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Smooth Structure of Some Symplectic Surfaces

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1. Introduction

McMullen and Taubes [MT] have constructed a remarkable simply connected smooth 4-manifold, denoted by X, starting from a 4-component link $K \subset S^3$ and four copies of the rational elliptic surface $E(1)$. The interest in the link K stems from the fact that it admits several inequivalent fibrations over S^1; these inequivalent fibrations give rise to two inequivalent symplectic structures on X, providing the first simply connected example of manifold with this property. The ingredients in the construction of [MT] are reminiscent of those used by Fintushel and Stern in defining a large class of smooth 4-manifolds, and it is natural to ask how these constructions are related. In this note we will compare the link surgery construction of [FS] and the McMullen–Taubes example in order to prove that the latter manifold is diffeomorphic to a Fintushel–Stern manifold. This analysis (further developed in [V]) will lead us to introduce a new presentation of X that allows us to identify a new symplectic structure on X. We will assume some familiarity with [FS] and [MT].

2. Construction of the 4-Manifolds

We start by recalling the link surgery construction of [FS], omitting (for the sake of brevity) full generality. Consider an n-component oriented link $K \subset S^3$. Let $p_i = -\sum_{j \neq i} lk(K_i, K_j)$. The closed manifold M_K obtained by performing p_i-surgery on the ith component has the property that the image m_i of each meridian $\mu(K_i)$ has infinite order in $H_1(M_K, \mathbb{Z})$ and is canonically framed; in $S^1 \times M_K$, the tori $S^1 \times m_i$ have self-intersection zero and are framed and essential in homology. Next take n copies of the simply connected elliptic surface without multiple fibers $E(m)$, each containing an elliptic fiber F_i, and construct, by normal connected sum, the manifold

$$E(m)_K = \coprod E(m)_i \#_{F_i = S^1 \times m_i} S^1 \times M_K. \tag{1}$$

The gluing is made so as to send the homology class of the normal circle to the ith torus $S^1 \times m_i$, represented by $p_i m_i + l_i$ (where l_i is the image of the preferred longitude $\lambda(K_i)$) to the class of a normal circle to the ith elliptic fiber. These

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prescriptions can be insufficient to uniquely define the manifold: the gluing map is defined up the action of $\mathrm{SL}(3,\mathbb{Z})$ matrices of the form

$$\begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{pmatrix}; \tag{2}$$

since F is in the neighborhood of a cusp fiber in $E(m)$, we can dispose of the indeterminacy corresponding to the upper left $\mathrm{SL}(2,\mathbb{Z})$ factor (due to the absence of a canonical choice for the basis of $H_1(F,\mathbb{Z})$) because any fiber- and orientation-preserving diffeomorphism of $\partial(E(m)\setminus \nu F)$ extends to a (fiber-preserving) diffeomorphism of $E(m)\setminus \nu F$ (see [GS, Chap. 8]); the symbol $\nu(\cdot)$ denotes the open neighborhood of an embedded submanifold. The remaining indeterminacy, however, cannot be disposed of in general. The manifold $E(m)_K$ is simply connected and has $b_2^+ \geq n$.

We will discuss now the example of McMullen and Taubes. Consider, in S^3, the 4-component oriented link K given by the union of the Borromean rings $K_1 \cup K_2 \cup K_3$ and the axis of $\mathbb{Z}_3$-symmetry K_4. Let $N := S^3 \setminus \nu K$. We recall the form of the Alexander polynomial $\Delta_K(x, y, z, t)$ of K; here x, y, z are the variables corresponding to the meridians of the Borromean rings and t corresponds to the meridian to the axis:

$$\begin{aligned}\Delta_K&(x,y,z,t)\\ &= -4 + (t+t^{-1}) + (x + x^{-1} + y + y^{-1} + z + z^{-1})\\ &\quad - (xy + (xy)^{-1} + yz + (yz)^{-1} + xz + (xz)^{-1}) + (xyz + (xyz)^{-1}).\end{aligned} \tag{3}$$

We have another description for N: perform 0-surgery on S^3 along the components of the Borromean rings; it is well known that this surgery yields T^3. We can thus write $N = S^3\setminus \nu K = T^3\setminus \nu L$, where L is a framed link in T^3 whose first three components give a basis of $H_1(T^3,\mathbb{Z})$. In fact, when we perform the 0-surgery on the Borromean rings, the three meridians $\mu(K_i)$ ($i = 1, 2, 3$) to the components of the Borromean rings go over longitudes m_i of L_i, while the preferred longitudes $\lambda(K_i)$ are sent to meridians l_i of L_i. The longitude of K_4 becomes a longitude to the component $L_4 \subset T^3$, which satisfies the relation $L_4 = L_1 + L_2 + L_3 \in H_1(T^3,\mathbb{Z})$; the meridian $\mu(K_4)$ of K_4 goes instead to a meridian m_4 of L_4 and is null-homotopic in T^3. It is instead nontrivial in $H_1(N,\mathbb{Z})$, where the four generators are given by the meridians. We have $H^1(N,\mathbb{Z}) \supset i^*H^1(T^3,\mathbb{Z}) = \mathbb{Z}\langle t\rangle^{\perp}$. Then define the normal connected sum

$$X = \coprod E(1)_i \#_{F_i = S^1\times L_i} S^1 \times T^3. \tag{4}$$

Again, the definition requires that the homology class of the normal circle to $S^1\times L_i$ be sent to the homology class of the normal circle to the ith elliptic fiber. The previous remarks on the ambiguity of the definition apply. This manifold is simply connected and has $b_2^+ > 1$.

We show now that both constructions appear as particular cases of a general construction: consider the exterior of an oriented n-component link $K \subset S^3$ together with the choice, in each boundary component, of an homology basis of simple curves (α_i, β_i) of intersection 1. We introduce the following definition.

DEFINITION 2.1. Take a link K as above with homology basis (α_i, β_i) and an elliptic surface $E(m)$. Define the manifold

$$E(m; \alpha_i, \beta_i) = \left(\coprod E(m)_i \setminus \nu F_i\right) \cup_{F_i \times S^1 = S^1 \times \alpha_i \times \beta_i} (S^1 \times (S^3 \setminus \nu K)), \qquad (5)$$

where the gluing is made by lifting a diffeomorphism between $S^1 \times \alpha_i$ and F_i to an orientation-reversing diffeomorphism of the boundary tori in such a way that the homology class of β_i is sent to the homology class of the normal circle to the ith elliptic fiber.

The gluing condition is not enough to define the manifold completely. As in the case of Fintushel–Stern manifolds, the ambiguity related to the absence of a chosen basis in $H_1(F_i, \mathbb{Z})$ is only apparent whereas the remaining ambiguity is effective. Moreover, the smooth manifold (as the notation suggests) can depend on the choice of the (α_i, β_i), with the noteworthy exception considered in the following lemma.

LEMMA 2.2. *Let $E(1; \alpha_i, \beta_i)$ be defined as before. Then the manifold is well-defined and moreover its definition depends uniquely on K; that is, it is unaffected by the choice of the basis on $\partial(S^3 \setminus \nu K)$.*

Proof. This follows from the fact that *any* orientation-preserving diffeomorphism of $\partial(E(1) \setminus \nu F)$, and not only the fiber-preserving ones, extends to an orientation-preserving diffeomorphism of $(E(1) \setminus \nu F)$ (see [GS]): on each boundary component we can reabsorb any orientation-preserving self-diffeomorphism of $S^1 \times \alpha_i \times \beta_i$ by an orientation-preserving self-diffeomorphism of $\partial(E(1)_i \setminus \nu F_i)$, which extends to $E(1)_i \setminus \nu F_i$. No matter how we glue the manifold $S^1 \times (S^3 \setminus \nu K)$ (in particular, for any choice of homology basis for the boundary), the resulting four manifolds are smoothly equivalent. $\square$

Analyzing the previous construction yields the following straightforward proposition.

PROPOSITION 2.3. *The Fintushel–Stern manifolds $E(m)_K$ and the McMullen–Taubes manifold X can be described via the construction in Definition 2.1.*

Proof. The definition of normal connected sum shows that the manifolds defined in equation (1) can be rewritten in the form

$$E(m)_K = \left(\coprod E(m)_i \setminus \nu F_i\right) \cup (S^1 \times (S^3 \setminus \nu K)), \qquad (6)$$

where the gluing is made by lifting a diffeomorphism between $S^1 \times \mu(K_i)$ and F_i to an orientation-reversing diffeomorphism of the boundary tori so that the homology class of $p_i \mu(K_i) + \lambda(K_i)$ is sent to the class of the normal circle to F_i. Hence the manifold $E(m)_K$ corresponds to the choice $(\alpha_i, \beta_i) = (\mu(K_i), p_i \mu(K_i) + \lambda(K_i))$.

Concerning the McMullen–Taubes example, an analysis of the definitions via normal connected sum of equation (4) (keeping track of the framing of L_i) shows, as $T^3 \setminus \nu L = S^3 \setminus \nu K$, that X corresponds to $m = 1$ and to the choice $(\alpha_i, \beta_i) = (\mu(K_i), \lambda(K_i))$ for $i = 1, 2, 3$ and $(\alpha_4, \beta_4) = (\lambda(K_4), -\mu(K_4))$. □

Note that the latter definition differs from the Fintushel–Stern one, applied to the same link, for the different choice of the homology basis. However, in this particular case, we have our next lemma.

LEMMA 2.4. *The McMullen–Taubes manifold X is diffeomorphic to the Fintushel–Stern manifold $E(1)_K$.*

Proof. This follows as particular case of Lemma 2.2. The same argument implies also that the manifold is well-defined. □

3. Symplectic Structures

We now want to compare the symplectic structure arising naturally from the different presentations of X. The proof of the existence of symplectic structures on X follows by application of Gompf's theorem on the symplectic normal connected sum between $\coprod_i E(1)$ and $S^1 \times M_K$ (resp., $S^1 \times T^3$) in the Fintushel–Stern (resp., McMullen–Taubes) construction. Both M_K and T^3 are fibered 3-manifolds obtained by Dehn filling of $S^3 \setminus \nu K$ along the different surgery curves. For any choice of a fiber Σ in M_K (resp., T^3) transverse to the image of the link, $E(1)_K$ (resp., X) inherits a natural symplectic structure induced from the closed, nondegenerate 1-form defining the fibration on $S^3 \setminus \nu K$. For any link K, fibrations on $S^3 \setminus \nu K$ are identified with the elements of $H^1(S^3 \setminus \nu K, \mathbb{Z})$ laying on the cones over some of the top-dimensional faces of the Thurston unit sphere. The latter is defined, for $\varphi \in H^1(S^3 \setminus \nu K, \mathbb{Z})$, by minimizing the quantity

$$\chi(\Sigma) = \sum_{\chi(\Sigma_i)<0} (-\chi(\Sigma_i)) \tag{7}$$

among properly embedded representatives Σ of the Poincaré dual of φ and then extending by linearity and continuity to real cohomology classes. The fibration on M_K restricts by construction (see [FS]) to the fibration of $S^3 \setminus \nu K$ with fiber given by the minimal spanning surface of the link K, that is, to the class $(1, 1, 1, 1) \in H^1(S^3 \setminus \nu K, \mathbb{Z})$. On T^3, as discussed in [MT], every fibration that restricts to the cone over the top-dimensional faces of the Thurston unit sphere on $i^*H^1(T^3, \mathbb{Z}) \subset H^1(S^3 \setminus \nu K, \mathbb{Z})$ induces a symplectic structure on X. We can relate the fibration of class $(1, 1, 1, 1)$ and the fibrations laying in $i^*H^1(T^3, \mathbb{Z})$: the analysis of the Thurston norm on $H^1(S^3 \setminus \nu K, \mathbb{Z})$, detailed in [MT], shows that $(1, 1, 1, 1)$ lies in the cone over the top-dimensional face identified by the dual vertex xyz (we use the same notation as equation (3)), a face that already contains fibered elements of $i^*H^1(T^3, \mathbb{Z})$. As a consequence, the canonical bundle corresponding to the symplectic structure induced on X by this fibration cannot be used to distinguish it from the ones exhibited in [MT].

Let's now discuss how we can produce a new symplectic structure that can be distinguished from the known ones by studying the canonical class. The unit sphere of the Thurston norm of $S^3 \setminus \nu K$ is given, as discussed in [MT], by the product of the unit sphere in the subspace $i^* H^1(T^3, \mathbb{Z})$ and the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ of the orthogonal subspace: every fibered face is determined by a dual vertex among the sixteen vertices of the Newton polyhedron of the Alexander polynomial. We can represent the orthogonal subspace to $i^* H^1(T^3, \mathbb{Z})$ as a pullback under inclusion of the first cohomology group of $S^1 \times S^2$: in fact, 0-surgery on the axis K_4 of the Borromean ring exhibits N as complement of a link $\hat{L}$ in $S^1 \times S^2$. The images of the meridians $\mu(K_i)$ for $i = 1, 2, 3$ are (null-homotopic) meridians to the components of $\hat{L}$ with the same index; $\mu(K_4)$ goes to a preferred longitude of $\hat{L}_4$. The longitudes $\lambda(K_i)$ for $i = 1, 2, 3$ go to preferred longitudes of the respective $\hat{L}_i$, while $\lambda(K_4)$ goes to a meridian to $\hat{L}_4$. The fiber of $S^1 \times S^2$ restricts to the fiber of $S^3 \setminus \nu K$ identified by the cohomology class $(0, 0, 0, 1) \in H^1(S^3 \setminus \nu K, \mathbb{Z})$ (a disk spanning the axis, pierced once by each component of the Borromean rings). We have now the following.

Definition 3.1. Consider the framed symplectic tori $S^1 \times \hat{L}_i \subset S^1 \times S^1 \times S^2$ of self-intersection zero together with four copies of the rational elliptic surface $E(1)$. We define the normal connected sum

$$Y = \coprod E(1)_i \#_{F_i = S^1 \times \hat{L}_i} S^1 \times S^1 \times S^2. \tag{8}$$

The definition of normal connected sum imposes that the homology class of the normal circle to $S^1 \times \hat{L}_i$ be sent over the homology class of the normal circle to the ith elliptic fiber.

This definition immediately yields our next proposition.

Proposition 3.2. *The manifold Y introduced in Definition 3.1 is a manifold of type $E(1; \alpha_i, \beta_i)$; it is, moreover, diffeomorphic to X and to the Fintushel–Stern manifold $E(1)_K$.*

Proof. The first statement follows by observing that the definition of Y corresponds to the choice $S^1 \times S^2 \setminus \nu\hat{L} = S^3 \setminus \nu K$ and to the homology basis $(\alpha_i, \beta_i) = (\lambda(K_i), -\mu(K_i))$ for $i = 1, 2, 3$ and $(\alpha_4, \beta_4) = (\mu(K_4), \lambda(K_4))$. The second statement is a corollary, as Lemma 2.4, of Proposition 2.2. □

The construction of X introduced in Definition 3.1 induces naturally a symplectic structure on the manifold: the fibration of $S^3 \setminus \nu K$ with class $(0, 0, 0, 1)$ has dual vertex t, as we can see by looking at the Alexander polynomial in equation (3). Theorem 3.4 of [MT] identifies the canonical bundle of this symplectic structure as the image of twice this vertex under the injective map $H_1(S^3 \setminus \nu K, \mathbb{Z}) \to H^2(X, \mathbb{Z})$. This canonical bundle has different valence, as vertex of the Newton polyhedron of the SW polynomial, than the canonical bundles obtained from the previous two construction of X (see [MT]) and so is combinatorially different. As a consequence, it lies in a different orbit with respect to the action of the diffeomorphism group of X that acts by preserving the Newton polyhedron. This proves the following.

THEOREM 3.3. *The symplectic structure induced by normal connected sum on Y is not equivalent* (*up to combination of pullback and homotopies*) *to the previous ones.*

The Seiberg–Witten polynomial of X is given by $\Delta_K(x^2, y^2, z^2, t^2)$; the new symplectic structure (and its conjugate), together with the fourteen constructed in [MT], exhaust the sixteen basic classes with coefficient ± 1.

In [V] we discuss how these constructions can be extended to obtain further generalizations of the Fintushel–Stern link surgery construction.

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