$ be the blow-up of $X$ at $\xi_\alpha$; for any $\alpha \in \Sigma_0$ consider $\xi_{\beta 1}$ ($\beta \in \Sigma_{\alpha 1}$), the centers of $v_i$ at $X_{\alpha 1}$, and let $X_{\beta 2}$ be the blow-up of $X_{\alpha 1}$ at the center $\xi_{\beta 1}$; and so on. We have a forest of integral schemes and proper birational morphisms and centers $\xi_{\delta h}$ of $v_i$ at $X_{\delta h}$, composed by trees, each of them corresponding with any different center in $X$.

Then we can associate to $B$ and $\mathbf{v}$ a weighted forest associated as just described. Vertices $P_{\delta h}$ correspond with any $X_{\delta h}$, and edges connect $X_{\delta h}$ with $X_{\lambda h+1}$ if $X_{\lambda h+1}$ comes from $X_{\delta h}$ by blowing up the center of some $v_j$. The weight at the vertex $P_{\delta h}$ is the pair

$$\big((v_{i_1}(\mathfrak{m}_{X_{\delta h}, \xi_{\delta h}}), \ldots, v_{i_s}(\mathfrak{m}_{X_{\delta h}, \xi_{\delta h}})), k(\xi_{\delta h})\big).$$

We call this forest *the Arf forest of* $B$ *with respect to* $\mathbf{v}$. Then, by Section 3, the Arf forest of $B$ is equivalent data to the Arf characteristic. In fact, at each vertex the coordinates of the first weight are the nonzero coordinates of the element $\underline{m}$ in $S(B')$ corresponding with the Arf closure $B'(\underline{m})$ (Proposition 2.1) of the blowing up of $B$ in that vertex. Moreover, the tree in each vertex of weight $\underline{m}$ corresponds

with the decomposition in local semigroups (see Section 3) of $S(B'(\underline{m}))$. Finally, the residual fields are obviously the same.

We remark that, since the center of any two valuations are different after a sufficient number of blowing ups, it follows that branches of the Arf forest correspond one-to-one with valuations $v_i$. Hence, for practical reasons the notation involving centers in the branch corresponding to a fixed valuation $v_i$ will be denoted by $\xi_{ih}$, $\mathfrak{m}_{ih}, \ldots$ rather than $\xi_{\delta h}$, $\mathfrak{m}_{X_{\delta h}}, \ldots$ as before, since $\delta$ is in fact determinated by $i$.

Then each branch of this forest corresponding with the valuation $v_i$ can be considered as the sequence

$$X \longleftarrow X_{i1} \longleftarrow X_{i2} \longleftarrow \cdots \longleftarrow X_{ih} \longleftarrow \cdots .$$

If we look at that branch in the Arf forest, the coordinate of the first weight of the $h$th point in the branch is $v_i(\mathfrak{m}_{X_{ih},\xi_{ih}})$. This means, in particular, that the Arf bamboo associated to $B$ and the single valuation $v_i$ with weights $(v_i(\mathfrak{m}_{X_{ih},\xi_{ih}}), k(\xi_{ih}))$ is nothing but the corresponding branch in the forest with weights depending only on $v_i$.

The geometric meaning of these Arf bamboos was studied in [2]. We review it in the sequel.

Now, assume that $k$ is an algebraically closed field, $X$ a separated integral algebraic scheme over $k$, and $\mathbf{v} = \{v_1, \ldots, v_d\}$ a set of divisorial valuations of the function field $k(X)$ over $k$ (i.e., such that $v_i(k) = 0$) with center at $X$. *Divisorial valuations* $v$ are those with $\mathrm{tr} \cdot \deg \cdot R_{\mathbf{v}}/\mathfrak{m}_{\mathbf{v}} = \dim(X) - 1$ or, equivalently, those such that there exists a projective model $X'$ for $k(X)$ and a smooth and codimension-1 point $\xi' \in X'$ such that $R_{\mathbf{v}} = \mathcal{O}_{X',\xi'}$.

Consider the center $\xi_i \in X$ of the valuation $v_i$ at $X$. That is, take an affine open $U \subset X$ such that $A = \Gamma(U, \mathcal{O}_X) \subset R_{\mathbf{v}_i}$ and let $\xi_i \in U$ be the point corresponding to the prime ideal $\mathfrak{p}_{\mathbf{v}_i} \cap A$. Consider the semilocal ring $B = \bigcap_{i=1}^{d} \mathcal{O}_{X,\xi_i}$. Then we call the Arf closure of $B$ the *Arf closure of $X$ relative to* $\mathbf{v}$. If we choose an affine open $U \subset X$ such that $\xi_i \in U$ for every $i$, then $A \subset B$ and the Arf closure of $A$ and $B$ relative to $\mathbf{v}$ is the same. Also, if $X_1$ is the blow-up scheme of a Noetherian ring at the common center of a set of valuations, then the Arf closure of $X_1$ with respect to such a set of valuations agrees with the one considered in Section 2.

We have associated to $B$ an Arf forest, which will be called the *Arf forest of $X$* or the *valuative Arf characteristic of $X$*. The residue fields $R_{v_i}/\mathfrak{p}_{v_i}$ of the valuations are finitely generated extensions of $k(\mathfrak{p})$. Since the valuations are pairwise distinct, this implies that the set $\mathbf{v}$ is in one-to-one correspondence with the set of branches in the forest and that the weights in the branch corresponding with $v_i$ are $(1, R_{v_i}/\mathfrak{m}_{v_i})$ for a large enough point in the branch. Note that the Arf ring at these points is trivial (coincides with $R_{v_i}$), so the Arf forest can be assumed to be finite, and each branch semigroup $S_i$ of $S(X)$ has a conductor. In fact, $S_i$ is of type

$$S_i = \{n_{i0}, n_{i0} + n_{i1}, \ldots, n_{i0} + n_{i1} + \cdots + n_{i(r-1)} + 1 + m,\ m \in \mathbb{N}\},$$

where $n_{i0}, n_{i1}, \ldots$ are the successive values of $\underline{n}$ in the bamboo and $n_{ir}$ is the first integer of the sequence such that $n_{ir} = 1$ (see [2]).

REMARK 4.1. If $\xi$ is a singular point of a variety $X$ and if $X$ is equipped with a finite set $\mathbf{v} = \{v_1, \ldots, v_d\}$ of divisorial valuations centered on $\xi$, then the semigroup $S$ in the Arf characteristic of $X$ with respect to $\mathbf{v}$ becomes a new numerical invariant for the singularity.

Our aim is to give a geometric interpretation for the valuative Arf characteristic associated with $X$ and $\mathbf{v}$. First, we describe the 1-dimensional case. Assume that $X$ is a curve, that the valuations $v_i$ correspond to branches of $X$ at certain closed points (the centers), and, as before, that all the valuations $v_i$ have the same center—say, the closed point $P$. The associated tree has $d$ branches corresponding exactly to the tree of infinitely near points determined by the curve $X$. The weights given by residue fields are trivial because all of them are isomorphic to $k$, and the weights

$$(n_{i_1}, \ldots, n_{i_s}) = (v_{i_1}(\mathfrak{m}_{ij}), \ldots, v_{i_s}(\mathfrak{m}_{ij}))$$

are the tuples of the multiplicities of the successive blowing ups of the branches of the curve $X$. In [7] there is a complete description of Arf closure for 1-dimensional rings.

Now consider a variety $X$ of dimension $> 1$. We will assume that, for any $v_i \in \mathbf{v}(\mathfrak{p})$, the residue field $R_{v_i}/\mathfrak{m}_{v_i}$ is a separably generated extension of $k(\mathfrak{p})$. We can reduce to the case of a unique center $\xi$. For any valuation $v_i$, blow up the center $\xi$ of $v_i$ at $X$ to obtain $X_{i1}$ and a new center $\xi_{i1}$ for $v_i$. Blowing up along the centers of any $v_i$, we obtain a tree of birational morphisms with $d$ branches

$$X \longleftarrow X_{i1} \longleftarrow X_{i2} \longleftarrow \cdots \longleftarrow X_{iq_i},$$

where $q_i$ is the least integer such that the center $\xi_{iq_i}$ is a smooth codimension-1 point of $X_{iq}$; thus, $R_v = \mathcal{O}_{X_{iq}, \xi_{iq_i}}$. To see that $q_i$ exists, see [2].

Consider the $k$-subschemes $W_{ij}$ of $X_{ij}$ corresponding to the centers $\xi_{ij}$. We have

$$\begin{array}{ccccccccc} X & \longleftarrow & X_{i1} & \longleftarrow & X_{i2} & \longleftarrow & \cdots & \longleftarrow & X_{iq_i} \\ \cup & & \cup & & \cup & & \cdots & & \cup \\ W & \longleftarrow & W_{i1} & \longleftarrow & W_{i2} & \longleftarrow & \cdots & \longleftarrow & W_{iq_i}, \end{array}$$

where $W_{iq}$ is a divisor defining $v_i$.

In the sequel, we relate the Arf characteristic with the geometry of curves in $X$ whose strict transforms in $X_{iq_i}$ are smooth and transversal to $W_{iq_i}$ (see [2] for the case of only one valuation).

Consider, for any $i$ $(1 \leq i \leq d)$, the Zariski open set $U_{iq_i} \subset X_{iq_i}$ of smooth points $P_{iq_i} \in W_{iq_i}$ that are smooth in $X_{iq_i}$ and such that all its images $P_{ij}$ in $W_{ij}$ $(0 \leq j \leq q_i)$ are smooth points of $W_{ij}$. Now consider, for any $i$, a smooth algebroid branch $\gamma_{iq_i} : \operatorname{Spec}(k[[t]]) \hookrightarrow X_{iq_i}$ with center at $P_{iq_i} \in W_{iq_i}$ and transversal to $W_{iq_i}$ at $P_{iq_i}$. Then, for any $i$, we have that $\gamma_{iq_i}$ induces an algebroid branch $\gamma_{ij}$ on $X_{ij}$ given by

$$\gamma_{iq_i} : \operatorname{Spec}(k[[t]]) \hookrightarrow X_{iq_i} \to X_{ij}$$

with center $P_{ij}$ in $X_{ij}$. We call the sequence $\gamma_i = (\gamma_{ij})_{j=0,\ldots,q_i}$ a *transversal arc* to $W_{iq_i}$ at $P_{iq_i}$. Note that $\gamma_{ij}$ is the strict transform of $\gamma_{il}$ in $X_{ij}$ for $l \leq j$. If all the

valuations $v_i$ are centered at a common point $P$, then the collection of branches $\gamma = \{\gamma_{i0}\}_{i=1,\dots,d}$ is a *transversal curve* in $X$ at $P \in U = \bigcap_{i=1}^{d} U_i \subset X$.

Set $e_{v_i}(\gamma_{ij}) = I_{P_{ij}}(\gamma_{ij}, W_{ij})$, the intersection multiplicity of $\gamma_{ij}$ and $W_{ij}$ at $P_{ij}$ and (by considering a local embedding) the integer given by

$$\min\{I_{P_{ij}}(\gamma_{ij}, H),\ H \supset W_{ij},\ H \text{ a smooth hypersurface}\}.$$

Note that the intersection multiplicity $I_{P_{ij}}(\gamma_{ij}, W_{ij})$ is given by the order

$$\mathrm{ord}_t(\gamma^*(\mathcal{I}_{W_{ij}, P_{ij}})),$$

where $\mathcal{I}_{W_{ij}, P_{ij}} \subset \mathcal{O}_{W_{ij}, P_{ij}}$ is the ideal defining the germ of the subvariety $W_{ij} \subset X_{ij}$.

The transversal arc $\gamma_i$ is a *completely transversal arc* if each $\gamma_{ij}$ is transversal to $W_{ij}$ at $P_{ij}$; likewise, a transversal curve $\gamma$ is completely transversal if all its branches are transversal, too. Note that, for any completely transversal branch $\gamma_i$, the multiplicity sequence $e_{v_i}(\gamma_i) = (e_{v_i}(\gamma_{ij}))_{j=0,\dots,q_i}$ consists of the multiplicities of the branches $\gamma_{ij}$, which are strict transforms of $\gamma_{i0}$.

THEOREM 4.1. *Consider $X$ and $\mathbf{v} = \{v_1, \dots, v_d\}$ as before.*

(i) *For any $i$ $(1 \le i \le d)$, there is a nonempty open set $V_{iq_i} \subset U_{iq_i}$ such that, for any $P_{iq_i} \in V_{iq_i}$ and any transversal arc $\gamma_i$ to $W_{iq_i}$ at $P_{iq_i}$, the multiplicity sequence $e_{v_i}(\gamma_j)$ is the sequence of the weights $(v_i(\mathfrak{m}_{X_{ij},\xi_{ij}}))_{j=0,\dots,q_i}$—that is, the integers $n_{ij}$ in the Arf tree of invariants.*

(ii) *For any set of points $\{P_{1q_1}, \dots, P_{dq_d}\}$ with $P_{iq_i} \in V_{iq_i}$, there exist completely transversal curves $\gamma \subset X$ such that their resolution multiplicity tree is the tree given by the Arf characteristic, forgetting the weights of the residue fields.*

*Proof.* This follows from Theorem 3 (for the case of one valuation in [2]) and from Section 1. □

Consider a local embedding of $X$ in a smooth variety $Z$, and let $Z_{ij}$ be the transform of $Z$ by the successive blowing ups along the support of the centers of the valuations $v_i$. We define a *normal completely transversal arc* (to $W_{iq_i}$) at $P_{iq_i}$ to be a completely transversal arc $\phi_i$ such that, for all $j$, $\phi_{ij}$ is contained in a smooth $H_{ij} \subset Z_{ij}$ transversal to $W_{ij}$ at $P_{ij}$ and of a complementary dimension. In the proof of the theorem in [2], one shows that normal completely transversal arcs exist at any point of $V_{iq_i}$. Thus, *normal completely transversal curves* can always be constructed.

To illustrate the construction in Theorem 4.1, we can describe the situation in which the valuations are precisely those that have a fixed codimension-1 (not necessarily smooth) center given by $\mathfrak{p}$. Then each $k(\mathfrak{p}_i)$ is a finite separable extension of $k(\mathfrak{p})$. For each $i$, the degree $r_i = [k(\mathfrak{p}_i) : k(\mathfrak{p})]$ is just the number of points $P_{iq_i} \in W_{iq_i}$, relative to the valuation $v_i$, mapping to the same $P \in W$. Hence at $P$ we have $r_i$ arcs with identical behavior, that is, with the same multiplicity sequence and resolution tree.

Moreover, if we intersect $X$ locally at $P$ with a complementary transversal smooth space $H \subset Z$ then we obtain a normal completely transversal curve $\gamma = \bigcup_{i=1,s=1}^{d,r_i} \gamma_{is}$, with $\gamma_{i1}, \dots, \gamma_{ir_i}$ passing through the same points of the resolution

tree and having the same multiplicity sequence. The same result holds for curves obtained in a similar way by transversal sections to the centers of the valuation $v_i$ at the general points $\mathfrak{p}_{ij}$.

REMARK 4.2. Theorem 4.1 shows how the valuative Arf characteristic is nothing but an algebraic way to look at the geometry of curves having as branches transversal curves to divisorial centers of the considered valuations (one such branch for each valuation).

We conclude this section by remarking that, in practice, one sometimes has—rather than a set of valuations—a valuative *cycle*: data of type $\sum_{i=1}^{r} a_i v_i$, where the $a_i$ are integers. Assume that the cycle is effective (i.e., that one has $a_i > 0$ for every $i$). Then, *the valuative Arf characteristic with respect to the set* $\mathbf{v} = \{v_i, \ldots, v_d\}$, *plus the information that each valuation* $v_i$ *is "counted"* $a_i$ *times, is nothing but a way of looking at the geometry of curves that have as branches transversal curves to divisorial centers of the considered valuations with exactly* $a_i$ *such branches for each valuation* $v_i$. One can consider completely transversal or normal completely transversal branches.

In other words, *valuative Arf characteristic relative to effective divisorial cycles* can be defined and geometrically interpreted.

A curve with $d$ (resp., with $\sum_{i=1}^{d} a_i$) branches as in Remark 3.1 is called a transversal curve to the set of valuations $\mathbf{v}$ (resp., divisorial cycle $\sum_{i=1}^{d} a_i v_i$). One can also talk about completely transversal or normal completely transversal curves to $\mathbf{v}$ (resp., $\sum_{i=1}^{d} a_i v_i$).

## 5. Arf Characteristic and Proximity

From now on we assume that the place of the ring $B$ is taken by a regular ring, which we will denote by $A$. In this case, the centers of the valuations of the set $\mathbf{v} = \{v_1, \ldots, v_d\}$, after successive blow-ups at previous centers, are regular points of the schemes to which they belong. These local rings at these centers correspond to regular local subrings of the quotient field $K$ of $A$. The order functions with respect to the maximal ideals of those regular local rings are new divisorial valuations, and they form a set of valuations $\mathbf{v}'$ that contains (properly, in general) the set $\mathbf{v}$. Moreover, the set of valuations $\mathbf{v}'$ keep the forest structure of the set of valuation centers, so it also respects the proximity relations among such centers (for the notion of proximity, see e.g. [3]).

In this section, we will show how the Arf characteristic relative to $\mathbf{v}'$ amounts to information of the proximity relations.

As before, and taking into account Remark 1.1, we can reduce to the case in which the centers of all the valuations are equal; hence, the ring $A$ can be assumed to be local regular and the center its maximal ideal. Assume also that $\dim(A) = n \geq 2$, and denote by $\prec$ the domination relation between local rings. One has $A \prec R_{v_i}$. For every $i$, there exists a chain of regular local rings

$$A = A_{i0} \prec A_{i1} \prec \cdots \prec A_{iq_i},$$

where $A_{iq_i} = R_{v_i}$ and $A_{ij+1}$ is the blowing up of $A_{ij}$ following the valuation $v_i$; that is,

$$A_{ij+1} = \mathcal{O}_{\mathrm{Bl}_{\mathfrak{m}_{ij}}(A_{ij}),\xi_{ij}},$$

where $\mathfrak{m}_{ij}$ is the maximal ideal of $A_{ij}$ and $\xi_{ji}$ is the center of $v_i$ at $\mathrm{Bl}_{\mathfrak{m}_{ij}}(A_{ij})$.

The local rings $A_{ij}$ are said to be the *infinitely near points* of $A$ following the set $\mathbf{v}$. One has $\dim(A_{ij}) \leq n$, and the set $\{A_{ij}\}$ is the vertex set of a tree with $d$ branches whose edges join vertices $A_{ij}$ and $A_{ij+1}$; note that some of the $A_{ij}$ can coincide for different values of the indices. This tree is called the *tree of infinitely near points* associated to $A$ and $\mathbf{v}$. For the sake of simplicity, we will also relabel the indices in the tree, so that a typical vertex will be denoted by $A_\alpha$, with $\mathfrak{m}_\alpha$ the maximal ideal of the regular local ring $A_\alpha$, $v_\alpha$ the $\mathfrak{m}_\alpha$-adic valuation, $R_{v_\alpha}$ the corresponding valuation ring, and $\xi_\alpha$ the center of $v_\alpha$ at $\mathrm{Bl}_{\mathfrak{m}_\alpha}(A_\alpha)$.

DEFINITION 5.1. Consider two infinitely near points $A_\alpha = A_{ij}$ and $A_\beta = A_{sh}$, where $A_\alpha \prec A_\beta$. Then $A_\beta$ is said to be *proximate* to $A_\alpha$, and we write $A_\beta \to A_\alpha$ if one has a chain

$$A_\alpha = A_{ij} = A_{sj} \prec A_{sj+1} \prec \cdots \prec A_{sh} = A_\beta$$

such that $\mathfrak{m}_{sj}A_{sj+1} = t^a A_{sj+1}$, with $a > 0$ and $t$ a regular parameter of $A_{sj+1}$ whose strict transform at $A_{sh} = A_\beta$ belongs to $\mathfrak{m}_\beta$. In other words, $A_\beta$ is proximate to $A_\alpha$ if and only if the ring $A_\beta$ is contained in the valuation ring $R_{v_\alpha}$ given by the $v_\alpha$-adic order.

Note that each $A_{ih+1}$ is proximate to $A_{ih}$ and that each $A_\beta$ is proximate to at most $n$ points. When all the $A_\alpha$ are $n$-dimensional (i.e., the centers $\xi_\alpha$ are closed points), one recovers the usual proximity relation among infinitely near points (see [3]). In the geometric situation, as in the preceding section, $A_\beta \to A_\alpha$ if the locus $V(\mathfrak{m}_\beta)$ is contained in the strict transform of the exceptional divisor created by the blowing up of $\mathfrak{m}_\alpha$ in $A_\alpha$.

We now define *the proximity matrix* associated with the tree, $P = (a_{\alpha\beta})$, to be the $m \times m$ matrix given by $a_{\alpha\alpha} = 1$, $a_{\alpha\beta} = -1$ if $A_\beta$ is proximate to $A_\alpha$ and $a_{\alpha\beta} = 0$ otherwise.

One has the following result.

PROPOSITION 5.1. *The matrix $P^{-1} = (b_{\alpha\beta})$ has positive entries. Moreover, one has $b_{\alpha\beta} = \mathrm{ord}_{A_\beta}(\mathfrak{m}_\alpha)$ if $A_\alpha \prec A_\beta$ and $b_{\alpha\beta} = 0$ otherwise.*

*Proof.* Consider $A_\gamma$ connected to $A_\alpha$ in the tree of infinitely near points, with $A_\gamma$ proximate to $A_\alpha = A_{\gamma_1}, \ldots, A_{\gamma_r}$. Any $A_{\gamma_j}$ lives in the branch between $A_\alpha$ and $A_\gamma$. Then we have $\mathfrak{m}_\alpha A_\beta = (t_1^{a_1} \cdots t_r^{a_r})A_\beta$, where the $t_j$ are part of a regular system of parameters and $a_j = \mathrm{ord}_{A_\beta}(\mathfrak{m}_{\gamma_j})$. In fact, the blowing up of $A_\alpha$ principalizes the maximal ideal $\mathfrak{m}_\alpha$ and yields a principal ideal of type $(t_1^{a_1})A_{\alpha 1}$, where $t_1$ is a regular parameter; therefore, successive blow-ups at points proximate to $A_\alpha$ lift the principal ideal $(t_1^{a_1})A_{\alpha 1}$ to one of type $(t_1^{a_1} \cdots t_r^{a_r})A_\beta$. Now every $A_\delta$ in the branch between $A_{\gamma_j}$ and $A_\beta$ is proximate to $A_{\gamma_j}$, and we have $b_{\gamma_j\beta} = \mathrm{ord}_{A_\beta}(\mathfrak{m}_{\gamma_j}) = \sum_{A_\delta \to A_{\gamma_j}} \mathrm{ord}_{A_\beta}(\mathfrak{m}_\delta)$. Then an easy argument shows that the matrix $(b_{\alpha\beta})$ is nothing but the inverse of the proximity matrix $P$. □

Associated with the valuations $v_1, \ldots, v_d$ and $A$, one has the tree of the valuative Arf characteristic of $A$. Coming back to the original notation, such valuative Arf invariants consist, in this case, of a tree with $d$ branches and weights $(\underline{n}_j, k(\mathfrak{p}_{ij}))$, where $\underline{n}_j = (n_{1j}, \ldots, n_{dj})$ and $n_{ij} = v_i(\mathfrak{p}_{ij})$. The $A_{ij}$ represent the multiplicity sequences $\{n_{i1}, \ldots, n_{iq_i}\}$ of the branches of a space curve having Arf tree as resolution tree.

THEOREM 5.1. *The proximity matrix for the tree of the infinitely near points associated with the set of valuations* $\mathbf{v}$ *can be obtained from the valuative Arf characteristic associated to it.*

*Proof.* For any valuation $v_i$, if $n_{ij} = n_{ij+1} + \cdots + n_{ih}$ then it follows that $A_{ij+1}, \ldots, A_{ih}$ are proximate to $A_{ij}$, so we get the matrix $P$. Then we recover $P^{-1}$ and the numbers $\operatorname{ord}_{A_\beta}(\mathfrak{m}_\alpha)$ for all pairs $(\alpha, \beta)$. □

Note that the $n_{ij}$ are the nonzero entries of the columns given by the last points of the branches, that is, the valuations $v_i$ for all $i$. Hence, from the proximity matrix we can obtain the values of the weights for the Arf tree.

The Arf closure with respect to valuations of the tree does not add new information; it follows from Arf closure of the top valuations in the tree.

REMARK 5.1. Notice that proximity relations are difficult to write down and, in practice, cannot be handled when $A$ is not a regular ring. The difficulties arise because, in general, the blowing up of each center creates several valuations (more precisely, a valuation cycle) instead of a single one. Only for rational surface singularities are the proximity relations understood and described (see [12]).

## 6. Arf Characteristic and Generic Complete Intersection Curves

As in the preceding section, we will keep the hypothesis of regularity on the ring $A$. Let $A$ be a regular local ring of dimension $n \geq 2$, $\mathfrak{m}$ its maximal ideal, and $I$ an $\mathfrak{m}$-primary ideal $A$. Consider the normalized blow-up

$$\pi : Y = \overline{\mathrm{Bl}_I(A)} \to X = \mathrm{Spec}(A),$$

where $\overline{\mathrm{Bl}_I(A)}$ is the normalization of the scheme $\mathrm{Proj}\big(\bigoplus_{n\geq 0} I^n/I^{n+1}\big)$. Then one has $I\mathcal{O}_Y = \mathcal{O}_Y(-D)$, where $D = \sum_{i=1}^d b_i E_i$ $(b_i > 0)$ is a Cartier divisor such that the codimension-1 components $E_i$ have exceptional support. Recall that the divisorial valuations $v_1, \ldots, v_d$ centered respectively at $E_1, \ldots, E_d$ are nothing but the Rees valuations associated to $I$. Notice that $b_i = v_i(I)$.

Consider the set $\mathbf{v} = \{v_1, \ldots, v_d\}$ of Rees valuations associated to $I$. Let $\sigma : Z \to X$ be the composition of blow-ups at the successive centers of the valuations of $\mathbf{v}$. That is, $\sigma$ is the composition of blow-ups at the subvarieties having the $\xi_\alpha$ as generic points, taking as many blow-ups as necessary to separate the centers of all $v_i$ and, for each $v_i$, to get a codimension-1 smooth center (see Section 4). The divisorial part of the ideal sheaf $I\mathcal{O}_Z$ is a Cartier divisor of type $D' = \sum_\alpha b_\alpha E_\alpha$,

where $E_\alpha$ is the strict transform at $Z$ of the exceptional divisor of the blowing up $\pi_\alpha$ of $\xi_\alpha$ and $a_\alpha = v_\alpha(I)$ for every $\alpha$. If $\alpha$ is the index corresponding to the couple $(q_i, i)$, then $b_\alpha = v_\alpha(I) = v_i(I) = b_i$.

Another way to look at the divisor $D'$ is by means of the proximity notion. In fact, if we denote by $E_\alpha^*$ the total transform at $Z$ of the exceptional divisor of the blow-up $\pi_\alpha$ (i.e., the Cartier divisor given by $\mathfrak{m}_\alpha \mathcal{O}_Z = \mathcal{O}_Z(-E_\alpha^*)$), then from the definition of proximity it follows that

$$E_\alpha = E_\alpha^* - \sum_{\beta \to \alpha} E_\beta^*$$

for every $\alpha$. Now one has $D' = \sum_\alpha m_\alpha E_\alpha$, where $m_\alpha = b_\alpha - \sum_{\alpha \to \gamma} b_\gamma$.

The last equality shows that $m_\alpha = v_\alpha(I^\alpha)$, where $I^\alpha$ is the weak tranform of $I$ at the local ring $A_\alpha$; by "weak transform" we mean the ideal obtained from $IA_\alpha$ by taking off its divisorial part (see [8]).

Notice that, according to Theorem 5.1, the proximity relation is given by the Arf closure relative to **v**. We have the expresions

$$b_\alpha = \sum_\beta b_{\alpha\beta} m_\beta,$$

where $(b_{\alpha\beta})$ is the inverse of the proximity matrix whose entries are the weights of the Arf tree. If the index $\beta$ corresponds to a valuation that is one of the $v_j$, then $b_\beta$ is nothing but the integer $b_j$ and so the preceding formula gives a geometrical interpretation of the multiplicities $b_j$ in the divisor $D$.

Next, assume that the local ring $A$ is a localization of a $k$-algebra of finite type, where $k$ is a field. Since $D$ is a Cartier divisor and since the sheaf $\mathcal{O}_Y(-D)$ is generated by global sections, one has a well-defined intersection number $(-D)^l V \geq 0$ for each $l$-dimensional irreducible subvariety of $Y$. Take general elements $f_1, \ldots, f_{n-1}$ in $I$ and consider the hypersurfaces $F_j$ in $X$ (given by the equations $f_j$) as well as their strict transforms $\widetilde{F_j}$ in $Y$. Note that $\pi^* f_j = \widetilde{F_j} + D$. Consider the curves $\tilde{C}$ and $C$ given, respectively, by $\widetilde{F_1} \cap \cdots \cap \widetilde{F_{n-1}}$ and $F_1 \cap \cdots \cap F_{n-1}$ (scheme-theoretic intersections).

PROPOSITION 6.1. *Assume that the characteristic of k is zero. Then, for a general choice of $f_1, \ldots, f_{n-1}$ in $I$, it follows that*:

(i) $(-D)^{n-1} \cdot E_i = \widetilde{F_1} \cdots \widetilde{F_{n-1}} \cdot E_i = \tilde{C} \cdot E_i$ *for each* $i$;
(ii) $\tilde{C}$ *is the strict transform of* $C$ *at* $Y$;
(iii) *each point* $Q$ *in* $\tilde{C} \cap E_i$, *for each* $i$, *does not belong to any other* $E_h$ $(h \neq i)$, *and both* $\tilde{C}$ *and* $E_i$ *are smooth and transversal to* $Q$; *and*
(iv) $(-D)^{n-1} \cdot E_i > 0$ *for* $i$.

*Proof.* Because the $f_1, \ldots, f_{n-1}$ are general elements, one has $\pi^* f_j = \widetilde{F_j} + D$ for every $j$. Thus,

$$(-D)^{n-1} \cdot E_i = (\widetilde{F_1} - \pi^* f_1) \cdots (\widetilde{F_{n-1}} - \pi^* f_{n-1}) \cdot E_i = \widetilde{F_1} \cdots \widetilde{F_{d-1}} \cdot E_i + S,$$

where $S$ is a sum of terms of the form

$$(\pi^* f_j) \cdot (E_i \cdot C);$$

here $C$ is a $(n-2)$-cycle. Since the cycle class $E_i \cdot C$ has a representative $c$ with support contained in $E_i$, by the projection formula one has

$$(\pi^* f_j) \cdot (E_i \cdot C) = (\pi^* f_j) \cdot c = f_j \cdot \pi^* c = 0.$$

Thus $S = 0$, which proves (i).

To prove (ii), (iii), and (iv), take a minimal reduction of the ideal $I$ [9, p. 112]—that is, an ideal $J \subset I$ generated by a regular sequence and such that the integral closure of $J$ and $I$ are the same (the *integral closure* of an ideal $I$ is the set of elements $x \in A$ satisfying an equality of type $x^m + a_1 x^{m-1} + \cdots + a_m = 0$ for some $m \geq 1$ and such that $a_l \in I^l$ for every $l$). Then the normalized blow-up of $I$ and $J$ are the same, and one has a factorization of the map $\pi$:

$$\overline{\mathrm{Bl}_I(A)} \longrightarrow \mathrm{Bl}_J(A) \xrightarrow{\pi_0} X.$$

By the the choice of $J$, the support $E$ of the exceptional divisor of $\pi_0$ is irreducible and isomorphic to the projective space $\mathbb{P}_k^{m-1}$. Moreover, since $\sigma$ is the normalization map, it induces surjective finite morphisms $\sigma_i : E_i \to E$ for every exceptional component $E_i$ of $\pi$. Now assume that the generic choice of $f_1, \ldots, f_{n-1}$ means that they are a part of a regular sequence generating such a $J$. Then, by definition of blow-up and the choice of $J$, it follows that the intersection $\widehat{G_1} \cap \cdots \cap \widehat{G_{n-1}}$, where $\widehat{G_i}$ is the strict transform of $f_i$ at $\mathrm{Bl}_J(A)$, is a smooth branch and is transversal to $E$ at a general point of $E$. Since $\sigma_i$ is finite, it now follows that $\hat{C}$ has exactly $\deg(\sigma_i)$ branches that are smooth and transversal to $E_i$ at $\deg(\sigma_i)$ general points of $E_i$. Since $\deg(\sigma_i) > 0$, it is clear that $(-D)^{n-1} \cdot E_i = E_i \cdot \hat{C} = \deg(\sigma_i) > 0$. This shows (ii), (iii), and (iv).

COROLLARY 6.1. *With assumptions and notation as before, the complete intersection curve of $n-1$ hypersurfaces given by general equations in the ideal $I$ is a transversal curve to the divisorial cycle $\sum_{i=1}^{r} a_i v_i$, where $a_i = (-D)^{n-1} \cdot E_i$.*

REMARK 6.1. The valuative Arf characteristic relative to the divisorial cycle described in Corollary 6.1 represents the geometry of the complete intersection curves of general elements of $I$.

Finally, we discuss the relation between the valuative Arf characteristic relative to the Rees valuations of the Jacobian ideal of an isolated hypersurface singularity and the polar invariants of Teissier [14]. Assume that the hypersurface $H$ is given in local coordinates by the equation

$$f(x_1, \ldots, x_n) = 0,$$

and consider the Jacobian ideal $J = (f, f_{x_1}, \ldots, f_{x_n})$ in the smooth ambient space. Then, a curve $C$ that is a complete intersection of $n-1$ hypersurfaces given by general elements of $J$ is nothing but a generic polar curve for $H$. The polar invariants for $f = 0$ are the rational numbers given by

$$I(\Gamma, H)/m(\Gamma),$$

where $\Gamma$ is a branch of $C$, $m(\cdot)$ denotes the multiplicity, and $I(\cdot,\cdot)$ denotes the intersection multiplicity.

Now, consider the Rees valuations $\mathbf{v} = \{v_1, \ldots, v_d\}$ of $J$ and the corresponding Arf invariants relative to the divisorial cycle in Corollary 6.1. One then concludes that *these invariants represent the geometry of the generic polar curve.* Thus, the information provided by this valuative Arf characteristic is much more complete than the information provided by the polar invariants. In fact, note that the values $m(\Gamma)$ are nothing but the weights at the root of the Arf tree. On the other hand, the intersection multiplicity $I(\Gamma, H)/m(\Gamma)$ can be computed by Noether's formula as

$$I(\Gamma, H) = \sum_{i=0}^{r} m(\Gamma_i)\,\mathrm{ord}(H_i),$$

where $\Gamma_i$ and $H_i$ are the $i$th strict transforms of $\Gamma$ and $H_i$ (respectively) by the successive blow-ups of the centers of the valuations.

Notice that, in order to obtain the polar invariants, we need only one transversal branch of the polar curve for each divisor. So, for the purpose of recovering polar invariants, it is enough to consider the Arf characteristic relative to the set $\mathbf{v}$ instead of the invariants relative to the divisorial cycle. □

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A. Campillo
Departamento de Algebra Geometria y Topologia
Universidad de Valladolid
47005 Valladolid
Spain

campillo@agt.uva.es

J. Castellanos
Departamento de Algebra
Facultad de Matematicas
Universidad Complutense
28040 Madrid
Spain

jcaste@alg.mat.ucm.es