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k-Plane Transforms and Related Operators on Radial Functions

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1. Introduction

In 1917, J. Radon proved that a smooth function in $\mathbb{R}^3$ is completely determined by its integrals over all the planes. This leads in a more general setting to consideration of the so-called k-plane transform. Let f be a smooth function in $\mathbb{R}^n$ and let $1 \le k < n$ be an integer. Denote by $G(n,k)$ the set (called the Grassmannian manifold) of all k-dimensional subspaces (or k-planes) of $\mathbb{R}^n$. The k-plane transform of f is defined as

$$Tf(x,\pi) = \int_\pi f(x-y)\,d\lambda_k(y)$$

for $x \in \mathbb{R}^n$ and $\pi \in G(n,k)$, where λ_k denotes the Lebesgue measure on π. When $k = 1$ this operator is usually named X-ray transform; when $k = n-1$, Radon transform. Such transformations have many practical and theoretical applications (see e.g. the references in [S]).

The properties of the k-plane transform depend on the properties of f. Here we are concerned with a size estimate measured in terms of a mixed norm inequality, namely,

$$\left(\int_{G(n,k)}\left(\int_{\pi^\perp}|Tf(x,\pi)|^q\,d\lambda_{n-k}(x)\right)^{r/q} d\gamma_{n,k}(\pi)\right)^{1/r} \le C_{p,q,r}\|f\|_p. \tag{1.1}$$

Here $\pi^\perp$ denotes the subspace orthogonal to π and $\gamma_{n,k}$ is the rotation-invariant measure on $G(n,k)$ (see [M, Chap. 3] for a construction of $\gamma_{n,k}$ and some of its properties). When inequality (1.1) holds for some p, the definition of the k-plane transform can be extended to $f \in L^p$ and $Tf(x,\pi)$ is finite for almost every translate of almost every k-plane.

A scaling argument replacing $f(x)$ by $f(\lambda x)$ shows that (1.1) is possible only if

$$\frac{n}{p} - \frac{n-k}{q} = k.$$

Moreover, checking the inequality against the characteristic function of a parallelepiped of sides $1 \times \delta \times \cdots \times \delta$, we can see that the condition

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$$\frac{n-k}{r} \geq \frac{1}{p'}$$

is also necessary.

In [DO], the X-ray transform appears when applying to potential-type operators a classical and useful tool in harmonic analysis, the method of rotations introduced by Calderón and Zygmund in 1956 to study homogeneous singular integral operators. Mixed norm inequalities are again needed, but now the order of the norms is reversed. For the k-plane transform, an inequality of this type would read as

$$\left(\int_{\mathbb{R}^n} \left(\int_{G(n,k)} |Tf(x,\pi)|^r \, d\gamma_{n,k}(\pi) \right)^{q/r} dx \right)^{1/q} \leq C_{p,q,r} \|f\|_p. \qquad (1.2)$$

In this case, the scaling argument gives

$$\frac{n}{p} - \frac{n}{q} = k$$

as a necessary condition. Moreover, taking as f the characteristic function of the unit ball, it follows that $|Tf(x,\pi)| \sim 1$ for large x when π is in a subset of $G(n,k)$ of a $\gamma_{n,k}$-measure $c|x|^{k-n}$ (use Lemma 3.11 of [M]). The integrability at infinity of the left-hand side of (1.2) gives the restriction

$$\frac{n-k}{r} > \frac{n}{q} = \frac{n}{p} - k.$$

More restrictions (which are not of interest for us in this paper) appear when using characteristic functions of parallelepipeds with some small sides. The case $k = 1$ was completely settled in the aforementioned paper.

When applied to the characteristic function χ_E of a set E, $T\chi_E(x,\pi)$ gives the k-dimensional Lebesgue measure of the intersection of E with the translate of π through x. Besicovitch constructed a plane set of measure zero that contains a unit segment in every direction, a construction that he later applied to solve the Kakeya needle problem. The existence of such irregular sets for higher dimensions and k-planes is an interesting question in geometric measure theory that has been only partially answered (see [F, Chap. 7]). In particular, a Besicovitch-type set shows that (1.1) must be false for $k = 1$ and $q = \infty$ because, for each $\varepsilon > 0$, we can construct a set of measure smaller than ε for which the left-hand side of (1.1) is at least 1.

The precise range of values of p, q, r for which inequality (1.1) holds is known only if $k \geq n/2$; partial results have been proved for $k < n/2$, but improving them seems to be a hard task (see [C] or the survey [Dr]; [W] contains a more recent result for $k = 1$). The aim of our paper is to study inequalities (1.1) and (1.2) for radial functions. On the one hand, we thus avoid sets and functions with irregularities in many directions; on the other hand, the range of validity of all the inequalities is larger when we consider radial functions. The only restriction required in (1.1) is given by the scaling argument, and for (1.2) the restriction imposed by the characteristic function of the ball must be added. Our first theorem states that both inequalities hold for the remainder values of p, q, and r.

THEOREM 1. *For radial functions, inequality* (1.1) *holds if and only if*

$$1 \le r \le \infty, \quad 1 \le p < \frac{n}{k}, \quad \frac{n}{p} - \frac{n-k}{q} = k;$$

inequality (1.2) *holds if and only if*

$$1 < p < \frac{n}{k}, \quad \frac{n}{p} - \frac{n}{q} = k, \quad \frac{n-k}{r} > \frac{n}{p} - k.$$

Actually, in [DO] the X-ray transform appeared as an element in a scale of potential-type directional operators whose counterpart over k-planes would be

$$T_\alpha f(x,\pi) = \int_\pi f(x-y)|y|^{\alpha-k}\, d\lambda_k(y)$$

for $0 < \alpha \le n$. We are interested in mixed norm inequalities of type (1.2) for T_α.

THEOREM 2. *For radial functions, inequality* (1.2) *holds for* T_α *if and only if*

$$1 < p < \frac{n}{\alpha}, \quad \frac{n}{p} - \frac{n}{q} = \alpha, \quad \frac{n-k}{r} > \frac{n}{p} - k.$$

Representing the points $(1/p, 1/r)$ for which a positive result holds in Theorem 2 inside the unit square, they describe a trapezoid if $\alpha < k$ and a triangle if $\alpha \ge k$.

There is a natural Hardy–Littlewood maximal function associated with the k-planes; it is defined as

$$Mf(x,\pi) = \sup_{R>0} \frac{1}{R^k} \int_{\{y\in\pi : |y|<R\}} |f(x-y)|\, d\lambda_k(y).$$

When $k = 1$, this operator corresponds to the directional maximal operator. Mixed norms in the cases $k = 1$ and $k = n-1$ were studied in [CDR], with partial results in the first case and complete results in the second. Directional maximal operators can be used to control a very interesting operator in harmonic analysis, the Kakeya maximal operator (see Section 5). A positive answer to the conjecture on mixed norm inequalities in the case $k = 1$ would solve the Kakeya operator problem, which is considered very hard. On the other hand, $Mf(x,\pi)$ appears to be a good substitute for $T_\alpha(x,\pi)$ when $\alpha = 0$. We once again restrict ourselves to radial functions and establish the following pointwise inequality, which is of independent interest and provides helpful inequalities to prove Theorem 2.

THEOREM 3. *Let E be a radial set of finite measure in $\mathbb{R}^n$, and let χ_E be its characteristic function. Then*

$$M\chi_E(x,\pi) \le C M_{\mathrm{HL}}\chi_E(x)^{k/n} \quad \forall x \in \mathbb{R}^n,\ \pi \in G(n,k),$$

where M_{HL} denotes the usual Hardy–Littlewood maximal operator in $\mathbb{R}^n$. The constant C depends only on n and k.

An immediate consequence of Theorem 3 is the following.

COROLLARY 4. *The operator* $f \mapsto \sup_\pi Mf(\cdot,\pi)$ *is bounded on* $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ *if* $p > n/k$, *and it is of restricted weak type for* $p = n/k$.

Here, "restricted weak type" means that it satisfies a weak-type inequality when restricted to characteristic functions (of radially symmetric sets, in our case). This is equivalent to saying that the operator applies $L^{n/k,1}_{\mathrm{rad}}$ into $L^{n/k,\infty}$. (For the definition of these Lorentz spaces and the equivalence with the restricted weak type see [SW], where the interpolation theorems used in this paper also appear.) Corollary 4 for $k = 1$ was proved in [CHS] using a different approach. The method we present here is simpler and extends better to $k > 1$. Notice that by using the Cartesian product of the previously mentioned Besicovitch-type set in the plane with the unit ball in $\mathbb{R}^{n-2}$, we obtain a counterexample to Corollary 4 for general functions.

We denote by L^p_{rad} the subspace of L^p formed by the radial functions, and we use the notation $A \sim B$ to indicate that the quotient A/B is bounded above and below by absolute positive constants depending only on k and n. The constant C can vary even within a single chain of inequalities.

2. Proof of Theorem 1

Inequality (1.1) holds trivially when $p = 1$, $q = 1$, and $r = \infty$, because for $f \ge 0$ we have

$$\int_{\pi^\perp} Tf(x,\pi)\,d\lambda_{n-k}(x) = \|f\|_1$$

using Fubini's theorem. On the other hand, we have the identity (see [S])

$$\int_{\mathbb{R}^n} g(x)\,dx = \int_{G(n,n-k)} \int_{\pi^\perp} |y|^{n-k} g(y)\,d\lambda_k(y)\,d\gamma_{n,n-k}(\pi).$$

The one-to-one correspondence $\pi \in G(n,k)$ with $\pi^\perp \in G(n,n-k)$ allows the identification of these manifolds and their associated measures up to a constant factor; this implies that

$$\int_{G(n,k)} Tf(x,\pi)\,d\gamma_{n,k}(\pi) = cI_k f(x),$$

where I_k is the Riesz potential of order k (i.e., the convolution operator with kernel $|x|^{k-n}$). From the well-known boundedness properties of this operator we deduce that (1.2) holds for $r = 1$, $1 < p < n/k$, and q given by the scaling relation. It is also known that, for $p = 1$ and $q = n/(n-k)$, a weak-type inequality holds. This result will be useful in this proof of Theorem 1. We remark that both results are true even if the function f is not radial.

The rest of the proof is based on an endpoint critical estimate that is the same in (1.1) and (1.2), namely, the case $p = n/k$, $q = \infty$, $r = \infty$. Although the inequality will not hold for every radial function f, it holds when f is the characteristic function of a set. Then we need only prove the following.

LEMMA 5. *Let E be a radially symmetric set in $\mathbb{R}^n$, and let Π be a translate of a k-plane of $\mathbb{R}^n$. Then there is a constant, depending only on k and n, such that*

$$\lambda_k(E \cap \Pi) \leq C|E|^{k/n}. \tag{2.1}$$

Proof.

Case $k = 1$. Although this case was already proved in [DO], we include here its elementary proof based on the following observation: the measure of the annulus $\{x : r < |x| < r + \varepsilon\}$ for a fixed ε is an increasing function of r.

Assume $\lambda_1(E \cap \Pi) = L$. If $0 \in \Pi$ then (by our observation) the measure of E is minimum when $E \cap \Pi$ is a segment of length L centered at the origin, so that $|E| \geq cL^n$. If $d = \operatorname{dist}(0, \Pi) > 0$, let x_0 be the point in Π closest to the origin. Only the part of E outside the ball $\{x : |x| < d\}$ intersects Π and, again in this case, the minimum measure corresponds to the case of a segment of length L centered at x_0 and contained in Π. Then $|E| \geq c[(d^2 + (L/2)^2)^{n/2} - d^n] \sim c \max(d^{n-2}L^2, L^n) \geq cL^n$.

Case $k \geq 2$. Using an approximation argument, we can assume without loss of generality that E is a finite union of open spherical annuli; that is,

$$E = \bigcup_{j=0}^{N} \{x : r_j < |x| < r_j + \varepsilon_j\}, \tag{2.2}$$

where $r_j + \varepsilon_j < r_{j+1}$ and $\varepsilon_j \leq r_j$ if $j \geq 1$ and where the term for $j = 0$ appears only if $r_0 = 0$. Then

$$|E| \sim \varepsilon_0^n + \sum_{j=1}^{N} r_j^{n-1} \varepsilon_j.$$

Let $d = d(0, \Pi)$. As in the case $k = 1$, we distinguish two possibilities: $d = 0$ and $d > 0$. For $d = 0$, the left-hand side of (2.1) is

$$\lambda_k(E \cap \Pi) \sim \varepsilon_0^k + \sum_{j=1}^{N} r_j^{k-1} \varepsilon_j.$$

Then we need to prove

$$\left(\sum_{j=1}^{N} r_j^{k-1} \varepsilon_j\right)^n \leq C\left(\sum_{j=1}^{N} r_j^{n-1} \varepsilon_j\right)^k. \tag{2.3}$$

The left-hand side of (2.3) can be written as

$$\sum_{j_1, \dots, j_n = 1}^{N} r_{j_1}^{k-1} \varepsilon_{j_1} r_{j_2}^{k-1} \varepsilon_{j_2} \dots r_{j_n}^{k-1} \varepsilon_{j_n},$$

which in turn is bounded by 2^n times

$$\sum_{j_1 \leq j_2 \leq \dots \leq j_n} r_{j_1}^{k-1} \varepsilon_{j_1} r_{j_2}^{k-1} \varepsilon_{j_2} \dots r_{j_n}^{k-1} \varepsilon_{j_n}.$$

Using that $r_j, \varepsilon_j \le r_m$ if $j < m$, we can replace the factors corresponding to the subscripts $j_1, \dots, j_{n-k}$ by $r_{j_{n-k+1}}^{n-k} \cdots r_{j_n}^{n-k}$ and so obtain part of the sum of the right-hand side of (2.3).

Assume now that $d > 0$. Only those parts of E outside the ball $\{x : |x| < d\}$ are of interest now. Let j_0 the smallest j for which $\{x : |x| > d,\ r_j < |x| < r_j + \varepsilon_j\}$ is not empty. Define s_j and δ_j as follows:

$$d^2 + s_j^2 = r_j^2, \qquad d^2 + (s_j + \delta_j)^2 = (r_j + \varepsilon_j)^2. \tag{2.4}$$

(If $r_{j_0} < d < r_{j_0} + \varepsilon_{j_0}$ we define $s_{j_0} = 0$.) Then $E \cap \Pi$ is a union of k-dimensional spherical annuli of inner radii s_j and width δ_j, so that

$$\lambda_k(E \cap \Pi) \sim \sum_{j=j_0}^{N} \max(s_j^{k-1}\delta_j, \delta_j^k).$$

From the definition of s_j and δ_j we have

$$2s_j\delta_j + \delta_j^2 \le 3r_j\varepsilon_j, \quad s_j, \delta_j \le Cr_j$$

and consequently $\max(s_j^{k-1}\delta_j, \delta_j^k) \le Cr_j^{k-1}\varepsilon_j$ since $k \ge 2$.

This ends the proof of the lemma. □

Fix $\pi \in G(n,k)$. Then the operator $f \mapsto Tf(\cdot, \pi)$ is bounded from $L^1(\mathbb{R}^n)$ to $L^1(\pi^\perp)$, as mentioned before, and from the Lorentz space $L^{n/k,1}_{\text{rad}}(\mathbb{R}^n)$ into $L^\infty(\pi^\perp)$ by Lemma 5. Then, using real interpolation for Lorentz spaces (see [SW]), we deduce that the operator is bounded from L^p_{rad} to L^q with p and q related by the scaling condition. Since the bounds are independent of π, we deduce the first part of Theorem 1 for $r = \infty$ and hence for all r.

To handle the second part of Theorem 1, we fix r $(1 < r < \infty)$ and let E be a radially symmetric set of finite measure. Using Lemma 5 then yields

$$\begin{aligned}
&\int_{G(n,k)} (T\chi_E(x,\pi))^r \, d\gamma_{n,k}(\pi) \\
&\quad \le \sup_{\pi \in G(n,k)} (T\chi_E(x,\pi))^{r-1} \int_{G(n,k)} T\chi_E(x,\pi)\, d\gamma_{n,k}(\pi) \\
&\quad \le C|E|^{(r-1)k/n} I_k\chi_E(x).
\end{aligned}$$

Using now the weak $(1, n/(n-k))$ inequality for the Riesz potential I_k, we deduce that

$$\left|\left\{x : \int_{G(n,k)} (T\chi_E(x,\pi))^r \, d\gamma_{n,k}(\pi) > t^r\right\}\right| \le Ct^{-\frac{rn}{n-k}}|E|^{1+\frac{kr}{n-k}}. \tag{2.5}$$

This is a weak-type inequality for the operator that sends f to

$$\left(\int_{G(n,k)} (Tf(x,\pi))^r \, d\gamma_{n,k}(\pi)\right)^{1/r}$$

(restricted to characteristic functions). If p_0 and q_0 are given by $(n-k)/r = (n/p_0) - k = n/q_0$, then (2.5) means that the operator is bounded from $L^{p_0,1}_{\text{rad}}$ into $L^{q_0,\infty}$. Since it is also bounded from $L^{n/k,1}_{\text{rad}}$ to L^∞, we can again use real interpolation to deduce for each r the result stated in Theorem 1.

We remark that our proof also gives endpoint results in Lorentz spaces.

3. Proof of Theorem 3

Given a, b such that $0 < a < b < \infty$, denote by $A_{a,b}$ the annulus $\{x : a < |x| < b\}$. Define the maximal function on annuli centered at the origin as

$$\mathcal{A}f(x) = \sup_{x \in A_{a,b}} \frac{1}{|A_{a,b}|} \int_{A_{a,b}} |f(y)|\, dy.$$

Given a set $D \subset \mathbb{R}^n$, we define its annular extension as

$$A[D] = \{x \in \mathbb{R}^n : |x| = |y| \text{ for some } y \in D\}.$$

We begin the proof of Theorem 3 by proving that, given a k-ball B lying on a translate of a k-plane Π and a radially symmetric set E in $\mathbb{R}^n$, there exists a constant C depending only on k and n such that

$$\frac{\lambda_k(B \cap E)}{\lambda_k(B)} \leq C\left(\frac{|A[B] \cap E|}{|A[B]|}\right)^{k/n}. \tag{3.1}$$

From this we deduce at once the pointwise inequality

$$\sup_{\pi \in G(n,k)} M\chi_E(x, \pi) \leq C(\mathcal{A}\chi_E(x))^{k/n}.$$

Theorem 3 will be a consequence of the following claim: If f is a radial function, then

$$M_{\mathrm{HL}} f(x) \sim \mathcal{A}f(x). \tag{3.2}$$

Proof of (3.1) *for* $k = 1$. In this case B is a line segment whose length we denote by L; let $\lambda_1(B \cap E) = \ell$. Then the left-hand side of (3.1) is ℓ/L. We use the geometric observation already stated in the proof of Theorem 1 that the minimum measure of $A[B] \cap E$ corresponds to the case when $B \cap E$ is a segment (of length ℓ and contained in B) that is as close to the origin as possible.

Assume first that $0 \in \Pi$ and let $d(0, B) = r$. If $r \leq L$, then $|A[B]| \sim L^n$ and $|A[B] \cap E| \geq c \max(r^{n-1}\ell, \ell^n) \geq c\ell^n$ so that (3.1) holds. If $r > L$, then $|A[B]| \sim r^{n-1}L$ and $|A[B] \cap E| \sim r^{n-1}\ell$. Since $\ell/L \leq 1$, (3.1) holds.

Let now $d = d(0, \Pi) > 0$ and let x_0 be the point in Π closest to the origin; let $D = d(x_0, B)$. If $L, D \leq d$, then $|A[B]| \sim d^{n-2}L \max(D, L)$ and $|A[B] \cap E| \geq cd^{n-2}\ell \max(D, \ell)$; if $D \leq d$ and $L > d$, then $|A[B]| \sim L^n$ and $|A[B] \cap E| \geq c \max(d^{n-2}\ell^2, \ell^n) \geq c\ell^n$. If $D > d$ and $L \leq D$, then $|A[B]| \sim D^{n-2}L^2$ and $|A[B] \cap E| \geq cD^{n-2}\ell^2$; finally, if $D > d$ and $L > D$, then $|A[B]| \sim L^n$ and $|A[B] \cap E| \geq c \max(D^{n-2}\ell^2, \ell^n) \geq c\ell^n$. In all cases, (3.1) holds. □

Proof of (3.1) *for* $k \geq 2$. Let B be the ball of center c_B and radius R contained in Π. Then $\lambda_k(B) \sim R^k$. As in the proof of Lemma 5, we assume that E can be written as a union of spherical annuli similar to (2.2). We again distinguish several cases.

Assume first that $0 \in \Pi$. If $|c_B| \leq 2R$ then $|A[B]| \sim R^n$, $\lambda_k(E \cap B) \leq C \sum_j' r_j^{k-1}\varepsilon_j$, and $|A[B] \cap E| \sim \sum_j' r_j^{n-1}\varepsilon_j$. (The symbol $\sum_j'$ means that only those values of j for which the annulus of index j intersects B are taken into account; the first and the last values can be adjusted to coincide with the inner and outer radii of $A[B]$.) A term of type ε_0^k and ε_0^n (respectively) corresponding to $j = 0$ could appear in each sum. Inequality (3.1) is then a consequence of inequality (2.3).

If $|c_B| > 2R$ then we have $|A[B]| \sim |c_B|^{n-1}R$, $\lambda_k(E \cap B) \sim \sum_j' R^{k-1}\varepsilon_j$, and $|A[B] \cap E| \sim \sum_j' |c_B|^{n-1}\varepsilon_j$. Since $\sum_j' \varepsilon_j \leq 2R$, (3.1) holds.

Let now $d = d(0, \Pi) > 0$ and let x_0 be the point in Π such that $d(0, x_0) = d$. Define s_j and δ_j as in (2.4). Then $\lambda_k(E \cap B) \sim \sum_j' \max(s_j^{k-1}\delta_j, \delta_j^k)$. Since for $r_j \geq 2d$ we have $s_j \sim r_j$ and $\delta_j \sim \varepsilon_j$, when $B \subset \{x : |x| \geq 2d\}$ the situation is reduced to the preceding one.

Write $D = d(x_0, c_B)$. If $D \geq 4d$ and $R \leq D/2$, then $B \subset \{x : |x| \geq 2d\}$ and the result is proved. Let $D \geq 4d$ and $R > D/2$. Then $|A[B]| \sim R^n$, so that (3.1) holds if

$$\left(\sum_j{}' \max(s_j^{k-1}\delta_j, \delta_j^k)\right)^n \leq C\left(\sum_j{}' r_j^{n-1}\varepsilon_j\right)^k.$$

This inequality follows from $\max(s_j^{k-1}\delta_j, \delta_j^k) \leq Cr_j^{k-1}\varepsilon_j$ and (2.3), as in Lemma 5. A similar proof applies when $D \leq 4d$ and $R > d$ because in this case again $|A[B]| \sim R^n$.

We are now left with the case $D \leq 4d$ and $R < d$, for which $|A[B]| = C[(d^2 + (D+R)^2)^{n/2} - (d^2 + (D-R)^2)^{n/2}] \leq Cd^{n-2}\max(DR, R^2)$. If $R \leq D/2$ then $\lambda_k(E \cap B) \leq CR^{k-1}\sum_j' \delta_j$ and $|A[B] \cap E| \sim d^{n-1}\sum_j' \varepsilon_j$. The required inequality is now

$$\left(\frac{\sum_j' \delta_j}{R}\right)^n \leq C\left(\frac{d\sum_j' \varepsilon_j}{DR}\right)^k;$$

but in this situation $s_j \geq D/2$ and $r_j \leq 10d$ $\big($in both cases for the terms in $\sum'\big)$, so that $\delta_j \leq 20dD^{-1}\varepsilon_j$; together with $\sum_j' \delta_j \leq 2R$, this gives the inequality. If $R > D/2$ then we need

$$\left(\frac{\sum_j' \max(s_j^{k-1}\delta_j, \delta_j^k)}{R^k}\right)^n \leq C\left(\frac{d\sum_j' \varepsilon_j}{R^2}\right)^k,$$

which is a consequence of $s_j, \delta_j \leq cR$ and $s_j\delta_j, \delta_j^2 \leq 3r_j\varepsilon_j \leq 30d\varepsilon_j$. □

Proof of (3.2). We prove first that there exists some C_1 such that $M_{\mathrm{HL}}f(x) \leq C_1\mathcal{A}f(x)$. Let B be the ball centered at x with radius R. Let $a = \max(0, |x| - R)$ and $b = |x| + R$. Our aim is to show that

$$\frac{1}{|B|}\int_B |f| \leq C\frac{1}{|A_{a,b}|}\int_{A_{a,b}} |f|. \tag{3.3}$$

If $|x| \leq 2R$, then $|A_{a,b}| \sim R^n$ and $B \subset A_{a,b}$ so that (3.3) holds. If $|x| > 2R$, then $|A_{a,b}| \sim |x|^{n-1}R$; rotating the ball B with respect to the origin, we can get a number N_1 of disjoint balls inside $A_{a,b}$. Geometric considerations show that it is possible to choose $N_1 \geq c_1(|x|/R)^{n-1}$ for some c_1 independent of $|x|$ and R; since f is radial, the integral on all the rotated balls is the same and again (3.3) holds.

We consider now the reverse inequality; that is, suppose there exists a C_2 such that $\mathcal{A}f(x) \le C_2 M_{\mathrm{HL}} f(x)$. Let $A_{a,b}$ be an annulus such that $x \in A_{a,b}$ and define $R = b - a$. If $a \le b/2$, then $|A_{a,b}| \sim b^n \sim R^n$ and $A_{a,b} \subset B(x, 4R)$ so that the converse of (3.3) holds with $B = B(x, 4R)$. If $a > b/2$, then $|A_{a,b}| \sim b^{n-1}R$. Let $B = B(x, 2R)$; we can cover $A_{a,b}$ with N_2 balls obtained by rotation of B in such a way that $N_2 \le c_2(|x|/R)^{n-1}$. Again, the fact that the integral of f on all of these balls is the same gives the converse of (3.3). □

4. Proof of Theorem 2

The proof of Theorem 2 for $k \ge 2$ is similar to the proof given in [DO] for $k = 1$. Here we sketch its main lines.

LEMMA 6. *Assume that f is nonnegative.*

(i) *Let $0 < \beta < \alpha < \gamma \le n$; then*

$$T_\alpha f(x,\pi) \le T_\beta f(x,\pi)^{1-s} T_\gamma f(x,\pi)^s, \quad \alpha = (1-s)\beta + s\gamma.$$

(ii) *Let $0 < \alpha < \gamma \le n$. Then there exists a constant C depending only on α, γ, and k such that*

$$T_\alpha f(x,\pi) \le CMf(x,\pi)^{1-\alpha/\gamma} T_\gamma f(x,\pi)^{\alpha/\gamma}.$$

Proof. Both inequalities are proved in a similar way. Write

$$T_\alpha f(x,\pi) = \int_{|y|<R} + \int_{|y|\ge R} f(x-y)|y|^{\alpha-k}\, d\lambda_k(y).$$

Use the elementary bounds $R^{\alpha-\beta} T_\beta f(x,\pi)$ and $R^{\alpha-\gamma} T_\gamma f(x,\pi)$, respectively, and choose R so that both bounds have the same size; this yields (i).

For (ii), use instead the bound $CR^\alpha Mf(x,\pi)$ for the first integral. It can be obtained decomposing the integration domain into annuli $\{y : 2^{-k-1}R \le |y| < 2^{-k}R\}$ for $k = 0, 1, 2, \ldots$ and then using on each annulus the bound $(2^{-k}R)^\alpha Mf(x,\pi)$. □

Proof of Theorem 2 for $\alpha > k$. Using part (i) of Lemma 6 (with $\beta = k$ and $\gamma = n$) together with Lemma 5, we have

$$T_\alpha \chi_E(x,\pi)^{\frac{n-k}{\alpha-k}} \le T_k \chi_E(x,\pi)^{\frac{n-\alpha}{\alpha-k}} T_n \chi_E(x,\pi) \le |E|^{\frac{n-\alpha}{\alpha-k}\cdot\frac{k}{n}} T_n \chi_E(x,\pi).$$

From here the boundedness from $L^{n/\alpha,1}_{\mathrm{rad}}$ to $L^\infty(L^{(n-k)/(\alpha-k)})$ is immediate, and the end of the proof is as in the second part of Theorem 1. □

Proof of Theorem 2 for $\alpha < k$. Use inequality (ii) of Lemma 6 with $f = \chi_E$ (which implies $M\chi_E(x,\pi) \le 1$) and $\gamma = k$, together with Lemma 5, to get $T_\alpha \chi_E(x,\pi) \le C|E|^{\alpha/n}$.

Use now the same inequality with Lemma 5 to obtain

$$T_\alpha \chi_E(x,\pi) \le CM\chi_E(x,\pi)^{1-\alpha/k} |E|^{\alpha/n},$$

which together with Corollary 4 implies the boundedness of T_α from $L^{n/k,1}_{\mathrm{rad}}$ to $L^{n/(k-\alpha),\infty}(L^\infty)$. The weak estimates for the values of $1/p$ and $1/r$ over the line joining the points $(1,1)$ and $(k/n,0)$ are obtained as before. For fixed r, real interpolation between Lorentz spaces gives Theorem 2. $\square$

5. Some Consequences of Theorem 3

The pointwise inequality of Theorem 3 leads to some interesting consequences for several operators acting on radial functions.

Let D be a set in $\mathbb{R}^n$, star-shaped with respect to the origin and with positive finite measure. If D is described in polar coordinates as $D \setminus \{0\} = \{(\rho,u) \in (0,\infty)\times S^{n-1} : 0<\rho<R(u)\}$, then the measure of D is $n^{-1}\int_{S^{n-1}} R(u)^n\,d\sigma(u)$ (here $d\sigma$ denotes the Lebesgue measure on the unit sphere). Then we have

$$\begin{aligned}\int_D |f(x-y)|\,dy &= \int_{S^{n-1}}\int_0^{R(u)} |f(x-\rho u)|\rho^{n-1}\,d\rho\,d\sigma(u)\\ &\le \int_{S^{n-1}} R(u)^n Mf(x,u)\,d\sigma(u) \le n|D| \sup_{u\in S^{n-1}} Mf(x,u).\end{aligned}$$

For $k=1$, $G(n,1)$ can be identified with a half-sphere and the measure $d\gamma_{n,1}$ with the Lebesgue measure on it. Define the maximal operator $\mathcal{M}$ as follows:

$$\mathcal{M}f(x) = \sup_D \frac{1}{|D|}\int_D |f(x-y)|\,dy,$$

where the supremum is taken over all sets D that are star-shaped with respect to the origin and with positive measure. If E is a radially symmetric set then using Theorem 3 and our previous calculation yields

$$\mathcal{M}\chi_E(x) \le C_n M_{\mathrm{HL}}\chi_E(x)^{1/n}.$$

COROLLARY 7. *The maximal function $\mathcal{M}$ is bounded on $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ for all $p>n$ and is of restricted weak-type (n,n).*

In particular, the Kakeya maximal operator defined as the supremum of averages of f over all parallelepipeds of sides $h\times h\times\cdots\times h\times Nh$ ($h>0$ variable, N fixed) is smaller than $\mathcal{M}$, so it is bounded on $L^p_{\mathrm{rad}}(\mathbb{R}^n)$ for $p>n$ with a constant that is independent of N. This result was obtained in [CHS]. For general functions, the best possible result is a logarithmic growth on N (known only for $n=2$).

A weighted version of Corollary 4 is also possible.

COROLLARY 8. *The operator $f \mapsto \sup_\pi Mf(\cdot,\pi)$ is bounded from $L^p_{\mathrm{rad}}(w)$ to $L^p(w)$ if $p>n/k$ and if w is in the Muckenhoupt class $A_{pk/n}$ and is of restricted weak type for $p=n/k$ with A_1 weights.*

As usual, A_p denotes the class of weights for which the Hardy–Littlewood maximal operator is bounded on $L^p(w)$ (the L^p space with respect to the measure

$w(x)\,dx)$ if $p > 1$ or satisfies a weak-type (1,1) inequality if $p = 1$. For a description of these classes of weights and their main properties, see [GR].

Proof of Corollary 8. From the boundedness of M_{HL} and the pointwise inequality, we deduce that the operator defined in the statement is bounded from $L^{q,1}_{\mathrm{rad}}(w)$ to $L^q(w)$ when $w \in A_{qk/n}$. By interpolation with L^∞ we deduce that, if $p > q$ and $w \in A_{qk/n}$, then the operator is bounded on $L^p_{\mathrm{rad}}(w)$. But the union of the classes $A_{qk/n}$ for all $q < p$ gives $A_{pk/n}$.

The second part of the statement is immediate. □

Notice that the same weighted result with $k = 1$ can be stated for the operator $\mathcal{M}$ considered in Corollary 7.

Finally, we give an application to a rough maximal operator with variable kernel. Let Ω be a function defined on $\mathbb{R}^n \times S^{n-1}$ and define the rough maximal function associated to Ω as

$$M_\Omega f(x) = \sup_{R>0} \frac{1}{R^n} \int_{|y|<R} |\Omega(x, y') f(x-y)|\, dy,$$

where y' denotes the projection of y over S^{n-1} and where

$$\sup_x \int_{S^{n-1}} |\Omega(x,u)|\, d\sigma(u) = A(\Omega) < \infty.$$

Using polar coordinates as before, we have

$$M_\Omega f(x) \le C_n A(\Omega) \sup_{u \in S^{n-1}} Mf(x,u).$$

COROLLARY 9. *The operator M_Ω is bounded from $L^p_{\mathrm{rad}}(w)$ to $L^p(w)$ for $p > n$ and $w \in A_{p/n}$, and it is of restricted weak type (n,n) for $w \in A_1$.*

When Ω is independent of x, the unweighted L^p boundedness is easy to prove (for all $p > 1$ and for general f). But even in such a case the weighted inequalities given in Corollary 9, when we restrict the operator to radial functions, are not always true if we consider general functions.

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