Dynamics of Polynomial Hamiltonian Vector Fields in $\mathbb{C}^{2k}$

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1. Introduction

The main purpose of this article is to provide a dynamical study of a natural class of holomorphic vector fields, namely polynomial Hamiltonian (or complex divergence-free) vector fields in $\mathbb{C}^2$. By "dynamical study" we mainly mean here "real-time study", in a situation where the complex orbits of the flow are well understood: these are only level sets of a polynomial in $\mathbb{C}^2$. We restrict ourselves to the polynomial case, which is relevant for many approximation problems (see Sections 4 and 5); this enables us to use the global geometry of level sets.

We also give a contribution to the study of Hamiltonian vector fields in $\mathbb{C}^{2k}$ ($k \geq 2$), again by first studying polynomial fields.

We introduce now some terminology (see Section 6 for the higher-dimensional case). Let p be an entire function in $\mathbb{C}^2$ (with coordinates (z, w)). The holomorphic vector field

$$X_p = \left(\frac{\partial p}{\partial w}, -\frac{\partial p}{\partial z} \right) = \frac{\partial p}{\partial w} \frac{\partial}{\partial z} - \frac{\partial p}{\partial z} \frac{\partial}{\partial w}$$

is called the *Hamiltonian* vector field associated to (the Hamiltonian) p and the symplectic form $\omega = dz \wedge dw$. This terminology is justified, as in the real case, by the relation $i_{X_p}\omega = dp$. For further information, see [F] and [FS1]. One sees readily that the flow of X_p preserves each level set $\{p = c\}$. We also recall from [F] that the real-time flow of a holomorphic vector field has a holomorphic extension to a neighborhood in $\mathbb{C}$ of its domain in the real axis.

The outline of this paper is as follows. In Sections 2–4 we give a rather complete picture of the dynamics of a generic class of polynomial Hamiltonian vector fields in $\mathbb{C}^2$. Note that, in order to speak of generic properties, one needs to fix the degree. We hope this can be used as an example for further study.

We also prove that the "quasi-ergodic hypothesis" is satisfied for polynomial and entire Hamiltonian vector fields. This gives a new proof of a result of [FS3].

In Section 5, we use the preceding work to study exploding orbits of holomorphic Hamiltonian vector fields [FG1; FG2]. One says that an orbit *explodes* if it reaches infinity in finite time. The following theorem is due to Fornæss and Grellier [FG1].

THEOREM 1.1 [FG1]. *There is a dense family $G \subset \mathcal{E}$ (vector space of entire functions in $\mathbb{C}^2$) such that, for any $F \in G$, there is a dense set of points with exploding orbits for the vector field X_F.*

With the help of a Fatou–Bieberbach domain we are able to prove a refinement of this theorem (we also think our proof is technically much simpler) that replaces "dense set of points" by "outside a union of real submanifolds".

In Section 6 we prove a version of this theorem in $\mathbb{C}^{2k}$, again using a Fatou–Bieberbach domain.

2. Level Sets of Generic Polynomial Hamiltonians

In this section we study the geometry of generic level sets of polynomials ("Hamiltonians") of two variables. Let $\mathcal{P}_d$ be the affine space of holomorphic polynomials of degree $\leq d$ in (z, w), provided with the standard basis $\{z^i w^j, i + j \leq d\}$. The following easy propositions show that, for a typical $p \in \mathcal{P}_d$, X_p has isolated zeros and the generic level set $\{p = c\}$ is irreducible and smooth up to the line at infinity. For background on complex algebraic curves, see for example [M].

PROPOSITION 2.1. *Let $d \geq 2$. There exists a Zariski dense open subset of Hamiltonians $p \in \mathcal{P}_d$ such that the Hamiltonian vector field X_p associated with p has isolated (and hence a finite number of) zeros.*

Proof. $Z(X_p)$ is the intersection of two algebraic sets:

$$Z(X_p) = \left\{ \frac{\partial p}{\partial w} = 0 \right\} \cap \left\{ \frac{\partial p}{\partial z} = 0 \right\}.$$

A sufficient condition for this algebraic set to be of exact dimension 0 at z_0 is

$$\det \left(\mathrm{Grad}\left(\frac{\partial p}{\partial z} \right)(z_0), \mathrm{Grad}\left(\frac{\partial p}{\partial w} \right)(z_0) \right) \neq 0; \tag{1}$$

that is, the determinant of the Hessian matrix $(\mathrm{Hess}\, p)(z_0)$ be nonzero.

Consider

$$f : \mathcal{P}_d \times \mathbb{C}^2 \to \mathbb{C}^3,$$

$$(p, z_0) \mapsto \left(\frac{\partial p}{\partial z}(z_0), \frac{\partial p}{\partial w}(z_0), \det(\mathrm{Hess}\, p)(z_0) \right).$$

A Hamiltonian p satisfies condition (1) at all zeros if, for all $z_0 \in \mathbb{C}^2$, $(p, z_0) \notin Z_f$; that is, $p \notin \pi(Z_f)$, where $\pi : \mathcal{P}_d \times \mathbb{C}^2 \to \mathcal{P}_d$ is the natural map.

Since $p \to (\partial p / \partial z)(z_0)$ and $p \to \det(\mathrm{Hess}\, p)(z_0)$ are homogeneous with respect to the coefficients of p, it follows that Z_f is a closed algebraic set in $\mathbb{P}\mathcal{P}_d \times \mathbb{C}^2$; the main theorem of elimination theory (see e.g. [M]) asserts that

$\pi(Z_f)$ is a closed algebraic subset of $\mathbb{PP}_d$. It only remains to show that $\pi(Z_f) \neq \mathbb{PP}_d$: take $p(z, w) = q(z) + w^2$, with q a degree-d polynomial such that q' has only simple roots. $\square$

REMARK.   Outside the zero set $Z(X_p)$, any $\{p = c\}$ is smooth because p is locally submersive.

Now we compactify $\mathbb{C}^2$ in $\mathbb{P}^2$ (projective 2-space) and consider the closure $\overline{p^{-1}(c)} = \Sigma$ of a level set in $\mathbb{P}^2$. Let us first fix some notation: The homogeneous coordinates of a point in $\mathbb{P}^2$ are denoted by $[Z : W : T]$, where $z = Z/T$ and $w = W/T$ are the affine coordinates in $\mathbb{C}^2$; also, $\tilde{p}(Z, W, T) = p_d(Z, W) + T p_{d-1}(Z, W) + \cdots + T^d p_0(Z, W)$ denotes the homogenization of p (we suppose p of exact degree d), and

$$\Sigma \cap \{T = 0\} = \{\tilde{p} = 0\} \cap \{T = 0\} = \{[Z, W, 0], \ p_d(Z, W) = 0\}.$$

These points at infinity are smooth points of $\overline{p^{-1}(c)}$ if the corresponding roots of p_d in $\mathbb{P}^1$ are simple (this is a transversality condition).

It is routine to check, as before, that the condition "p has exact degree d and the roots of p_d in $\mathbb{P}^1$ are simple" is valid on a Zariski open subset of $\mathcal{P}_d$.

PROPOSITION 2.2.   *There exists a Zariski dense open subset of $\mathcal{P}_d$ consisting of irreducible polynomials. For such a polynomial p, $p - c$ is irreducible for all but finitely many $c \in \mathbb{C}$.*

Proof. Consider, for any $1 \leq k \leq d$, the map $f : \mathcal{P}_k \times \mathcal{P}_{d-k} \to \mathcal{P}_d$ defined by $f(q, r) = qr$. This map can obviously be given a projective sense; that is, one has a factorization $\tilde{f} : \mathbb{PP}_k \times \mathbb{PP}_{d-k} \to \mathbb{PP}_d$. By the proper mapping theorem, the image of $\tilde{f}$ is an algebraic set in $\mathbb{PP}_d$ that has, of course, nonempty complement.

On the other hand, if p is irreducible then the set $\{c \in \mathbb{C}, \ p - c \text{ is reducible}\}$ is a proper algebraic subset of $\mathbb{C}$ and hence is finite. $\square$

3. Real Orbits in a Generic Level Set

We wish to perform a study of the (real) dynamics of the vector field $X_p = (\partial p/\partial w, -\partial p/\partial z)$ ($d = \deg p \geq 4$) in a single level set $p^{-1}(c)$ that is invariant by its flow. In this section we fix generic p and c as in the foregoing propositions; that is, $\overline{p^{-1}(c)}$ is a smooth compact Riemann surface of genus $(d-1)(d-2)/2 > 1$, so X_p extends as a meromorphic vector field on $\Sigma = \overline{p^{-1}(c)}$ without any zeros on its affine part. Throughout this article, the term "orbit" means the real orbit of a point x_0 and is denoted by $O(x_0)$ ($O^+(x_0)$ is the positive orbit).

We first study the behavior of the vector field near a point at infinity of Σ.

PROPOSITION 3.1.   *There exists a holomorphic chart near any point at infinity of Σ, where X_p has the form $u^{-(d-3)} \partial/\partial u$.*

Without any loss of generality we can assume $\tilde{p}[1, 0, 0] \neq 0$ (with notation as in the preceding section), and then we settle in the affine chart $\{Z \neq 0\}$. We now put

$x = W/Z$ and $y = T/Z$ ($\{T = 0\}$ becomes $\{y = 0\}$ in the new chart). One easily checks that, in this new chart, the vector field expresses as

$$X_p(x, y) = -\left(xy\frac{\partial p}{\partial w}\left(\frac{1}{y}, \frac{x}{y}\right) + y\frac{\partial p}{\partial z}\left(\frac{1}{y}, \frac{x}{y}\right)\right)\frac{\partial}{\partial x} - y^2\frac{\partial p}{\partial w}\left(\frac{1}{y}, \frac{x}{y}\right)\frac{\partial}{\partial y};$$

if we let $\hat{p}(x, y) = p(1/y, x/y)$ ($= p(z, w)$) whenever the two expressions make sense), it becomes

$$X_p(x, y) = y^3\left(\frac{\partial \hat{p}}{\partial y}(x, y)\frac{\partial}{\partial x} - \frac{\partial \hat{p}}{\partial x}(x, y)\frac{\partial}{\partial y}\right).$$

We claim that X_p has poles of order $d - 3$ in y at the points at infinity of Σ. Indeed we have $\hat{p}(x, y) = 1/y^d p_d(1, x) + \cdots + p_0(1, x)$, so

$$\frac{\partial \hat{p}}{\partial y}(x, y) = \frac{-d}{y^{d+1}}p_d(1, x) + \frac{-d + 1}{y^d}p_{d-1}(1, x) + \cdots;$$

but the first term $p_d(1, x)$ vanishes at any point of $\Sigma \cap \{y = 0\}$.

Proof of Proposition 3.1. We have chosen p_d with simple roots in the line at infinity; in coordinates (x, y), this means that $p_d(1, x)$ has only simple roots. Hence, if x_0 is one of these then $(\partial/\partial x)(p_d(1, x))(x_0) \neq 0$. By the implicit function theorem, y can thus serve as a local coordinate for $\Sigma = \{y^d\hat{p}(x, y) = cy^d\}$ near $(x, y) = (x_0, 0)$. By virtue of the previous claim, in an appropriate chart of Σ around this point, X_p takes the form $X(v) = v^{-(d-3)}f(v)(\partial/\partial v)$, $f(0) \neq 0$.

It remains to reach the normal form $u^{-(d-3)}(\partial/\partial u)$; that is, we must find a local holomorphic diffeomorphism ψ of $(\mathbb{C}, 0)$ such that

$$\psi_*X(v) = (\psi^{-1}(v))^{-(d-3)}f(\psi^{-1}(v))\psi'(\psi^{-1}(v))\frac{\partial}{\partial v} = v^{-(d-3)}\frac{\partial}{\partial v}.$$

Let $u = \psi^{-1}(v)$. Then one has $u^{-(d-3)}f(u)\psi'(u) = \psi(u)^{-(d-3)}$, that is,

$$\frac{d}{du}\left(\frac{\psi(u)^{d-2}}{d-2}\right) = \frac{u^{d-3}}{f(u)} = u^{d-3}\left(\frac{1}{f(0)} + \cdots\right).$$

Let $G(u) = u^{d-2}/((d-2)f(0)) + \cdots$ be a primitive of $u^{d-3}/f(u)$. It suffices then to take $k(u)$ a branch of the $(d-2)$th root of

$$\frac{(d-2)G(u)}{u^{d-2}} = \left(\frac{1}{f(0)} + \cdots\right),$$

and $\psi(u) := uk(u)$ gives the solution. □

A brief study of this vector field near 0 gives a picture like Figure 1. There are $2(d-2)$ separatrices, that is, integral curves that attain the pole 0 in finite forward or backward time. Those will respectively be called stable and unstable separatrices (stable and unstable manifolds of the singularity).

This means in particular that only a finite number of orbits on Σ reach infinity in finite (positive or negative) time.
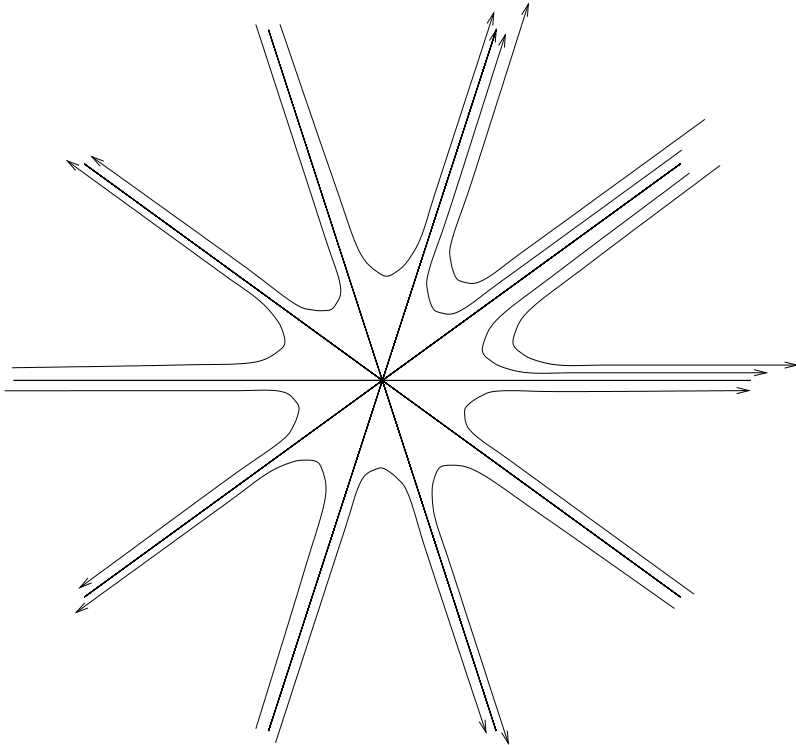
Figure 1   Integral curves of $u^{-4}\partial/\partial u$ near 0

REMARK.   Multiplying X_p by a C^∞ function with zeros of sufficiently high order at the poles yields a C^k field $\overline{X_p}$ with the same trajectories as X_p on Σ. This enables us to apply Poincaré–Bendixson theory for smooth flows on surfaces, where the fixed points of the flow are the poles of X_p.

We now recall some basic ideas concerning the dynamics of holomorphic vector fields on an abstract open Riemann surface S [FS1; F; MV]. Let X be a holomorphic vector field without zeros on S (S will be the affine part of Σ). Suppose the flow $\mathcal{X}(t, x)$ of X is defined on an open interval $J \subset \mathbb{R}$; then it extends as a holomorphic map on a neighborhood of $J \times \{x\}$ in $\mathbb{C} \times S$. The inverse of the local diffeomorphism $\phi\colon t \mapsto \mathcal{X}(t, x)$ is a "holomorphic flow box chart", that is, $(\phi^{-1})_* X = \partial/\partial t$. The flow box chart can be extended as soon as $t \to \mathcal{X}(t, x)$ is injective ("long flow box chart"). In this chart, the flow is a horizontal translation. Observe also that, by holomorphy, the vertical direction and length unit are preserved under change of flow box charts (the horizontal direction is given by the vector field); because the vertical Lebesgue measure is invariant under horizontal translations, on each small transversal we have a measure preserved by the Poincaré map, that is, an invariant transverse measure (see also [MV]).

The following proposition follows then from Poincaré–Bendixson theory, the absence of zeros on the affine part of Σ, and the identity theorem (see [FS1]).

PROPOSITION 3.2.   *If a positive orbit $O^+(x)$ on Σ is bounded in $\mathbb{C}^2$, then it is periodic. Each periodic orbit is embedded in a ring domain where the flow is conjugate to a rotation. The corresponding maximal ring domain is bounded by a cycle of separatrices (a "graph" or "polycycle").*

From the finite number of separatrices on Σ we deduce the following corollary.

COROLLARY 3.3.   *There are only finitely many annuli on Σ where the flow is conjugate to a rotation.*

These results are part of the theory of quadratic differentials [Str]: if the vector field X expresses as $f(z)(\partial/\partial z)$ in the local coordinate z, then the expression $q(z) = (1/f(z)^2)dz^2$ is well-defined on Σ. By definition, q is a quadratic differential; its trajectories are the leaves of the foliation tangent to the field of lines defined by the condition $(1/f(z)^2)dz^2 > 0$. As a foliation, the integral curves of X are the trajectories of q.

The flow of X preserves the measure

$$\frac{1}{|f(z)|^2} idz \wedge d\bar{z}$$

associated to q. In the case of X_p on Σ, this measure has finite mass since there are no zeros and Σ is compact. It is absolutely continuous with respect to Lebesgue measure. Note that $(1/|f(z)|^2)idz \wedge d\bar{z}$ is only the coordinate-invariant writing of the Lebesgue measure in flow box charts.

Recall that if f is a measure-preserving transformation of a finite measure space M then the Poincaré recurrence theorem asserts that, for almost every point x of M, $O^+(x)$ intersects any set of positive measure infinitely many times.

The following proposition is a consequence of the theory of quadratic differentials, but for the convenience of the reader we include here a proof that uses only the theory of smooth flows of surfaces (see e.g. [PM; NZ]). Recall that an orbit is said to be *positively recurrent* if it is contained in its ω-limit set. Any periodic orbit is recurrent and, by Poincaré–Bendixson theory and holomorphy, a nonperiodic recurrent orbit on Σ must be unbounded (since it must cluster on a singular point of the vector field).

PROPOSITION 3.4.   1. *All orbits of the vector field X_p, except stable separatrices, are positively recurrent.*
    2. *The ω-limit set of any nontrivial (i.e. nonperiodic) ω-recurrent trajectory has nonempty interior.*

By reversing time, one obtains similar conclusions for α-limit sets.

Proof of Proposition 3.4. 1. We use the finite invariant measure introduced previously (Lebesgue measure in flow box charts). The union of stable separatrices is a nullset and, in its complement, all orbits are defined in positive time. The

Poincaré recurrence theorem then asserts that almost every orbit is positively recurrent (including unstable separatrices that are not saddle connections).

This would be sufficient for our purposes, but we want to show also that all non-recurrent orbits are separatrices. Suppose $x_0 \in \Sigma$ is not ω-recurrent. Let T be a small open interval transverse to X_p at x_0, and let

$$D = \{x \in T \mid O^+(x) \text{ cuts } T\}$$

be the domain of the Poincaré first return map. Since D is an open subset of T of full measure, it follows that $D = \bigcup_n I_n$ is a union of open intervals. Suppose x_1 is a boundary point of such an interval on T. Then it is a classical result (see e.g. [PM, pp. 145–146]; we sketch a proof shortly) that $\omega(x_1)$ is a single saddle point, that is, x_1 is on a stable separatrix. Because $O^+(x_1)$ does not cut T again, there are only finitely many such points and x_0 is one of them.

We now sketch a proof of the fact that $\omega(x_1)$ is a saddle point (see [PM] for details). First, through any point of a nontrivial recurrent orbit γ there is a circle transversal to X_p: fix a small transversal, cover the segment of γ between two consecutive intersections by two long flow box charts, and then use straight lines transverse to X_p in these charts. Since almost all orbits are nontrivial recurrent, there is a transverse circle through any point of Σ.

Now suppose there is a regular point a in $\omega(x_1)$, and let C be a transverse circle through a. If $x_1 \in \partial I$ as before and if $y \in I$, then the number of times $O^+(y)$ cuts C between two consecutive intersections with T is a constant N on I, as it is locally constant and I is connected. But $O^+(x_1)$ cuts C infinitely many times without cutting T again—a contradiction.

2. Let A be the union of the annuli of periodic orbits. All orbits in $\Sigma \setminus \bar{A}$ are nontrivial (i.e. nonperiodic) ω-recurrent except stable separatrices; this is the first part of the proposition. Let us show that the union of the unstable separatrices is dense in $\Sigma \setminus \bar{A}$ (in fact we show this for stable separatrices, but it suffices to reverse time to achieve the desired result). Let U be an open subset of $\Sigma \setminus A$ and suppose that no stable separatrix cuts U. Then $\mathcal{X}(t, x)$ is defined for all $t > 0$ and $x \in U$, $\mathcal{U} = \bigcup_{t>0} \mathcal{X}(t, U)$ is a hyperbolic open set (recall that genus$(\Sigma) > 1$), $\mathcal{U} \cap A = \emptyset$, and the flow is a holomorphic endomorphism of $\mathcal{U}$.

Let us show that the time-1 map $\mathcal{X}_1 \colon \mathcal{U} \to \mathcal{U}$ is not onto. Suppose it is; then the time-(-1) of the flow is well-defined on $\mathcal{U}$; that is, the vector field X_p is $\mathbb{R}$-complete when restricted to $\mathcal{U}$. We now consider the closure G of the 1-parameter group of automorphisms generated by the flow in $\mathcal{U}$. Since $\mathcal{U}$ is hyperbolic and since orbits do not tend uniformly to the boundary (by recurrence), G is compact (see [FS1]). In particular, orbits are compactly contained in $\mathcal{U}$. By Poincaré–Bendixson theory there is a periodic orbit or a critical point in $\mathcal{U}$, which is a contradiction.

Now the time-1 map $\mathcal{X}_1$ is a contraction for the Kobayashi metric, and it is a classical result [CG, Lemma 4.2.2] that all orbits tend to a single point of $\bar{\mathcal{U}}$. Thus no orbit in $\mathcal{U}$ can be nontrivial recurrent, which is a contradiction.

Hence there are q unstable separatrices S_i whose ω-limit sets $\omega(S_i)$ have nonempty interior and cover $\Sigma \setminus A$. The S_i are recurrent because they cannot be saddle connections. Let $x \in \omega(S_i)$ be a recurrent point; Maier's theorem [NZ, p. 30] asserts that, if x and y are recurrent and $\omega(y) \ni x$, then $\omega(x) \ni y$ (you

should be convinced by drawing a picture, but the proof is not that obvious). In our case, $\omega(x) \supset S_i$ and we are done. $\qquad\square$

REMARK. We also quote the following result from [Str, p. 166], which we shall use later: If the ω-limit set $\omega(x)$ has nonempty interior, then $\partial(\omega(x))$ is a union of saddle connections (orbits that are stable and unstable separatrices). Moreover, except for at most countably many θ, the vector field $e^{i\theta}X_p$ has no saddle connections and all orbits except possibly separatrices are dense on Σ.

4. Neighborhood of a Periodic Orbit in $\mathbb{C}^2$. Generic Properties

We want first to link the dynamics between nearby level sets in the neighborhood of a closed orbit $O^+(x_0)$. We still take generic p of degree ≥ 4 as in Section 2, and we suppose $\{p = p(x_0)\}$ is smooth, irreducible, and does not carry any zero of X_p.

PROPOSITION 4.1. *Let x_0 be as before, with period T_0 for the flow of X_p. Then, either*:

(a) *there exists a neighborhood of $O^+(x_0)$ in $\mathbb{C}^2$ where the flow is conjugate to a rotation with period T_0; or*

(b) *$O^+(x_0)$ has a neighborhood where periodic orbits with period near T_0 all lie in $\{p = p(x_0)\}$; or*

(c) *there exists a real 1-parameter family S of values of c near $p(x_0)$ such that each periodic orbit near $O^+(x_0)$ with period T near T_0 lies in a $\{p = c\}$, $c \in S$.*

Proof. Recall from Proposition 3.2 that, if x_0 is T_0-periodic, then it has an annular neighborhood A in $\{p = p(x_0)\}$ where the flow is conjugate to a rotation. Hence there is a neighborhood U of $O^+(x_0)$ in $\mathbb{C}^2$ that is biholomorphic to $A \times D$ (unit disc in $\mathbb{C}$) and where, after a suitable change of coordinates, A lies on the z-axis and each nearby level set is a graph over A. We assume furthermore that X_p does not vanish on level sets intersecting U.

The domain of the flow $(t, x) \mapsto \mathcal{X}(t, x)$ is an open set in $\mathbb{C} \times \mathbb{C}^2$ containing $\mathbb{R} \times O^+(x_0)$. By reducing U if necessary, we can assume that $\mathcal{X}(t, x)$ is well-defined on $V \times U$, where V is a neighborhood of $[0, T_0]$ in $\mathbb{C}$. Any closed orbit sufficiently close to $O^+(x_0)$ with period near T_0 is contained in U. Conversely, we claim that each closed orbit that remains in U has period T near T_0. This is a consequence of the local picture of U given previously: since any orbit contained in U lies in annulus over A, a closed orbit cannot wind $k > 1$ times (this argument is in [FS1]).

Let (with $V(T_0)$ a neighborhood of T_0 in $\mathbb{C}$)

$$\psi : V(T_0) \times U \to \mathbb{C}^2,$$

$$(t, x) \mapsto \mathcal{X}(t, x) - x.$$

The foregoing discussion shows that any periodic orbit contained in U is in $Z(\psi) = \{\psi = 0\}$, which is an analytic set. Since $O^+(x_0) \subset Z(\psi)$, it follows that $\dim Z(\psi) \geq 1$.

If dim $Z(\psi) = 1$, then $U \cap \{p = p(x_0)\}$ is isolated within annuli of periodic orbits of period near T_0.

If dim $Z(\psi) = 2$, as

$$d_{(t,x)}\psi \cdot (\tau, \xi) = \tau X_p(\mathcal{X}(t, x)) + (d^x_{(t,x)}\mathcal{X} - \mathrm{id}) \cdot \xi$$

and X_p is nonvanishing, one sees readily that $Z(\psi)$ is smooth (ψ is submersive near $O^+(x_0)$) and the projection on the x-coordinate is locally onto near $O^+(x_0)$. If $Z(\psi) \subset \{t = T_0\}$, then $O^+(x_0)$ has a neighborhood of T_0-periodic orbits. If not, it means that, for $x \in U$ and $x = \pi(T(x), x)$ with $(T(x), x) \in Z(\psi)$ and by reducing U again, we can suppose that $T(x)$ is holomorphic and depends only on $p(x)$, because if x has a $T(x)$-periodic orbit $(T(x) \in \mathbb{C})$ then it is surrounded on $\{p = p(x)\}$ by $T(x)$-periodic orbits. It remains to note that if $T(x) \in \mathbb{C} \setminus \mathbb{R}$ then $O^+(x)$ spirals outside U. ☐

REMARK.   If a point has a complex and nonreal period, its real orbit can still be closed—that is, have another real period. For example, on the torus $\mathbb{C}/\mathbb{Z}[i]$, for any $k \in \mathbb{Z}$ we have that any point is $(1 + ki)$-periodic for the flow of the (complete) vector field induced by $\partial/\partial u$.

We are now in position to prove the main theorem concerning the behavior of generic polynomial Hamiltonian vector fields.

THEOREM 4.2.   *Let $d \geq 4$. There exists a set E of zero measure in $\mathcal{P}_d$ such that, for any $p \in \mathcal{P}_d \setminus E$, the set of points with bounded positive orbit for X_p is contained in an at most countable union Σ of real hypersurfaces in $\mathbb{C}^2$. If $x \in \mathbb{C}^2 \setminus \Sigma$, then either*:

(a) *the flow $\Phi(t, x)$ does not exist for $t \in \mathbb{R}^+$, that is, $\Phi(t, x)$ tends to infinity in finite positive time (x belongs to the set of separatrices); or*
(b) *the flow $\Phi(t, x)$ exists for $t \in \mathbb{R}^+$, in which case $O^+(x)$ is recurrent and the limit set of x has nonempty interior in $\{p = p(x)\}$.*

Moreover, this latter set has full measure in $\mathbb{C}^2 \setminus \Sigma$.

Proof. Propositions 2.1 and 2.2 show that there is a Zariski closed subset (hence of zero measure) $E_0 \subset \mathcal{P}_d$ such that, for $p \in \mathcal{P}_d \setminus E_0$, X_p has finitely many zeros and such that p is irreducible, of exact degree d, and has only simple roots on the line at infinity. Hence, for all but finitely many $c \in \mathbb{C}$, the results of Section 3 are valid on $\{p = c\}$. For $p_0 \in \mathcal{P}_d \setminus E_0$ we want to show that, for all but at most countably many $e^{i\theta} \in S^1$, we have:

(i) $e^{i\theta} p_0 \in \mathcal{P}_d \setminus E_0$; and
(ii) $X_{e^{i\theta} p_0}$ has no open set of bounded (= periodic) orbits.

The first point is obvious, and the second is a theorem of [FS1] whose proof goes as follows. Let Σ_0 be the (finite) set of nongeneric (i.e., reducible or critical) level lines and let $(\Delta_n)_{n \geq 1}$ be a neighborhood basis of $\mathbb{C}^2 \setminus \Sigma_0$. Let $\Phi_\theta(t, x)$ be the flow of $e^{i\theta} X_{p_0} = X_{e^{i\theta} p_0}$, and set

$$I_n = \{e^{i\theta} \in S^1 \mid \forall x \in \Delta_n, \ \forall t \in \mathbb{R}^+, \ \Phi_\theta(t, x) \in B(0, n)\};$$

if $\theta_0 \in I_n$, then $\Phi_{\theta_0}(t, x)$ is conjugate to a rotation on the open set $\tilde{\Delta}_n$ saturated of Δ_n by the flow. Then, for $0 < \theta < \varepsilon$, the orbit of points in Δ_n under $X_{e^{i(\theta_0+\theta)}p_0}$ spiral outside $B(0, n)$; that is, $\theta_0 + \theta \notin I_n$. Hence I_n and $\bigcup_{n \geq 1} I_n$ are at most countable.

Fubini's theorem then asserts that there is a set $E = E_0 \cup E_1$ of zero measure in $\mathcal{P}_d$ such that, for $p_0 \in \mathcal{P}_d \backslash E$, there are no open sets of bounded (= periodic) orbits. We claim that for such a p_0 the conclusions of the theorem are valid.

Let $p_0 \in \mathcal{P}_d \backslash E$ as before. Let $A_n = \{$periodic points in $\bar{B}(0, n)$ with period $T \in [1/n, n]\}$ (we avoid zero because there may be fixed points in $B(0, n)$). Then A_n is contained in a finite union of real hypersurfaces. Indeed, since $p_0 \notin E$, there is no open set of periodic orbits and hence (by Proposition 4.1) for all $(x, T_0) \in \bar{B}(0, n) \times [1/n, n]$ there exists $V_1 \times V_2 \in \mathrm{Neigh}(x, T_0)$ such that either:

(i) no point of V_1 is T-periodic for all $T \in V_2$; or

(ii) all points of V_1 that are T-periodic ($T \in V_2$) are contained in a real hypersur-face, possibly a single level set.

Compactness of $\bar{B}(0, n) \times [1/n, n]$ then implies the desired result. We deduce that bounded orbits are contained in $\bigcup_n A_n$, which is a countable union of real hypersurfaces. The behavior of unbounded orbits on generic level lines was detailed in Section 3 (see esp. Proposition 3.4). $\qquad \square$

REMARKS.   1. By reversing time, one obtains similar conclusions for negative and total orbits.

2. The proof shows that, if N is the dimension of $\mathcal{P}_d$, then E has σ-finite $(N-1)$-dimensional Hausdorff measure.

3. In the case of degree 3, generic level sets are complex tori and X_p has neither zeros nor poles. Thus X_p is induced by a constant vector field, and its dynamics is well known: orbits are all periodic or all dense. The dynamics in the genus-0 case is described in [MV].

With the terminology of [FS3], we say that the quasi-ergodic hypothesis is satisfied if, in generic level sets, there is a dense orbit. We prove here that the quasi-ergodic hypothesis is valid for polynomial Hamiltonian vector fields in $\mathbb{C}^2$.

THEOREM 4.3.   *Let $d \geq 4$. Then there exists a dense subset G of full measure of $\mathcal{P}_d$ with the following property: for any $p \in G$, there is a dense G_δ subset $\mathcal{C}_p \subset \mathbb{C}$ such that, for $c \in \mathcal{C}_p$, there is a dense orbit on $\{p = c\}$.*

REMARK.   It follows from the proof of Corollary 4.4 that G is a G_δ.

Proof of Theorem 4.3. We first exhibit the set G. Let p be a generic degree-d polynomial as in Theorem 4.2, and let E be as in this theorem. Pick a dense sequence $\{c_n, n \in \mathbb{N}\}$ in $\mathbb{C}$. We know (see the remark after Proposition 3.4) that for each $p^{-1}(c_n)$, except for a set Θ_n consisting of at most countably many θ, the vector field $e^{i\theta} X_p$ has all orbits except separatrices dense on $p^{-1}(c_n)$ (this is a matter of breaking all saddle connections).

Thus we have that, for each $p \in \mathcal{P}_d \backslash E$, there exists a countable set Θ such that, for $\theta \notin \Theta$:

(i) on each level line $\{p = c_n\}$, $n \in \mathbb{N}$, all orbits $X_{e^{i\theta}p}$ except separatrices are dense; and

(ii) $e^{i\theta}p \in \mathcal{P}_d \backslash E$ (i.e., $e^{i\theta}p$ is generic in the sense of Theorem 4.2).

By Fubini's theorem, the set $G \subset \mathcal{P}_d \backslash E$ of polynomials with dense orbits on $\{p = c_n, n \in \mathbb{N}\}$ is of full measure.

It remains to show that if $p \in G$ then the set

$$\{x \in \mathbb{C}^2 \mid O^+(x) \text{ is dense in } p = p(x)\}$$

is a G_δ; this easily implies the same property for values $c \in \mathbb{C}$. First we claim that there exists a family of (nonconnected) open sets U_n "transverse" to the singular foliation $\{p^{-1}(c), c \in \mathbb{C}\}$ in the following sense:

(i) for each noncritical value $c \in \mathbb{C}$ of p and for all $n \in \mathbb{N}$, $p^{-1}(c) \cap U_n \neq \emptyset$;

(ii) for each noncritical c, $\{p^{-1}(c) \cap U_n, n \in \mathbb{N}\}$ is a neighborhood basis of $p^{-1}(c)$.

We now sketch the construction of this family. Above each noncritical value c of p (there are finitely many critical values by Bezout's theorem), p is a topological fibration with fiber isomorphic to a genus-g surface minus d points. Hence, above a small neighborhood V of c we easily get a family $U_n(V)$ satisfying the desired assumptions. Then it suffices to cover $\mathbb{C} \backslash \{$critical values$\}$ by a countable number of such V.

Next, let $\mathcal{U}_k$ be the open set of x such that $O^+(x)$ hits U_k. We know that

$$\mathcal{U}_k \supset \{p = c_n, n \in \mathbb{N}\};$$

therefore, $\mathcal{U}_k$ is dense and $\bigcap_k \mathcal{U}_k$ is the desired G_δ. $\qquad\square$

Let $\mathcal{E} = \mathcal{O}(\mathbb{C}^2)$ be the space of entire functions in $\mathbb{C}^2$. The following corollary was proved for entire Hamiltonian vector fields in [FS3] with quite different methods. Theorem 4.3 allows us to provide a new proof of this result.

COROLLARY 4.4.   *There exists a dense G_δ and a G' of Hamiltonians in $\mathcal{E}$ satisfying the quasi-ergodic hypothesis. In other words, for $H \in G'$ there is a dense G_δ subset $\mathcal{C}_H \subset \mathbb{C}$ such that, for $c \in \mathcal{C}_H$, there is a dense orbit on $\{p = c\}$.*

Proof. We know by Theorem 4.3 that the quasi-ergodic hypothesis is valid on a dense subset of $\mathcal{E}$; we need only check that it is valid on a G_δ.

For this, take $p \in \mathcal{P}_d$ satisfying the quasi-ergodic hypothesis and let $\Sigma = p^{-1}(c)$ with a dense orbit. Cover $\Sigma \cap \bar{B}(0, N)$ by a finite number of balls $\{U_j\}_{1 \leq j \leq q(N)}$ of radius $< 1/m$ (m and N are positive integers). Let $\mathcal{U} = \mathcal{U}(p, c, \varepsilon, m, N) \subset \mathcal{E}$ be the open set of Hamiltonians H such that:

(i) $\|p - H\|_{L^\infty(\bar{B}(0, N))} < \varepsilon$;

(ii) there exists a neighborhood $V_1(c)$ of c such that $c' \in V_1(c)$ implies

$$H^{-1}(c') \cap B(0, N) \subset \bigcup_1^{q(N)} U_j;$$

and

(iii) there exists a neighborhood $V_2(c)$ of c such that, for $c' \in V_2(c)$ and for all j_1, j_2, there is an orbit on $H^{-1}(c')$ intersecting U_{j_1} and U_{j_2}.

It is clear that $\mathcal{U}$ is a neighborhood of p in $\mathcal{E}$.

Now the union—of these $\mathcal{U}(p, c, \varepsilon, m, N)$ for p in $\bigcup_{d \geq 4} \mathcal{P}_d$ satisfying the quasi-ergodic hypothesis and c such that there is a dense orbit on $p^{-1}(c)$—is an open dense subset $\mathcal{U}(\varepsilon, m, N)$ of $\mathcal{E}$. The intersection $G' = \bigcap_{\varepsilon, m, N} \mathcal{U}(\varepsilon, m, N)$ provides the desired G_δ (we pick a sequence $\varepsilon_n \to 0$).

Indeed, if $H \in G'$, let $\mathcal{V}_{m,N}$ be the set of $c \in \mathbb{C}$ such that there is a covering of $H^{-1}(c) \cap B(0, N)$ by finitely many balls of radius $1/m$ and an orbit of X_H that intersects all these balls. Then $\mathcal{V}_{m,N}$ is open and dense in $\mathbb{C}$. And if $c \in \bigcap_{m,N} \mathcal{V}_{m,N}$, then the flow of X_H is topologically transitive on $H^{-1}(c)$.   □

5. Exploding Orbits

First, we need to recall well-known facts concerning Fatou–Bieberbach (F.B.) domains. An F.B. domain is an open $\Omega \subset \mathbb{C}^2$, biholomorphic to $\mathbb{C}^2$ and such that $\bar{\Omega} \neq \mathbb{C}^2$. Such domains frequently arise in holomorphic dynamics of automorphisms of $\mathbb{C}^2$.

Consider a so-called Hénon mapping

$$f \colon \mathbb{C}^2 \to \mathbb{C}^2,$$

$$(z, w) \mapsto (aw + q(z), az),$$

with $|a| < 1$ and where q is a polynomial of degree at least 2. Suppose that 0 is an attracting point; then its immediate basin of attraction Ω is an F.B. domain, which is Runge and has the following remarkable property [BS]: Ω intersects each algebraic curve on a nonempty relatively compact set.

This fact can be used to construct vector fields with many exploding orbits (see [F]); we recall here a basic example. Let $\phi \colon \mathbb{C}^2 \to \Omega$ be the F.B. map, and consider for instance a constant vector field V in Ω: all real orbits are real lines and so cut $\partial \Omega$ in finite time. Thus, all real orbits of $(\phi^{-1})_* V$ (where $(\cdot)_*$ denotes the usual push-forward on vector fields) explode.

Let $\mathcal{E}$ be the space of entire functions in $\mathbb{C}^2$, which is seen as the space of holomorphic Hamiltonians.

THEOREM 5.1.   *There is a dense subset $G \subset \mathcal{E}$ such that, for every $H \in G$, the set of points with nonexploding orbits for X_H is contained in an at most countable union of real hypersurfaces.*

Proof. With notation as before, the operator $\phi^* \colon \mathcal{O}(\Omega) \to \mathcal{O}(\mathbb{C}^2) = \mathcal{E}$, $\phi^*(g) = g \circ \phi$, is a continuous (topology of uniform convergence on compact sets) isomorphism. Let $\mathcal{P}$ be the space of holomorphic polynomials of two variables, $\mathcal{P} = \bigcup_{d \geq 1} \mathcal{P}_d$. Since Ω is Runge, $\phi^*(\mathcal{P})$ is a dense subset of $\mathcal{E}$, and if $\mathcal{P}'$ is dense in $\mathcal{P}$ then so is $\phi^*(\mathcal{P}')$ in $\mathcal{E}$. Take as a $\mathcal{P}'$ the union for $d \geq 4$ of the full measure subsets of Theorem 4.2; of course, $\mathcal{P}'$ is dense in $\mathcal{P}$. We claim that $G = \phi^*(\mathcal{P}')$ satisfies the assertion of the theorem.

First, note that if the eigenvalues of the Hénon mapping f at 0 are nonresonant (e.g., $0 < |\lambda_1| < |\lambda_2|$ and $|\lambda_2|^2 < |\lambda_1|$) then the F.B. map ϕ is $\lim_{n\to\infty}(d_0 f)^{-n} f^n$ and is of constant Jacobian determinant 1 (this is also true without the nonresonance assumption [RR; Ste]), so ϕ preserves the symplectic form: $\phi^*(dz \wedge dw) = dz \wedge dw$.

Hence ϕ preserves Hamiltonian vector fields [AM]; that is, $\phi_*(X_L) = X_{L \circ \phi^{-1}}$. In particular, if $L = p \circ \phi \in \phi^*(\mathcal{P}')$ then $\phi_*(X_L) = X_p$. This means that the (real) integral curves γ_t of X_L are the $\phi^{-1} \circ \sigma_t$, where σ_t are the integral curves of X_p. Then we are done: γ_t explodes if and only if σ_t reaches $\partial \Omega$ in finite time, and because $\{p = c\} \cap \Omega$ is bounded for any c, this happens if σ_t is unbounded and so the theorem follows from Theorem 4.2. $\square$

REMARK.   We cannot expect to obtain (Baire-) generic properties in $\mathcal{E}$, owing to Corollary 4.4.

6. Exploding Orbits in $\mathbb{C}^{2k}$

Here $\mathcal{E}$ denotes the space of entire functions in $\mathbb{C}^{2k}$. We want to give a proof of the following theorem.

THEOREM 6.1.   *There exists a dense subset $G \subset \mathcal{E}$ such that, if $H \in G$, then the set of points in $\mathbb{C}^{2k}$ with exploding orbits for X_H is a dense G_δ.*

We recall that in $\mathbb{C}^{2k}$ (coordinates $(z_1, \ldots, z_k, w_1, \ldots, w_k)$) provided with the symplectic form $\omega = \sum_i dz_i \wedge dw_i$, the holomorphic Hamiltonian vector field associated with the Hamiltonian H is

$$X_H = \left(\frac{\partial H}{\partial w_1}, \ldots, \frac{\partial H}{\partial w_k}, -\frac{\partial H}{\partial z_1}, \ldots, -\frac{\partial H}{\partial z_k} \right);$$

here X_H is tangent to the hypersurfaces $\{H = c\}$.

The following theorem is due to Fornæss and Sibony [FS2].

THEOREM 6.2.   *There exists a dense G_δ subset $G' \subset \mathcal{E}$ such that, if $H \in G'$, then the set of points in $\mathbb{C}^{2k}$ with unbounded orbits for X_H is a dense G_δ.*

It is an easy exercise to see that in this theorem one can replace $\mathcal{E}$ by the space $\mathcal{P}_d$ of holomorphic polynomials of degree $\leq d$ in $\mathbb{C}^{2k}$ ($d \geq 3$, d fixed). Indeed, the main argument of the proof is that bounded open sets of bounded orbits are unstable under small perturbations (replacing X_p by $X_{e^{i\theta}p}$) of the Hamiltonian.

Proof of Theorem 6.1.

Step 1: A Fatou–Bieberbach domain. The technique of proof is the same as in Section 5. Let f be the following polynomial automorphism of $\mathbb{C}^{2k}$:

$$f : (z_1, \ldots, z_k, w_1, \ldots, w_k) \mapsto (aw_1 + q(z_1), \ldots, aw_k + q(z_k), az_1, \ldots, az_k),$$

with $|a| < 1$, q a polynomial of degree ≥ 2, and 0 as an attracting fixpoint. Note that f is a regular automorphism [S] and the basin of attraction of 0 is a Fatou–Bieberbach domain Ω. In homogeneous coordinates in $\mathbb{P}^{2k}$, $[Z_1 : \cdots : Z_k : W_1 : \cdots : W_k : T]$, we have

$$\bar{\Omega} \cap \{T = 0\} = I^+(\bar{f}) = [0 : \cdots : 0 : W_1 : \cdots : W_k : 0] =: I^+$$

[S]; it is a $(k - 1)$-dimensional linear subspace of $\{T = 0\} \cong \mathbb{P}^{2k-1}$. If for each $d \geq 3$ we find a G_δ dense subset $G_d \subset \mathcal{P}_d$ such that, for $p \in G_d$, the generic real

orbits of X_p cut $\partial\Omega$ in finite (positive and negative) time, then Theorem 6.1 will be proven (see Section 5).

Step 2: The complex orbits. We study here the leaves of the singular holomorphic foliation by curves $\mathcal{F}$ on $\mathbb{P}^{2k}$ generated by the vector field X_p for generic $p \in \mathcal{P}_d$. In what follows, L_x denotes the leaf of $\mathcal{F}$ through x; if $x \in \Omega$, then Λ_x is the connected component of $L_x \cap \Omega$ containing x and $O^+(x)$ is the real-time positive orbit through x. We want to show that, near I^+, $\mathcal{F}$ has a nice behavior for generic p.

We compute the expression of X_p after a change of coordinates near $\{T = 0\}$, for example, in the chart $\{W_1 \neq 0\}$. One has

$$z_1 = Z_1/T, \ldots, z_k = Z_k/T \quad \text{and} \quad w_1 = W_1/T, \ldots, w_k = W_k/T;$$

set

$$x_1 = \frac{Z_1}{W_1}, \ldots, x_k = \frac{Z_k}{W_k}, \quad y_2 = \frac{W_2}{W_1}, \ldots, y_k = \frac{W_k}{W_1}, \quad t_1 = \frac{T}{W_1}.$$

The generic point is

$$m = (z_1, \ldots, z_k, w_1, \ldots, w_k) = (x_1/t_1, \ldots, x_k/t_1, 1/t_1, y_2/t_1, \ldots, y_k/t_1),$$

where the two expressions make sense. The expression of $X_p(m)$ in the new chart is

$$\sum_{i=1}^{k} \left[t_1 \frac{\partial p}{\partial w_i}(m) + x_i t_1 \frac{\partial p}{\partial z_1}(m) \right] \frac{\partial}{\partial x_i} + \sum_{j=2}^{k} \left[-t_1 \frac{\partial p}{\partial z_j}(m) + y_j t_1 \frac{\partial p}{\partial z_1}(m) \right] \frac{\partial}{\partial y_j}$$

$$+ t_1^2 \frac{\partial p}{\partial z_1}(m) \frac{\partial}{\partial t_1}.$$

To obtain a holomorphic vector field near $\{t_1 = 0\}$, we need to multiply X_p by t_1^{d-2} for generic p (outside a proper algebraic subset of $\mathcal{P}_d$). Then the singularities of $\mathcal{F}$ in $\{T = 0\}$ are the solutions of the homogeneous system of generically independent $2k - 1$ equations

$$\frac{1}{z_1}\frac{\partial p_d}{\partial w_1} = \cdots = \frac{1}{z_k}\frac{\partial p_d}{\partial w_k} = \frac{1}{w_1}\frac{\partial p_d}{\partial z_1} = \cdots = \frac{1}{w_k}\frac{\partial p_d}{\partial z_k},$$

where p_d is the highest-degree term of p.

As a consequence we get that, for (Zariski-) generic p (see Section 2 for methods):

(i) $\{T = 0\}$ and $\{T = 0\} \cap \{\tilde{p} = 0\}$ are $\mathcal{F}$-invariant;
(ii) $\mathrm{Sing}(\mathcal{F})$ is 0-dimensional and $\mathrm{Sing}(\mathcal{F}) \cap I^+ = \emptyset$; and
(iii) there are no tangencies between $\mathcal{F}$ and I^+.

For the last property, note that I^+ is the $(k-1)$-dimensional subset of $\{T = 0\} \cong \mathbb{P}^{2k-1}$ parametrized by $[0 : \cdots : 0 : W_1 : \cdots : W_k : 0]$; moreover, in an affine chart of $\{T = 0\}$, $\mathcal{F}$ is generated by a vector field $X = (X_1, \ldots, X_{2k-1})$ as before.

Now the set of points where X is parallel to I^+ is $\{X_1 = \cdots = X_k = 0\}$, which is again $(k-1)$-dimensional. These two sets generically do not intersect.

REMARK.   It is possible to show that the indeterminacy set $I^+(\bar{f})$ of the projectivization of a polynomial automorphism of $\mathbb{C}^{2k}$ preserving ω has dimension $\geq k - 1$.

Step 3. There exists a neighborhood V of I^+ in $\mathbb{C}^{2k}$ such that, for Baire-generic p, there is an open V_1 dense in V such that all real orbits of X_p starting at $V_1 \cap \Omega$ cut $\partial\Omega$ in finite positive and negative time.

Take p Zariski-generic satisfying the conclusions of step 2, and fix a neighborhood of I^+ free of singular points. If $x \in I^+$ then L_x is transverse to I^+; we can straighten $\mathcal{F}$ in a neighborhood N of x in $\mathbb{P}^{2k}$. In the straightened chart, I^+ is a submanifold nowhere tangent to the foliation, $\mathcal{F}$ is parallel to $\{T = 0\}$, and $\bar{\Omega} \cap \{T = 0\} = I^+$. This ensures that if $y \in \Omega$ is close enough to x then the connected component Λ_y of $L_y \cap \Omega$ satisfies $\Lambda_y \subset\subset N$ and $\overline{\Lambda_y} \cap \{T = 0\} = \emptyset$; hence Λ_y is bounded in $\mathbb{C}^{2k}$ and biholomorphic to a bounded domain in $\mathbb{C}$.

Because the leaves of $\mathcal{F}$ are straightened in N, we can use the methods of Sections 3 and 4 here (beware that, in the foliated chart, the vector field X_p is tangent to a set of straight lines but not constant). If the positive orbit of $y_1 \in \Lambda_y$ does not cut $\partial\Omega$ in finite time then it is periodic, since Λ_y is biholomorphic to a bounded open set in $\mathbb{C}$ (remember, there are no critical points in V). Also, for all but countably many θ, $X_{e^{i\theta}p}$ has no open sets of periodic orbits in V and periodic orbits remain in a union of real submanifolds of codimension ≥ 1.

The complement of this union of submanifolds is not open a priori, and it remains to check that the dense set of points of $V \cap \Omega$ with orbits leaving Ω contains an open V_1: just note that the condition "$O^+(x)$ leaves $\bar{\Omega}$" is open. We can do the same for negative orbits. This proves step 3.

To conclude the proof of Theorem 6.1, take (Baire-generic) p satisfying steps 2 and 3. Let $x \in \mathbb{C}^{2k}$; then $\Lambda_x \subset\subset \mathbb{C}^{2k}$. Indeed, if $L_x \cap \Omega$ had an unbounded component then it would reach any neighborhood of I^+, which contradicts the local picture given in step 3: near I^+, the leaves are parallel to the hyperplane at infinity and the connected components Λ_y are bounded. Hence, if $O^+(x)$ is unbounded then it cuts $\partial\Omega$ in finite time. So if p is taken in the Baire-generic family of Theorem 6.2—which comes down to breaking open sets of periodic orbits by multiplying by $e^{i\theta}$—then we are done.   □

REMARKS.   1. Following these techniques, one can easily show the same kind of results for volume-preserving vector fields in $\mathbb{C}^n$.

2. As in the C^∞ case [AM], it is possible to find an invariant measure on generic level sets for a polynomial Hamiltonian vector field (at least for generic p). Indeed, assume $\{X_p = 0\} \cap \{p = 0\} = \emptyset$; then, by Hilbert's Nullstellensatz, there exists a polynomial holomorphic $(2k - 1, 0)$-form σ such that $\sigma \wedge dp = (1 - Ap)\omega^k$, where A is a polynomial. One then easily checks that $\sigma|_{p=0}$ is an X_p-invariant form (this is a local condition) and that $i\sigma \wedge \bar{\sigma}$ is an invariant volume form on $\{p = 0\}$.

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