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F-Rational Rings and the Integral Closures of Ideals

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1. Introduction

The history of the Briançon–Skoda theorem and its ensuing avatars in commutative algebra have been well documented in many papers (see e.g. [AH1; LS]). We will therefore only briefly review the relevant concepts and theorems. First recall the definitions of the integral closure of an ideal.

DEFINITION 1.1. Let R be a ring and let I be an ideal of R. An element $x \in R$ is *integral over* I if x satisfies an equation of the form $x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_j \in I^j$ for $1 \leq j \leq n$. The *integral closure* of I, denoted by $\bar{I}$, is the set of all elements integral over I. This set is an ideal.

Let R^o be the set of all elements of R not in a minimal prime. An equivalent though less standard (but for our purposes a more useful) definition of integral closure is the following.

EQUIVALENT DEFINITION 1.1. Let R be a Noetherian ring and let I be an ideal of R. An element $x \in R$ is *integral over* I if there exists an element $c \in R^o$ such that $cx^n \in I^n$ for all $n \gg 0$.

A theorem proved by Briançon and Skoda [BS] for convergent power series over the complex numbers and generalized to arbitrary regular local rings by Lipman and Sathaye states as follows.

THEOREM 1.2 [BS; LS]. *Let R be a regular local ring and let I be an ideal generated by ℓ elements. Then, for all $n \geq \ell$,*

$$\overline{I^n} \subseteq I^{n-\ell+1}.$$

This was partially extended to the class of pseudo-rational rings by Lipman and Teissier [LT]. However, they were unable to recover the full strength of Theorem 1.2.

THEOREM 1.3 [LT, (2.2)]. *Let R be a Noetherian local ring and assume that the localization R_P is pseudo-rational for every prime ideal P in R. Suppose that I has a reduction J such that* $\dim R_P \leq \delta$ *for every associated prime P of J^n. Then*

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$$\overline{I^{n+\delta-1}} \subseteq J^n.$$

In particular, if J can be generated by a regular sequence of length δ, then the above containment holds for all $n \geq 1$.

The present two authors, as well as Lipman, have pushed the original theorem further by introducing "coefficients"; see [AH1; AH2; AHT; L]. The methods used by the present authors have relied on the theory of tight closure. These improvements, however, have been valid only in regular rings, and the question of whether the statement of Theorem 1.2 remains valid in arbitrary pseudo-rational rings has remained open since 1981. Recent progress was made by Hyry and Villamayor [HyV], who proved (among other things) that if R is local Gorenstein and essentially of finite type over a field of characteristic 0, then $\overline{I^{n+\ell-1}} \subseteq I^n$ for an arbitrary ideal I with ℓ generators. In this paper we will use tight closure methods to prove that Theorem 1.2 is valid for F-rational rings (the definition is in Section 2). In characteristic p, Smith [Sm] proved that F-rational implies pseudo-rational, but it can be stronger in general. However, for affine algebras in equicharacteristic 0, the concepts of rational singularity, pseudo-rational singularity, and F-rational type all agree, owing to work of Lipman and Teissier [LT] for the equivalence of rational singularity and pseudo-rational singularity, and of Smith [Sm] and Hara [Ha] and independently Mehta and Srinivas [MS] for the equivalence of rational singularity and F-rational type (Smith proved that rational implies F-rational type and the other authors have just recently proved the converse). It follows from these equivalences that, in equicharacteristic 0, we are able to prove Theorem 1.2 for rational singularities.

The basic idea of this paper is inspired by the proof of a cancellation theorem (see [Hu1]). The key idea is to relate an arbitrary ideal I to a system of parameters in a manner that closely approximates the structure of the powers of I. We do this by using first a basic construction and then a theorem that relates the integral closure of powers of I with the tight closure of the system of parameters. In the next section we briefly discuss tight closure; see [HH1; Hu2] for more references and information.

2. Tight Closure

We begin with the definition.

DEFINITION 2.1. Let R be a Noetherian ring of characteristic $p > 0$. Let I be an ideal of R. An element $x \in R$ is said to be in the *tight closure* of I if there exists an element $c \in R^o$ such that $cx^q \in I^{[q]}$ for all large $q = p^e$, where $I^{[q]}$ is the ideal generated by the qth powers of all elements of I.

Every ideal in a regular ring is tightly closed. We say that elements $x_1, \dots, x_n$ in R are *parameters* if the height of the ideal generated by them is at least n (we allow them to be the whole ring, in which case the height is said to be ∞). If the ideal they generate is proper, then the Krull height theorem says that the height is exactly n.

Definition 2.2. A Noetherian ring R of characteristic $p > 0$ is said to be *F-rational* if the ideals generated by parameters are tightly closed.

This definition arose from the work of Fedder and Watanabe [FW] because of the apparent connection to the concept of rational singularities.

The concept of pseudo-rationality was introduced in [LT], partly as a substitute for the notion of rational singularities in positive and mixed characteristic, where desingularizations are not known to exist in general. Their definition is as follows (see [LT, Sec. 2]).

Definition 2.3. Let $(R, \mathbf{m})$ be a d-dimensional local Noetherian ring. The ring R is said to be *pseudo-rational* if it is normal, Cohen–Macaulay, and analytically unramified and if, for every proper birational map $\pi : W \to X = \operatorname{Spec}(R)$ with W normal and closed fiber $E = \pi^{-1}(\mathbf{m})$, the canonical map

$$H^d_{\mathbf{m}}(\pi_*(\mathcal{O}_W)) = H^d_{\mathbf{m}}(R) \to H^d_E(\mathcal{O}_W)$$

is injective.

In [LT] it is proved that, for a local ring essentially of finite type over a field of characteristic 0, the notions of pseudo-rational and rational singularity agree. In [Sm] it is shown that, in positive characteristic, F-rational implies pseudo-rational. Smith uses this to prove that rings of finite type over a field of characteristic 0 that are F-rational type have rational singularities. Here "F-rational type" essentially means that characteristic-p models of the variety are F-rational. More precisely, we next introduce the idea of a model.

Let R be a ring that is finitely generated over a field of characteristic 0, say $R = k[X_1, \dots, X_n]/I$. Then we can choose generators for the ideal I and, by collecting coefficients of those generators, find a finitely generated $\mathbf{Z}$-algebra $A \subseteq k$ such that defining $R_A = A[X_1, \dots, X_n]/(I \cap A[X_1, \dots, X_n])$ yields $R = k \otimes_A R_A$. We call the map $A \to R_A$ a family of *models* of R. We sometimes insist that the map $A \to R_A$ be flat, which one can always obtain by expanding A by localizing at a single element. A typical closed fiber of R_A over A is a characteristic-p model of R.

Definition 2.5. Let R be a finitely generated algebra over a field of characteristic 0. Then R is said to have *F-rational type* if R admits a family of models $A \to R_A$ in which a Zariski dense set of closed fibers are F-rational. (This does not depend on the choice of models.)

The theorem in [Sm] states that, if X is a scheme of finite type over a field of characteristic 0, then if X has F-rational type it has only rational singularities. Recently, the converse has been proved by Hara [Ha] and independently by Mehta and Srinivas [MS].

3. F-Rational Rings and Tight Closure

In this section we first discuss a basic construction that will play a crucial role in the paper. Given an ideal I in a Noetherian local ring $(R, \mathbf{m})$, a minimal reduction

J of I—say, $J = (a_1, \ldots, a_\ell)$—and an integer N, we wish to construct an ideal $\mathfrak{A}$ generated by parameters such that $J \equiv \mathfrak{A}$ modulo $\mathbf{m}^N$ and such that $\mathfrak{A}$ is closely related to I and its powers. For example, one would like $I \subseteq \mathfrak{A}$, but this is in general not possible since I may not be contained in any ideal generated by parameters. We record what we need in Proposition 3.2. We need the following lemma from [AHT].

LEMMA 3.1. *Let $(R, \mathbf{m})$ be a local ring with infinite residue field and let $I \subseteq R$ be an ideal of analytic spread ℓ. Let $J \subseteq I$ be a minimal reduction of I. Then there exists a "basic" generating set $a_1, \ldots, a_\ell$ for J such that*

(1) *if P is a prime ideal containing I and $\operatorname{ht} P = i \leq \ell$ then $(a_1, \ldots, a_i)_P$ is a reduction of I_P, and*
(2) $\operatorname{ht}((a_1, \ldots, a_i)I^n : I^{n+1} + I) \geq i + 1$ *for all $n \gg 0$.*
(3) *If $c_i \equiv a_i$ modulo I^2, then* (1) *and* (2) *hold with c_i replacing a_i.*

Proof. The first two statements are found in [AHT, Lemma 7.2]. The last statement follows from the proof of Lemma 7.2 in [AHT]. The choice of a basic generating set depends only on the images of the a_i in the associated graded ring $G = R/I \oplus I/I^2 \oplus \cdots$. In particular, since c_i and a_i have the same leading forms in G, (3) follows. □

PROPOSITION 3.2. *Let $(R, \mathbf{m})$ be an equidimensional and catenary local ring with infinite residue field and let $I \subseteq R$ be an ideal of analytic spread ℓ. Let $J \subseteq I$ be a minimal reduction of I. We assume that $\operatorname{ht} I = g$ and $J = (a'_1, \ldots, a'_\ell)$, a basic generating set for J as in Lemma 3.1. Let N and w be fixed integers, and suppose that for $g + 1 \leq i \leq \ell$ we are given finite sets of primes $\Lambda_i = \{Q_{ji}\}$ all containing I and of height i. Then there exist elements $a_1, \ldots, a_\ell$ and $t_{g+1}, \ldots, t_\ell$ such that the following hold* (*we set $t_i = 0$ for $i \leq g$ for convenience*):

(1) $a_i \equiv a'_i$ *modulo I^2*;
(2) *for $g + 1 \leq i \leq \ell$, $t_i \in \mathbf{m}^N$*;
(3) *$b_1, \ldots, b_g, b_{g+1}, \ldots, b_\ell$ are parameters, where $b_i = a_i + t_i$*;
(4) *if R/I is equidimensional then the images of $t_{g+1}, \ldots, t_\ell$ in R/I are parameters*;
(5) *there is an integer M such that $t_{i+1} \in (J_i^t I^M : I^{M+t})$ for all $0 \leq t \leq w + \ell$, where $J_i = (a_1, \ldots, a_i)$*;
(6) *$t_{i+1} \notin \bigcup_j Q_{ji}$, where the union is over the primes in Λ_i.*

Proof. We choose the a_i and t_i inductively. We first modify $a'_1, \ldots, a'_g$ to $a_1, \ldots, a_g$ in such a way that these elements form parameters. We can do this with $a_i \equiv a'_i$ modulo I^2 for $1 \leq i \leq g$. Suppose we have chosen $a_1, \ldots, a_i$ and $t_1, \ldots, t_i$ so that all six of the listed statements are true for these elements. Fix the minimal primes $P_1, \ldots, P_k$ (all necessarily of height i) above $B_i = (b_1, \ldots, b_i)$. Divide them into two sets: let $P_1, \ldots, P_n$ be the ones that contain I, and let $P_{n+1}, \ldots, P_k$ be those that don't contain I. We first change a'_{i+1} to an element $a_{i+1} \equiv a'_{i+1}$ modulo J^2 such that $a_{i+1} \notin \bigcup_{j=n+1}^k P_j$. This choice is possible because the nilradical of J is the same as the nilradical of I. Next choose M_i such that the height of $I + (J_i I^{M_i} : I^{M_i+1})$ is at least $i+1$, and choose M to be the maximum of the M_i. (This is possible by Lemma 3.1.) This choice forces all $(J_i^t I^M : I^{M+t}) + I$ to be of height at least $i+1$ for all $t \geq 0$. For suppose that $(J_i^t I^M : I^{M+t}) + I \subseteq Q$, where Q is a prime of height at most i. Since $I \subseteq Q$, this forces $(J_i I^M : I^{M+1}) \nsubseteq Q$, and after localization at Q we have $(I^{M+1})_Q = (J_i I^M)_Q$. But this forces $(I^{M+t})_Q = (J_i^t I^M)_Q$ for all integers t, and so $(J_i^t I^M : I^{M+t}) \nsubseteq Q$, a contradiction. Using prime avoidance, choose

$$t_{i+1} \in \bigcap_{t=0}^{w+\ell} (J_i^t I^M : I^{M+t}) \cap \mathbf{m}^N \cap \left(\bigcap_{j=n+1}^{k} P_j\right)$$

and

$$t_{i+1} \notin \left(\bigcup_{j=1}^{n} P_j\right) \cup \left(\bigcup_j Q_{ji}\right).$$

This is possible because I is contained in each of the primes in the second line and all these primes have height i, while the height of $I + (J_i^t I^M : I^{M+t})$ is at least $i+1$. We set $b_{i+1} = a_{i+1} + t_{i+1}$. We claim this choice proves (1)–(6) for these new elements. Our choice of a_{i+1} and t_{i+1} make statements (1), (2), (5), and (6) trivial. To prove (3) we need only prove $b_{i+1} \notin \bigcup_{j=1}^{k} P_j$. If $j \leq n$, then $a_{i+1} \in I \subseteq P_j$ while $t_{i+1} \notin P_j$. Hence $b_{i+1} \notin P_j$. If $j \geq n+1$, then $a_{i+1} \notin P_j$ while $t_{i+1} \in P_j$. Again $b_{i+1} \notin P_j$, proving (3). Statement (4) follows from (3). Clearly the height of $(I, b_{g+1}, \ldots, b_{i+1})$ is at least that of $b_1, \ldots, b_{i+1}$ and hence at least $i+1$. But $(I, b_{g+1}, \ldots, b_{i+1}) = (I, t_{g+1}, \ldots, t_{i+1})$. Since R is equidimensional and catenary, it follows that the images of the t_j in R/I form parameters. □

Theorem 3.3. *Let $(R, \mathbf{m})$ be an equidimensional and catenary local ring of characteristic p having infinite residue field. Let I be an ideal of analytic spread ℓ and positive height g. Let J be a minimal reduction of I. Fix $w, N \geq 0$. Choose a_i and t_i as in Proposition 3.2. Set $\mathfrak{A} = B_\ell = (b_1, \ldots, b_g, \ldots, b_\ell)$. Then*

$$\overline{I^{\ell+w}} \subseteq (\mathfrak{A}^{w+1})^*.$$

Proof. Our choice of elements means that $a_1, \ldots, a_g, a_{g+1} + t_{g+1}, \ldots, a_{i+1} + t_{i+1}$ is part of a system of parameters. Fix the notation as in Proposition 3.2. By our choice of the t_j we have that $t_j I^{M+k} \subseteq J_{j-1}^k I^M$ for all $1 \leq k \leq w + \ell$. We first claim that this implies

$$t_j^n I^{M+nk} \subseteq J_{j-1}^{nk} I^M$$

for all $n \geq 1$. Assume this is true for a fixed n, and multiply by $t_j I^k$. We obtain that $(t_j I^k)(t_j^n I^{M+nk}) \subseteq (t_j I^k) J_{j-1}^{nk} I^M$. Since $t_j I^{M+k} \subseteq J_{j-1}^k I^M$, we now have

$$t_j^{n+1} I^{M+(n+1)k} \subseteq J_{j-1}^{nk} J_{j-1}^k I^M$$

as required. Fix $c \in I^M \cap R^o$. Note that the above containment shows that, for all $n \geq 1$,

$$c t_j^{n+1} I^{(n+1)k} \subseteq J_{j-1}^{(n+1)k}. \tag{3.4}$$

Set $B_i = (b_1, \ldots, b_i)$. Let $g \leq i \leq \ell$ and $w \geq r \geq 0$. We show by induction that $c^{i-g} J_i^{(i+r)q} \subseteq (B_i^{r+1})^{[q]}$. The base case is when $i = g$ and $r \leq w$ is arbitrary.

In this case $J_g^{(g+r)q} \subseteq (J_g^{r+1})^{[q]} = (B_g^{r+1})^{[q]}$. The first inclusion in the above line follows at once from [HH1, proof of (5.4)].

Assume now that we are given r and $i > g$ and that the claim is true either for $i' < i$ (with $r' \leq w$ arbitrary) or for $i' = i$ (with $r' < r \leq w$). By our choice of c and of the t_j,

$$\begin{aligned} c^{i-g}J_i^{(i+r)q} &\subseteq c^{i-g}J_i^{[q]}J_i^{(i+r-1)q} \\ &\subseteq c^{i-g}[J_g^{[q]}J_i^{(i+r-1)q} + a_{g+1}^q J_i^{(i+r-1)q} + \cdots + a_i^q J_i^{(i+r-1)q}] \\ &= c^{i-g-1}[cJ_g^{[q]}J_i^{(i+r-1)q} + ca_{g+1}^q J_i^{(i+r-1)q} + \cdots + ca_i^q J_i^{(i+r-1)q}]. \end{aligned}$$

Consider a typical term in this sum, $ca_j^q J_i^{(i+r-1)q}$, where $g+1 \leq j \leq i$. Since $b_j = a_j + t_j$, we can write this term as

$$ca_j^q J_i^{(i+r-1)q} = cb_j^q J_i^{(i+r-1)q} - ct_j^q J_i^{(i+r-1)q}.$$

Using (3.4) (note $i + r - 1 \leq w + \ell$), we obtain

$$ca_j^q J_i^{(i+r-1)q} \subseteq cb_j^q J_i^{(i+r-1)q} + J_{j-1}^{(i+r-1)q}$$

and so

$$\begin{aligned} c^{i-g}J_i^{(i+r)q} \subseteq c^{i-g-1}[cJ_g^{[q]}J_i^{(i+r-1)q} &+ (cb_{g+1}^q J_i^{(i+r-1)q} + J_g^{(i+r-1)q}) \\ &+ \cdots + (cb_i^q J_i^{(i+r-1)q} + J_{i-1}^{(i+r-1)q})], \end{aligned}$$

which by the induction hypothesis is contained in

$$\begin{aligned} J_g^{[q]}(B_i^r)^{[q]} + b_{g+1}^q(B_i^r)^{[q]} + (B_b^{r+i-g})^{[q]} + \cdots + b_i^q(B_i^r)^{[q]} + (B_{i-1}^{r+1})^{[q]} \\ \subseteq (B_i^{r+1})^{[q]}. \end{aligned}$$

In particular, note that

$$c^{\ell-g}J_\ell^{(\ell+r)q} \subseteq (B_\ell^{r+1})^{[q]} \tag{3.5}$$

for all $r \leq w$.

We now prove that $\overline{I^{\ell+w}} \subseteq (\mathfrak{A}^{w+1})^*$. Let $u \in \overline{I^{\ell+w}}$. Choose an element $d \in R^o$ such that $du^q \subseteq J^{(\ell+w)q}$. Then $c^{\ell-g}du^q \in c^{\ell-g}J^{(\ell+w)q} \subseteq (B_\ell^{w+1})^{[q]}$ by (3.5). It follows that $u \in (B_\ell^{w+1})^* = (\mathfrak{A}^{w+1})^*$. □

REMARK. Theorem 3.3 is still valid even if $\operatorname{ht}(I) = 0$. In this case, choose $c_1 \in I^M$ and c_2 in the intersection of the minimal primes of 0 that do not contain I and avoiding those that do contain I. Thus $c_2I^N = 0$ for $N \gg 0$ and $c = c_1 + c_2 \in R^o$ satisfies equation (3.4).

An almost immediate consequence is one of our main theorems.

THEOREM 3.6. *Let $(R, \mathbf{m})$ be an F-rational local ring of positive characteristic p, and let $I \subseteq R$ be an ideal generated by ℓ elements. Then $\overline{I^{\ell+w}} \subseteq I^{w+1}$ for all $w \geq 0$.*

Proof. There is no loss of generality in assuming that R has an infinite residue field. We can replace I by a minimal reduction of itself; suppose that J is that minimal reduction. The number of generators of J is at most ℓ, so without loss of generality we may assume ℓ is the number of generators of J. Fix an integer N. We think of w as fixed and choose $t_{g+1}, \dots, t_\ell$ and $a_1, \dots, a_\ell$ as in Proposition 3.2. In particular, $t_i \in \mathbf{m}^N$ for all i. By (3.3), $\overline{I^{\ell+w}} \subseteq (\mathfrak{A}^{w+1})^* = \mathfrak{A}^{w+1} \subseteq J^{w+1} + (t_{h+1}, \dots, t_\ell) \subseteq J^{w+1} + \mathbf{m}^N$. The equality $(\mathfrak{A}^{w+1})^* = \mathfrak{A}^{w+1}$ above follows from [A, Thm. 1.1]. By the Krull intersection theorem we obtain that $\overline{I^{\ell+w}} \subseteq \bigcap_N (J^{w+1} + \mathbf{m}^N) = J^{w+1}$. □

This characteristic-p theorem allows us to prove the same result in equicharacteristic 0.

THEOREM 3.7. *Let R be an algebra of finite type over a field of characteristic 0 and having only rational singularities. Let $I \subseteq R$ be an ideal generated by ℓ elements. Then $\overline{I^{\ell+w}} \subseteq I^{w+1}$ for all $w \geq 0$.*

Proof. By the work of both Hara [Ha] and Mehta and Srinivas [MS], R is of F-rational type. It is straightforward to prove in this case that, if the conclusion holds in a dense open set of fibers in some family of models $A \to R_A$ of R, it also holds in R. Hence we may pass to positive characteristic and assume that R is finitely generated over a field of characteristic $p > 0$ such that R_P is F-rational for all primes P. The conclusion will follow if we prove it locally, since the number of generators can only drop after localization. It follows that we can reduce to the local F-rational case and apply Theorem 3.6 to finish the proof. □

4. F-Rational Gorenstein Rings

Our next theorem is new, even for R regular, as far as we know. The proof is based on a careful analysis of the proof of Theorem 3.5 together with the ideas behind the cancellation theorem of [Hu1] (see also [CP] for further cancellation results). Our main theorem applies to rings that are F-rational and Gorenstein. It is known [HH2, (3.4), (4.7)] that F-rational and F-regular are the same when the base ring is Gorenstein. A ring R is *F-regular* if every ideal is tightly closed in every localization of R. Of course, all regular rings are F-regular, but the class of F-regular rings is considerably broader than that of regular rings.

THEOREM 4.1. *Let $(R, \mathbf{m})$ be an F-rational Gorenstein local ring of dimension d and having positive characteristic. Suppose that I is an ideal of height g and analytic spread $\ell > g$ with R/I Cohen–Macaulay. Then, for any reduction J of I, $\overline{I^{\ell-1}} \subseteq J$.*

Proof. There is no loss of generality in assuming that R has an infinite residue field and that J is a minimal reduction. Fix an integer N and set $w = 0$ in the notation of Proposition 3.2 and Theorem 3.3. We will prove that $\overline{I^{\ell-1}} \subseteq J + \mathbf{m}^N$. An application of the Krull intersection theorem then finishes the proof.

We choose $t_{g+1}, \dots, t_\ell$ and $a_1, \dots, a_\ell$ as in Proposition 3.2, with N fixed as before. Let $b_i = a_i + t_i$ for $1 \le i \le \ell$. Choose $\mathbf{x} = x_{\ell+1}, \dots, x_d$ so that $b_{g+1}, \dots, b_\ell, \mathbf{x}$ is a regular sequence on R/I and set $\mathfrak{A} = (b_1, \dots, b_\ell, \mathbf{x})$. We set $D = J_g : t_{g+1}$ and $K = (J_g, b_{g+2}, \dots, b_\ell, \mathbf{x})$.

Let $Q = (I, b_{g+2}, \dots, b_\ell, \mathbf{x}) + K : D$. We claim that $\mathfrak{A} : t_{g+1} \subseteq Q$. Suppose that

$$t_{g+1}u = w + vb_{g+1}, \tag{4.2}$$

where $w \in K$. Then $t_{g+1}(u - v) \in (J_{g+1}, b_{g+2}, \dots, b_\ell, \mathbf{x})$ and hence

$$\begin{aligned} u - v \in (J_{g+1}, b_{g+2}, \dots, b_\ell, \mathbf{x}) : b_{g+1} &\subseteq (I, b_{g+2}, \dots, b_\ell, \mathbf{x}) : b_{g+1} \\ &\subseteq (I, b_{g+2}, \dots, b_\ell, \mathbf{x}) \end{aligned}$$

since R/I is Cohen–Macaulay. Hence $u - v \in Q$ and to prove $u \in Q$ it suffices to show that $v \in K : D$. Let $d \in D$ and consider dv. Using (4.2), we obtain that $t_{g+1}du = dw + dvb_{g+1}$ and hence $dvb_{g+1} \in (J_g, b_{g+2}, \dots, b_\ell, \mathbf{x})$. Thus

$$Dv \subseteq (J_g, b_{g+2}, \dots, b_\ell, \mathbf{x}) : b_{g+1} = (J_g, b_{g+2}, \dots, b_\ell, \mathbf{x}) = K.$$

This proves our claim and in particular proves that $\mathfrak{A} : Q \subseteq \mathfrak{A} : (\mathfrak{A} : t_{g+1})$.

We next claim that $\overline{I^{\ell-1}} \subseteq \mathfrak{A} : Q$. First observe that $(I, b_{g+2}, \dots, b_\ell, \mathbf{x}) \cdot \overline{I^{\ell-1}} \subseteq I \cdot \overline{I^{\ell-1}} + \mathfrak{A}$ and, by Theorem 2.6, $I \cdot \overline{I^{\ell-1}} \subseteq \mathfrak{A}$ (using that R is F-rational). Hence it remains only to prove that $\overline{I^{\ell-1}} \cdot (K : D) \subseteq \mathfrak{A}$. We use a lemma.

LEMMA 4.3. *With the same notation as before,*

$$t_{g+1} \cdot \overline{I^{\ell-1}} \subseteq J_g.$$

Proof. Let $z \in \overline{I^{\ell-1}}$ and choose an element $d \in R^o$ such that $dz^n \in I^{n(\ell-1)}$ for all n. Choose $c \in I^M$ nonzero as in (3.4). Using (3.4), we then obtain

$$dct_{g+1}^q z^q \in ct_{g+1}^q I^{q(\ell-1)} \subseteq t_{g+1}^q I^{q(\ell-1)+M} \subseteq J_g^{q(\ell-1)} \subseteq J_g^{[q]},$$

where the last containment follows because $\ell - 1 \ge g$ and J_g has g generators. Hence $t_{g+1}z \in (J_g)^*$. Since R is F-rational, $t_{g+1}z \in J_g$, proving the lemma. □

Lemma 4.3 proves that $\overline{I^{\ell-1}} \subseteq D$. Hence $\overline{I^{\ell-1}}((J_g, b_{g+2}, \dots, b_\ell, \mathbf{x}) : D) \subseteq \mathfrak{A}$. We have proved that $\overline{I^{\ell-1}} \subseteq \mathfrak{A} : Q$.

By local duality, we have $\overline{I^{\ell-1}} \subseteq \mathfrak{A} : Q \subseteq \mathfrak{A} : (\mathfrak{A} : t_{g+1}) \subseteq (J_{g+1}, t_{g+1}, b_{g+2}, \dots, b_\ell, \mathbf{x}) \subseteq (J, t_{g+1}, \dots, t_\ell, \mathbf{x}) \subseteq J + \mathbf{m}^N$. □

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