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The Complexity of Bounded Quantifiers in Some Ordered Abelian Groups

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Abstract This paper obtains lower and upper bounds for the number of alternations of bounded quantifiers needed to express all formulas in certain ordered Abelian groups admitting elimination of unbounded quantifiers. The paper also establishes model-theoretic tests for equivalence to a formula with a given number of alternations of bounded quantifiers.

1 Introduction

Working with the language $\mathcal{L} = \{\leq, +, -, 0\}$ of ordered Abelian groups, [9] introduces a notion of bounded quantifier and shows that certain ordered Abelian groups not admitting elimination of quantifiers admit elimination of unbounded quantifiers: every $\mathcal{L}$-formula is equivalent in the group to a bounded formula (a formula with bounded quantifiers only). The present paper discusses the complexity, in terms of quantifier alternations, of the resulting bounded formulas. In some, but not all, ordered Abelian groups admitting elimination of unbounded quantifiers, there is a finite upper bound to the number of alternations of bounded quantifiers needed to represent all $\mathcal{L}$-formulas. There is also a family $\{G_k\}_{k\geq 0}$ of ordered Abelian groups such that every $\mathcal{L}$-formula is equivalent in G_k to a bounded formula with at most n_k alternations of bounded quantifiers, but $\{n_k\}_{k\geq 0}$ cannot have a finite upper bound.

Section 2 introduces classes $b\Sigma_n$ and $b\Pi_n$ of bounded formulas analogous to the prefix classes Σ_n and Π_n and establishes model-theoretic tests for equivalence to a formula in $b\Sigma_n$ or $b\Pi_n$, modulo a theory. These tests resemble the Łoś-Tarski test, Chang-Łoś-Suszko test, and Keisler sandwich theorem, both in statement and proof, but exploit a different kind of embedding; convexity of substructures plays a large role here. Section 2 also applies the tests to particular examples.

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Section 3 establishes lower bounds on the number of alternations of bounded quantifiers needed to represent all $\mathcal{L}$-formulas in groups like the Hahn products $\overleftarrow{\prod}_{0<i<\omega^k}\mathbb{Z}$.[1] Exploiting the quantifier-elimination procedure of [7], Part 2, as in [9], Section 2, one may link the prefix complexity of axioms for $(\omega^k, \leq, 0)$ to the prefix complexity of axioms for the $\mathcal{L}$-theory of $\overleftarrow{\prod}_{0<i<\omega^k}\mathbb{Z}$. This prefix complexity is related to the membership of formulas in the classes $b\Sigma_n$ or $b\Pi_n$ because [9], Section 5, proves that the $\mathcal{L}$-theory of $\overleftarrow{\prod}_{0<i<\omega^k}\mathbb{Z}$ may be axiomatized by the $\mathcal{L}$-theory of ordered Abelian groups together with axioms of the form $\forall x_1 \exists x_2 \beta(x_1, x_2)$, where β is a bounded formula. One finally obtains a lower bound on bounded-quantifier complexity of $\mathcal{L}$-formulas in $\overleftarrow{\prod}_{0<i<\omega^k}\mathbb{Z}$ by exploiting results of Fraïssé ([4], [6], [5]) on the prefix complexity of axioms for $(\omega^k, \leq, 0)$.

Upper bounds on the number of alternations of bounded quantifiers needed to represent all $\mathcal{L}$-formulas appear in Section 4. These bounds apply to many of the Hahn products $\overleftarrow{\prod}_{0<i<\theta}G_i$ studied in Section 5 of [9] and are obtained by combining the quantifier-elimination procedure of Gurevich [7], the quantifier elimination of Doner, Mostowski, and Tarski [2] for the first-order theory of well-orderings, and a count of bounded quantifiers in the bounded-quantifier formulas of Lemma 5.1 in [9]. These techniques also allow one to address a question left open in [9] about the theories of certain Hahn products $\overleftarrow{\prod}_{0<i<\theta}G_i$ not admitting elimination of unbounded quantifiers.

The Conclusion, Section 5, discusses possible improvements to the results of previous sections. This paper stays relatively short only by relying on [9], to which the reader will need to make frequent reference.

2 Some Tests for Bounded Formulas

Let $\mathcal{M}$ be a first-order language and $u \lhd v$ be a quantifier-free $\mathcal{M}$-formula with at most u and v free. A quantifier in an $\mathcal{M}$-formula is said to be bounded just in case it occurs in a context $\forall x \lhd t\varphi =$

$$\forall x(x \lhd t \to \varphi)$$

or in a context $\exists x \lhd t\varphi =$

$$\exists x(x \lhd t \wedge \varphi),$$

where t is an $\mathcal{M}$-term not containing x; an $\mathcal{M}$-formula is bounded just in case all of its quantifiers are bounded. The bounded formulas of Section 1 arise when $\mathcal{M}$ is the language $\mathcal{L}$ of ordered Abelian groups and $u \lhd v$ is

$$(0 \leq v \wedge u, -u \leq v) \vee (v \leq 0 \wedge v \leq u, -u)$$

(i.e., $|u| \leq |v|$).

One may define classes $b\Sigma_n$ and $b\Pi_n$ of bounded $\mathcal{M}$-formulas by recursion as follows: $b\Sigma_0 = b\Pi_0$ is the class of quantifier-free $\mathcal{M}$-formulas; $b\Sigma_{n+1}$ is the class of formulas

$$\exists x_1 \lhd t_1 \ldots \exists x_k \lhd t_k\varphi,$$

where $\varphi \in b\Pi_n$; and $b\Pi_{n+1}$ is the class of formulas

$$\forall x_1 \lhd t_1 \ldots \forall x_k \lhd t_k\varphi,$$

where $\varphi \in b\Sigma_n$. Formulas in $b\Sigma_n$ ($b\Pi_n$) are called bounded-Σ_n (bounded-Π_n) formulas, and one may speak also of bounded-universal, bounded-existential, and bounded-$\forall\exists$ formulas (instead of bounded-Π_1, bounded-Σ_1, and bounded-Π_2 formulas).

A notion of convex substructure ([9], Section 3) plays a role in certain model-theoretic tests for membership in these classes. If $\mathcal{A}$, $\mathcal{B}$ are $\mathcal{M}$-structures, say that $\mathcal{A}$ is a convex substructure of $\mathcal{B}$—in symbols, $\mathcal{A} \leq \mathcal{B}$—just in case $\mathcal{A} \subseteq \mathcal{B}$ and A is downward-closed with respect to $\lhd^{\mathcal{B}}$: that is, if $a \in A$, $b \in B$, and $b \lhd^{\mathcal{B}} a$, then $b \in A$. By induction on logical complexity one may show that if $\mathcal{A} \leq \mathcal{B}$, $\varphi(x_1, \ldots, x_n)$ is a bounded $\mathcal{M}$-formula, and $\overline{a} \in A^n$, then

$$\mathcal{A} \models \varphi[\overline{a}] \text{ just in case } \mathcal{B} \models \varphi[\overline{a}].$$

If $\mathcal{C}$ is any $\mathcal{M}$-structure, $X \subseteq C$, and either $\mathcal{M}$ contains a constant symbol or $X \neq \varnothing$, there is a least $\mathcal{D} \leq \mathcal{C}$ with $X \subseteq D$. $\mathcal{D}$ is the intersection of all convex substructures of $\mathcal{C}$ that contain X and consists of all $f \in C$ for which there are $\mathcal{M}$-terms $t_1(x_1, \ldots, x_k), \ldots, t_{l+1}(x_1, \ldots, x_{k+l})$, $e_1, \ldots, e_k \in X$, and $e_{k+1}, \ldots, e_{k+l} \in C$ with

$$e_{k+1} \lhd^{\mathcal{C}} t_1^{\mathcal{C}}[e_1, \ldots, e_k], e_{k+2} \lhd^{\mathcal{C}} t_2^{\mathcal{C}}[e_1, \ldots, e_{k+1}], \ldots, e_{k+l} \lhd^{\mathcal{C}} t_l^{\mathcal{C}}[e_1, \ldots, e_{k+l-1}],$$

and $f = t_{l+1}^{\mathcal{C}}[\overline{e}]$. Call $\mathcal{D}$ the convex hull $\mathcal{C}(X)$ of X in $\mathcal{C}$, with domain $D = \overline{X}$.

Theorem 2.1 *Suppose T is an $\mathcal{M}$-theory, $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{M}$-formula, and $\mathcal{M}$ contains a constant symbol if $n = 0$. The following conditions are equivalent:*

(i) *$\varphi(\overline{x})$ is equivalent modulo T to a bounded $\mathcal{M}$-formula $\psi(\overline{x})$.*
(ii) *If $\mathcal{A}, \mathcal{B} \models T$, $\mathcal{C} \leq \mathcal{A}, \mathcal{B}$, and $\overline{c} \in C^n$, then*

$$\mathcal{A} \models \varphi[\overline{c}] \textit{ just in case } \mathcal{B} \models \varphi[\overline{c}].$$

Proof The proof is exactly that of Theorem 3.1 in [9], but applied only to the single formula $\varphi(\overline{x})$. □

A version of the Łoś-Tarski test for bounded formulas goes as follows.

Theorem 2.2 *Suppose T is an $\mathcal{M}$-theory, $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{M}$-formula, and $\mathcal{M}$ contains a constant symbol if $n = 0$. The following conditions are equivalent:*

(i) *$\varphi(\overline{x})$ is equivalent modulo T to a bounded-universal formula $\psi(\overline{x})$.*
(ii) *If $\mathcal{A}, \mathcal{B} \models T$, $\mathcal{C} \leq \mathcal{A}$, $\mathcal{C} \subseteq \mathcal{B}$, and $\overline{c} \in C^n$, then*

$$\mathcal{B} \models \varphi[\overline{c}] \textit{ only if } \mathcal{A} \models \varphi[\overline{c}].$$

Proof To establish the harder direction from (ii) to (i), let $\Gamma(\overline{x})$ be the set of all bounded-universal formulas $\psi(\overline{x})$ implied by $\varphi(\overline{x})$, modulo T; the assumption regarding $\mathcal{M}$ implies that $\Gamma(\overline{x}) \neq \varnothing$. If $T \cup \Gamma(\overline{x}) \models \varphi(\overline{x})$, the compactness theorem provides a finite $\Delta(\overline{x}) \subseteq \Gamma(\overline{x})$ with $T \cup \Delta(\overline{x}) \models \varphi(\overline{x})$, and $\varphi(\overline{x})$ is equivalent modulo T to $\wedge\Delta(\overline{x})$, which is logically equivalent to a bounded-universal formula.

If $T \cup \Gamma(\overline{x}) \not\models \varphi(\overline{x})$, there are $\mathcal{A} \models T$ and $\overline{c} \in A^n$ with

$$\mathcal{A} \models (\Gamma \cup \{\neg\varphi\})[\overline{c}].$$

Let $\mathcal{C}$ be the convex hull of $\{c_i : 1 \leq i \leq n\}$ in $\mathcal{A}$. $T \cup \mathrm{diag}(\mathcal{C}) \models \neg\varphi(\overline{c})$: for if $\mathcal{B} \models T \cup \mathrm{diag}(\mathcal{C})$, then one may assume that $\mathcal{C} \subseteq \mathcal{B} \upharpoonright \mathcal{M}$, and by (ii)

$$\mathcal{B} \models \varphi[\overline{c}] \Rightarrow \mathcal{A} \models \varphi[\overline{c}].$$

The compactness theorem provides $\delta_1, \ldots, \delta_l \in \mathrm{diag}(\mathcal{C})$ with $T \cup \{\delta_1, \ldots, \delta_l\} \models \neg\varphi(\overline{c})$ and

$$T \models \varphi(\overline{c}) \rightarrow \vee_{i=1}^{l} \neg\delta_i .$$

If the new constants in $\vee_{i=1}^{l} \neg\delta_i$ other than the cs are $f_1, \ldots, f_m \in C$, then for each $i = 1, \ldots, m$ there are $\mathcal{M}$-terms $t_{i,1}(x_1, \ldots, x_n), \ldots, t_{i,l_i+1}(x_1, \ldots, x_{n+l_i})$ and $g_{i,1}, \ldots, g_{i,l_i} \in A$ with

$$g_{i,1} \lhd^{\mathcal{A}} t_{i,1}^{\mathcal{A}}[\overline{c}], g_{i,2} \lhd^{\mathcal{A}} t_{i,2}^{\mathcal{A}}[\overline{c}, g_{i,1}], \ldots, g_{i,l_i} \lhd^{\mathcal{A}} t_{i,l_i}^{\mathcal{A}}[\overline{c}, g_{i,1}, \ldots, g_{i,l_i-1}],$$

and

$$f_i = t_{i,l_i+1}^{\mathcal{A}}[\overline{c}, \overline{g}_i].$$

Quantifying out all the new constants, one concludes that modulo T, $\varphi(\overline{x})$ implies

$$\forall x_{1,1} \lhd t_{1,1}(\overline{x}) \ldots \forall x_{1,l_1} \lhd t_{1,l_1}(\overline{x}, x_{1,1}, \ldots, x_{1,l_1-1}) \ldots \\ \forall x_{n,l_n} \lhd t_{n,l_n}(\overline{x}, x_{n,1}, \ldots, x_{n,l_n-1})\theta,$$

where θ results from $\vee_{i=1}^{l} \neg\delta_i$ when each f_i is replaced by $t_{i,l_i+1}(\overline{x}, x_{i,1}, \ldots, x_{i,l_i})$ and each c_j is replaced by x_j. The displayed formula therefore belongs to $\Gamma(\overline{x})$, and $\overline{c}$ obeys this formula in $\mathcal{A}$. By setting each $x_{i,j}$ equal to $g_{i,j}$ in $\mathcal{A}$ one finds that $\mathcal{A}_C \models \vee_{i=1}^{l} \neg\delta_i$, contrary to the definition of $\mathrm{diag}(\mathcal{C})$. So $T \cup \Gamma(\overline{x}) \models \varphi(\overline{x})$ and the argument is complete. $\square$

One immediately obtains a test for bounded-existential formulas.

Corollary 2.3 *Suppose T is an $\mathcal{M}$-theory, $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{M}$-formula, and $\mathcal{M}$ contains a constant symbol if $n = 0$. The following conditions are equivalent:*

(i) *$\varphi(\overline{x})$ is equivalent modulo T to a bounded-existential formula $\psi(\overline{x})$.*

(ii) *If $\mathcal{A}, \mathcal{B} \models T$, $\mathcal{C} \leq \mathcal{A}$, $\mathcal{C} \subseteq \mathcal{B}$, and $\overline{c} \in C^n$, then*

$$\mathcal{B} \models \varphi[\overline{c}] \text{ if } \mathcal{A} \models \varphi[\overline{c}].$$

In the following examples these tests are applied to the $\mathcal{L}$-theories of ordered Abelian groups (where $u \lhd v$ is as above). $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \equiv \mathbb{Q} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \mathbb{Z}$ are polyregular of rank two, and so admit elimination of unbounded quantifiers ([9], Theorem 5.2, Lemma 5.4).[2] All groups in their canonical polyregular systems ([1], p. 106) are $\varnothing$-definable ([1], Corollary 3.5). Let $\psi_1(x)$ be a definition of the first nontrivial element $\rho_1(G)$ of this system; in $\mathbb{Q} \overleftarrow{\times} \mathbb{Z}$, $\psi_1(x)$ defines $\mathbb{Q} = \mathbb{Q} \overleftarrow{\times} \{0\}$, while in $\mathbb{Q} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \mathbb{Z}$, $\psi_1(x)$ defines $\mathbb{Q} \overleftarrow{\times} \mathbb{Q} = \mathbb{Q} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \{0\}$. One may assume that $\psi_1(x)$ is a bounded formula; in fact, the proof of Lemma 3.2 in [1] shows that one may let $\psi_1(x)$ be

$$\forall |y| \leq |x| \exists |z| \leq |y| (y = 2z),$$

a bounded-$\forall\exists$ formula. But since the embedding

$$(a, b) \mapsto (a, b, 0)$$

of $\mathbb{Q} \overleftarrow{\times} \mathbb{Z}$ in $\mathbb{Q} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \mathbb{Z}$ sends $(0, 1) \notin \rho_1(\mathbb{Q} \overleftarrow{\times} \mathbb{Z})$ to $(0, 1, 0) \in \rho_1(\mathbb{Q} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \mathbb{Z})$, Theorem 2.2 implies that $\psi_1(x)$ is not equivalent over $\mathbb{Q} \overleftarrow{\times} \mathbb{Z}$ to a bounded-universal formula.

$\mathbb{Q} \overleftarrow{\times} \mathbb{Z}$ is also elementarily equivalent to $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q}$, and $\mathbb{Q} = \mathbb{Q} \overleftarrow{\times} \{0\} \leq \mathbb{Q} \overleftarrow{\times} \mathbb{Z}$ may be embedded in $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q}$ by the map

$$f : (a, 0) \mapsto (0, 0, a).$$

Because it sends $(1, 0) \in \rho_1(\mathbb{Q} \overleftarrow{\times} \mathbb{Z})$ to $(0, 0, 1) \notin \rho_1(\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q})$, Corollary 2.3 implies that $\psi_1(x)$ is not equivalent over $\mathbb{Q} \overleftarrow{\times} \mathbb{Z}$ to a bounded-existential formula.

If $\mathcal{D} \subseteq \mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q}$ has domain $\{0\} \times \{0\} \times \mathbb{Q} = \operatorname{ran} f$, one may show that tuples from $\mathcal{D}$ satisfy the same bounded-existential formulas in $\mathcal{D}$ and in $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q}$. Clearly, one need show merely that if $\overline{d}$ from D satisfies the bounded-existential formula $\theta(\overline{x})$ in $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q}$, then $\mathcal{D} \models \theta[\overline{d}]$. But if $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q} \models \theta[\overline{d}]$, then $\mathbb{Q} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \mathbb{Q} \models \theta[\overline{d}]$ because $\theta(\overline{x})$ is existential and $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q} \subseteq \mathbb{Q} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \mathbb{Q}$; so $\mathcal{D} \models \theta[\overline{d}]$ because

$$\mathcal{D} = \{0\} \overleftarrow{\times} \{0\} \overleftarrow{\times} \mathbb{Q} \preccurlyeq \mathbb{Q} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \mathbb{Q}$$

by quantifier elimination for divisible ordered Abelian groups. This absoluteness of bounded-existential formulas between $\mathcal{D}$ and $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q}$ implies that $\psi_1(x)$ is not equivalent over $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q} \equiv \mathbb{Q} \overleftarrow{\times} \mathbb{Z}$ to any propositional combination of bounded-existential formulas: for while $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \mathbb{Q} \not\models \psi_1[f(1, 0)]$, $\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \models \psi_1[(1, 0)]$, and f is an isomorphism between $\mathbb{Q} \overleftarrow{\times} \{0\} \leq \mathbb{Q} \overleftarrow{\times} \mathbb{Z}$ and $\mathcal{D}$.

The following test for bounded-$\forall\exists$ formulas resembles the Chang-Łoś-Suszko test.

Theorem 2.4 *Suppose T is an $\mathcal{M}$-theory, $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{M}$-formula, and $\mathcal{M}$ contains a constant symbol if $n = 0$. The following conditions are equivalent:*

(i) *$\varphi(\overline{x})$ is equivalent modulo T to a bounded-$\forall\exists$ formula $\psi(\overline{x})$.*

(ii) *Let $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$ be a chain of $\mathcal{M}$-structures with $\mathcal{A}_i \leq \mathcal{B}_i \models T$ for all $i \in \mathbb{N}$. If $\overline{a} \in A_0^n$, $\mathcal{B}_i \models \varphi[\overline{a}]$ for all $i \in \mathbb{N}$, and $\cup_i \mathcal{A}_i \leq \mathcal{D} \models T$, then $\mathcal{D} \models \varphi[\overline{a}]$.*

Proof Assume (i) is true, and let $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$ be a chain of $\mathcal{M}$-structures with $\mathcal{A}_i \leq \mathcal{B}_i \models T$ always. If $\overline{a} \in A_0^n$, $\mathcal{B}_i \models \varphi[\overline{a}]$ always, and $\cup_i \mathcal{A}_i \leq \mathcal{D} \models T$, then $\mathcal{B}_i \models \psi[\overline{a}]$ always because $\mathcal{B}_i \models T$, $\mathcal{A}_i \models \psi[\overline{a}]$ always because $\psi(\overline{x})$ is bounded, and $\cup_i \mathcal{A}_i \models \psi[\overline{a}]$ because $\psi(\overline{x})$ is logically equivalent to an $\forall\exists$-formula. Since $\cup_i \mathcal{A}_i \leq \mathcal{D}$ and $\psi(\overline{x})$ is bounded, $\mathcal{D} \models \psi[\overline{a}]$; so since $\mathcal{D} \models T$, $\mathcal{D} \models \varphi[\overline{a}]$.

Assume now that (ii) is true. Let $\Gamma(\overline{x})$ be the set of all bounded-$\forall\exists$ formulas implied by $\varphi(\overline{x})$, modulo T. The assumption about $\mathcal{M}$ implies that $\Gamma(\overline{x}) \neq \varnothing$. If $T \cup \Gamma(\overline{x}) \models \varphi(\overline{x})$, then the compactness theorem provides a finite $\Delta(\overline{x}) \subseteq \Gamma(\overline{x})$ for which $T \cup \Delta(\overline{x}) \models \varphi(\overline{x})$, and $\varphi(\overline{x})$ is equivalent modulo T to $\wedge\Delta(\overline{x})$. Since $\wedge\Delta(\overline{x})$ is logically equivalent to a bounded-$\forall\exists$ formula, the desired result holds if $T \cup \Gamma(\overline{x}) \models \varphi(\overline{x})$.

If one introduces new constant symbols $c_1, \ldots, c_n$ and replaces the free occurrences of each x_i in $\Gamma(\overline{x}) \cup \{\varphi(\overline{x})\}$ by c_i, one wants to show that $T \cup \Gamma(\overline{c}) \models \varphi(\overline{c})$. By making this change of notation from the start, one may assume that $\mathcal{M}$ contains at least one constant symbol, that $\varphi(\overline{x}) = \varphi$ is a sentence, and that $\Gamma(\overline{x}) = \Gamma$ consists of all bounded-$\forall\exists$ sentences implied by φ modulo T. Since $\mathcal{M}$ contains constant symbols, any subset X of an $\mathcal{M}$-structure $\mathcal{A}$ has a convex hull $\mathcal{A}(X) \leq \mathcal{A}$. When $X = \varnothing$, let $b\mathcal{A}$—the bounded part of $\mathcal{A}$—be $\mathcal{A}(\varnothing)$.

Suppose $\mathcal{A}_0 \models T \cup \Gamma$, and let $\operatorname{diag}_{b\forall}(b\mathcal{A}_0)$ be the set of all bounded-universal $\mathcal{M}_{bA_0}$-sentences true in $b\mathcal{A}_0$. If

$$T \cup \operatorname{diag}_{b\forall}(b\mathcal{A}_0) \cup \{\varphi\}$$

has no model, then by the compactness theorem there are $\delta_1, \ldots, \delta_l \in \operatorname{diag}_{b\forall}(b\mathcal{A}_0)$ with

$$T \cup \{\delta_1, \ldots, \delta_l\} \models \neg\varphi$$

and so $T \models \varphi \to \vee_{i=1}^{l} \neg\delta_i$. If the constant symbols of $\mathcal{M}_{bA_0} - \mathcal{M}$ in $\vee_{i=1}^{l} \neg\delta_i$ correspond to $a_1, \ldots, a_n \in bA_0$, then one may show as in the proof of Theorem 2.2 that modulo T, φ implies

$$\forall x_{1,1} \lhd t_{1,1} \ldots \forall x_{1,l_1} \lhd t_{1,l_1}(x_{1,1}, \ldots, x_{1,l_1-1}) \ldots \forall x_{n,l_n} \lhd t_{n,l_n}(x_{n,1}, \ldots, x_{n,l_n-1})\theta,$$

where θ results from $\vee_{i=1}^{l} \neg\delta_i$ when for each j, the constant symbol corresponding to a_j is replaced by $t_{j,l_j+1}(x_{j,1}, \ldots, x_{j,l_j})$. The displayed sentence is logically equivalent to a bounded-$\forall\exists$ sentence, and so belongs to Γ and is true in $\mathcal{A}_0$. But by arguing as in the proof of Theorem 2.2 one finds that the truth of this sentence in $\mathcal{A}_0$ would imply the truth of $\vee_{i=1}^{l} \neg\delta_i$ in $b\mathcal{A}_0$, though each δ_i belongs to $\mathrm{diag}_{b\forall}(b\mathcal{A}_0)$.

Thus $T \cup \mathrm{diag}_{b\forall}(b\mathcal{A}_0) \cup \{\varphi\}$ has a model $(\mathcal{A}_1)_{bA_0}$, and $b\mathcal{A}_0 \subseteq b\mathcal{A}_1$. One may assume that $A_0 \cap (bA_1 - bA_0) = \varnothing$.

If $\mathrm{Th}((\mathcal{A}_0)_{A_0}) \cup \mathrm{diag}(b\mathcal{A}_1)$ has no model, then by the compactness theorem there are $\delta_1, \ldots, \delta_l \in \mathrm{diag}(b\mathcal{A}_1)$ for which

$$\mathrm{Th}((\mathcal{A}_0)_{A_0}) \models \vee_{i=1}^{l} \neg\delta_i.$$

If the constant symbols of $\mathcal{M}_{bA_1,A_0} - \mathcal{M}_{A_0}$ in $\vee_{i=1}^{l} \neg\delta_i$ correspond to $a_1, \ldots, a_n \in bA_1$, then, since $A_0 \cap bA_1 \subseteq bA_0$, any constant symbols of $\mathcal{M}_{A_0}$ in $\vee_{i=1}^{l} \neg\delta_i$ correspond to elements of bA_0. As in the proof of Theorem 2.2 one may show that $\mathrm{Th}((\mathcal{A}_0)_{A_0})$ implies

$$\forall x_{1,1} \lhd t_{1,1} \ldots \forall x_{1,l_1} \lhd t_{1,l_1}(x_{1,1}, \ldots, x_{1,l_1-1}) \ldots \forall x_{n,l_n} \lhd t_{n,l_n}(x_{n,1}, \ldots, x_{n,l_n-1})\theta,$$

where θ results from $\vee_{i=1}^{l} \neg\delta_i$ when for each j, the constant symbol corresponding to a_j is replaced by $t_{j,l_j+1}(x_{j,1}, \ldots, x_{j,l_j})$. The displayed sentence belongs to $\mathrm{diag}_{b\forall}(b\mathcal{A}_0)$ and so is true in $\mathcal{A}_1$, but since the δ_is belong to $\mathrm{diag}(b\mathcal{A}_1)$, one may argue as in the proof of Theorem 2.2 that the displayed sentence fails in $\mathcal{A}_1$. This contradiction implies that $\mathrm{Th}((\mathcal{A}_0)_{A_0}) \cup \mathrm{diag}(b\mathcal{A}_1)$ has a model $(\mathcal{A}_2)_{A_0,bA_1}$, and thus $\mathcal{A}_0 \preccurlyeq \mathcal{A}_2$ and $b\mathcal{A}_0 \subseteq b\mathcal{A}_1 \subseteq b\mathcal{A}_2$.

Repeating this argument countably many times, one obtains models $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ of T such that $\{b\mathcal{A}_i\}_{i\in\mathbb{N}}$ is a chain, $\{\mathcal{A}_{2i}\}_{i\in\mathbb{N}}$ is an elementary chain, and every $\mathcal{A}_{2i+1} \models \varphi$. If

$$\mathcal{A}_\infty = \cup_i \mathcal{A}_{2i},$$

then $\mathcal{A}_0 \preccurlyeq \mathcal{A}_\infty$ and

$$b\mathcal{A}_\infty = \cup_i b\mathcal{A}_{2i} = \cup_i b\mathcal{A}_{2i+1}.$$

So $\mathcal{A}_\infty \models T$, $\mathcal{A}_\infty \models \varphi$ by (ii), and $\mathcal{A}_0 \models \varphi$ if $\mathcal{A}_0 \models T \cup \Gamma$. Thus $T \cup \Gamma \models \varphi$ and the argument is complete. $\square$

One may illustrate Theorem 2.4 by looking at definable sets in the ordered Abelian group $H = \mathbb{Q} \overleftarrow{\times} \overleftarrow{\Pi}_{0<i\leq\omega}\mathbb{Z}$. In this polyregular group of rank $\omega + 1$, every $\mathcal{L}$-formula is equivalent to a bounded formula by Theorem 5.2 and Lemma 5.4 of [9]. The convex subgroup $\mathbb{Q} \overleftarrow{\times} \overleftarrow{\Pi}_{0<i<\omega}\mathbb{Z}$ of H is $\varnothing$-definable ([9], Theorem 5.6), and so $K = H - \mathbb{Q} \overleftarrow{\times} \overleftarrow{\Pi}_{0<i<\omega}\mathbb{Z}$ is $\varnothing$-definable. A bounded formula will do the trick—in fact, a $b\Pi_4$-formula will work; see Section 4—and if one does not confine attention to bounded formulas, one may use $\varphi(x) =$

$$\forall y(|y| > |x| \to \exists |z| \leq |x| \exists |w| \leq |2y|(y = z + 2w)),$$

which is logically equivalent to an essentially universal formula.[3] But one may show that no $b\Pi_2$-formula defines K in H. For $i \geq 1$, let

$$H_i = \mathbb{Q} \overleftarrow{\times} \overleftarrow{\Pi}_{0<j\leq i}\mathbb{Q} \overleftarrow{\times} \overleftarrow{\Pi}_{i<l\leq\omega}\mathbb{Z} :$$

that is, replace the first i $\mathbb{Z}$-factors of H by $\mathbb{Q}$. $H \subseteq H_1 \subseteq H_2 \subseteq \cdots$ are models of $\mathrm{Th}(H)$ ([9], Theorem 4.1) in which the element

$$a = (0, 0, \ldots, 1)$$

obeys $\varphi(x)$. The union

$$H \cup \bigcup_{i\geq 1} H_i = \overleftarrow{\Pi}_{0<j<\omega}\mathbb{Q} \overleftarrow{\times} \mathbb{Z}$$

of this chain is convex in

$$\overleftarrow{\Pi}_{0<j<\omega}\mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \overleftarrow{\Pi}_{0<l\leq\omega}\mathbb{Z},$$

a model of $\mathrm{Th}(H)$ ([9], Theorem 4.1) in which the element $(0, 0, \ldots, 1, 0, 0, \ldots)$ corresponding to a does not obey $\varphi(x)$. So by Theorem 2.4 $\varphi(x)$ is not equivalent in H to a $b\Pi_2$-formula, though $\varphi(x)$ is equivalent in H to a bounded formula and to a Π_2-formula.

One may devise model-theoretic tests for formulas in arbitrary classes $b\Pi_k$ or $b\Sigma_k$. Since the rest of this paper will not need such tests they will not be proved here but merely illustrated with a version of the Keisler sandwich theorem. Start by defining a bounded n-sandwich to be a collection of $\mathcal{M}$-structures $\mathcal{B}_0 \subseteq \mathcal{A}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{A}_1 \subseteq \cdots \mathcal{A}_{n-1} \subseteq \mathcal{B}_n$ for which there are $\mathcal{M}$-structures $\mathcal{A}'_i \geq \mathcal{A}_i$ (for $0 \leq i < n$) and $\mathcal{B}'_i \geq \mathcal{B}_i$ (for $0 \leq i \leq n$) with $\mathcal{A}'_0 \preccurlyeq \cdots \preccurlyeq \mathcal{A}'_{n-1}$ and $\mathcal{B}'_0 \preccurlyeq \cdots \preccurlyeq \mathcal{B}'_n$. Given this definition, one may state the following result.

Theorem 2.5 *Let $\mathcal{M}$ be a language with at least one constant symbol and $T \cup U$ be a set of $\mathcal{M}$-sentences. The following conditions are equivalent:*

(i) *Modulo T, U is equivalent to a set of $b\Pi_{2n}$-sentences.*

(ii) *If $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ is a $b\Sigma_{2n-1}$-chain[4] of models of $T \cup U$ and $\cup_{i\in\mathbb{N}}\mathcal{A}_i \models T$, then $\cup_{i\in\mathbb{N}}\mathcal{A}_i \models U$.*

(iii) *If $\mathcal{A} \models T \cup U$, $\mathcal{B} \models T$, and there is a bounded n-sandwich $\mathcal{B}_0 \subseteq \mathcal{A}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{A}_1 \subseteq \cdots \mathcal{A}_{n-1} \subseteq \mathcal{B}_n$ with $\mathcal{B} = \mathcal{B}'_0$ and $\mathcal{A} \preccurlyeq \mathcal{A}'_0$, then $\mathcal{B} \models U$.*

Analogous results hold for $b\Pi_{2n+1}$-sentences and for $b\Sigma_k$-sentences.

3 Lower Bounds on Bounded-Quantifier Complexity

For many ordinals θ, the $\mathcal{L}$-structure $\overleftarrow{\Pi}_{0<i<\theta}\mathbb{Z}$ admits elimination of unbounded quantifiers but the relevant bounded formulas cannot be too simple, that is, cannot be restricted to $b\Sigma_n \cup b\Pi_n$ if n is too small. As noted in Section 1, one may calculate such n by linking the complexity of bounded formulas in $\overleftarrow{\Pi}_{0<i<\theta}\mathbb{Z}$ with the prefix complexity of axioms for $\overleftarrow{\Pi}_{0<i<\theta}\mathbb{Z}$, linking the prefix complexity of such axioms with the prefix complexity of axioms for $(\theta, \leq, 0)$, and invoking Fraïssé's lower bounds for such complexity ([4], 5.5, 5.3.2; [5], Note 1 on p. 60; and [6], 5.2.2). One may link the prefix complexity of axioms for $\overleftarrow{\Pi}_{0<i<\theta}\mathbb{Z}$ with the prefix complexity of axioms for $(\theta, \leq, 0)$ by making a detour through the two-sorted theories of ordered Abelian groups described in [9] and noting that the relative quantifier-elimination procedure of Gurevich in general does not increase prefix complexity. To link the complexity of bounded formulas in $\overleftarrow{\Pi}_{0<i<\theta}\mathbb{Z}$ with the prefix complexity of axioms for $\overleftarrow{\Pi}_{0<i<\theta}\mathbb{Z}$, one exploits Corollary 5.3 of [9], which shows that in the cases of interest $\overleftarrow{\Pi}_{0<i<\theta}\mathbb{Z}$ has axioms built in a special way from bounded formulas.

As in [9], Section 2, let $\mathrm{EXL}^{\#}$ be an expansion of the two-sorted language EXL of [7] by symbols purely of subgroup sort, and let EXT_1 be the first-order EXL-theory in which the relative quantifier-elimination of Gurevich succeeds.

Lemma 3.1 *Let $\varphi(\overline{x}, y, \overline{Z})$ be a Σ_n- (Π_n-) formula of $\mathrm{EXL}^{\#}$ without group-sort quantifiers. Then $\exists x\varphi$ is equivalent modulo EXT_1 to a Σ_n- (Σ_{n+1}-) formula of $\mathrm{EXL}^{\#}$ without group-sort quantifiers.*

Proof One reaches the desired conclusion by applying Gurevich's quantifier-elimination procedure ([7], Part 2). Though arbitrary applications of his Lemmas 7.2 and 7.3 could increase prefix complexity, here they do not because he applies these lemmas only to quantifier-free formulas β_i or β. Applications of his Lemma 7.4 here adjoin additional existential quantifiers on subgroup variables to the start of a formula. □

$\mathrm{ELL} = \{+, -, <, 0\}$ is the group-sort reduct of $\mathrm{EXL}^{\#}$.

Corollary 3.2 *A Σ_n- (Π_n-) formula of ELL is equivalent modulo EXT_1 to a Σ_n- (Π_n-) formula of $\mathrm{EXL}^{\#}$ without group-sort quantifiers.*

Proof The argument, based on Lemma 3.1, goes by induction on n. □

Again as in [9], $\widetilde{\mathrm{EXL}}^{\#}$ is an expansion of $\mathrm{EXL} - \{^+\}$ by symbols purely of subgroup sort, and $\widetilde{\mathrm{EXT}}_1$ is a corresponding first-order theory in which the relative quantifier-elimination procedure of Gurevich succeeds.

Lemma 3.3 *Let $\varphi(\overline{x}, y, \overline{Z})$ be a Σ_n- (Π_n-) formula of $\widetilde{\mathrm{EXL}}^{\#}$ without group-sort quantifiers. If $n \geq 2$ ($n \geq 1$), then $\exists x\varphi$ is equivalent modulo $\widetilde{\mathrm{EXT}}_1$ to a Σ_n- (Σ_{n+1}-) formula of $\widetilde{\mathrm{EXL}}^{\#}$ without group-sort quantifiers.*

Proof The argument is just the same as in Lemma 3.1 but yields a weaker conclusion because the function symbol $^+$ need not be available. So when Gurevich uses the formula

$$X \subset X^+$$

on pp. 216–17 of [7], one must write

$$\exists Y(X \subset Y \wedge \forall Z(X \subset Z \rightarrow Y \subseteq Z)) \vee \forall Y(Y \subseteq X)$$

instead, and this Σ_2-formula will boost prefix complexity unless n is sufficiently large. □

Let $\widetilde{\mathrm{EXL}}^{\#}_s$ be the set of symbols of $\widetilde{\mathrm{EXL}}^{\#}$ purely of subgroup sort, as in [9], Section 2.

Corollary 3.4 *When $n \geq 2$, a Σ_n- (Π_n-) formula of ELL is equivalent modulo $\widetilde{\mathrm{EXT}}_1$ to a Σ_n- (Π_n-) formula of $\widetilde{\mathrm{EXL}}^{\#}$ without group-sort quantifiers. There is such a Σ_n- (Π_n-) formula which is a positive propositional combination of atomic ELL-formulas, $\pm D(p, s, k, t)$, $\pm E(p, s, c, t)$, $\pm tRl \bmod A(t)$ ($R \in \{=, <, >\}$, $l \neq 0$), and formulas obtained from Σ_n- (Π_n-) formulas of $\widetilde{\mathrm{EXL}}^{\#}_s$ through replacement of certain free variables of subgroup sort by terms $A(t)$ or $F(p, s, t)$.*

Proof One establishes the first claim by induction on n as in Corollary 3.2, but with the help of Lemma 3.3 instead of Lemma 3.1. The second claim follows from the first by the proof of Theorem 2.2 in [9] together with the fact that since $n \geq 2$, one may assume that terms of subgroup sort that contain terms of group sort are terms $A(t)$ or $F(p, s, t)$ (see the remarks about unnested formulas in [8], pp. 58–59). $\square$

The next results concern the two-sorted theories $\mathrm{Lex}(T_{\mathrm{v}}, \mathcal{F})$ introduced in Section 4 of [9]. Given the $\mathcal{L}_{\mathrm{v}}$-theory T_{v} of linearly ordered sets with least element 0 and the family $\mathcal{F}$ of $\mathcal{L}_{\mathrm{v}}$-formulas $Z(\alpha)$ and $I_{p,n}(\alpha)$ (p prime, $n \geq 0$), one may define an expansion $(\mathcal{L}_{\mathrm{v}}^2)\hat{}$ of $\mathcal{L}_{\mathrm{v}}^2$ and an $(\mathcal{L}_{\mathrm{v}}^2)\hat{}$-theory $\mathrm{Lex}(T_{\mathrm{v}}, \mathcal{F})\hat{}$ extending $\mathrm{Lex}(T_{\mathrm{v}}, \mathcal{F})$ as follows. $(\mathcal{L}_{\mathrm{v}}^2)\hat{}$ is the disjoint union of $\mathcal{L}_{\mathrm{v}}^2$ with the symbols of $\widetilde{\mathrm{EXL}} - (\mathrm{ELL} \cup \{\varnothing, \subset, \in, A(\_\_), F(p, s, \_\_), \_\_ = 0 \bmod \dots, \_\_ < 0 \bmod \dots, \_\_ > 0 \bmod \dots\})$: so to $\mathcal{L}_{\mathrm{v}}^2$ one adds the relation symbols $D(p, s, k, \_\_)$, $E(p, s, c, \_\_)$, $E(\dots)$, $p(s, k, \dots) > r$, and $\_\_Rl \bmod \dots$, where p is prime, $1 \leq k \leq s$, $c \in \mathbb{Z}$, $r \geq 0$, $R \in \{=, <, >\}$, $l \in \mathbb{Z} - \{0\}$, and $\_\_$ $(\dots)$ is an argument place for terms of group (value) sort. $\mathrm{Lex}(T_{\mathrm{v}}, \mathcal{F})\hat{}$ is the definitional expansion of $\mathrm{Lex}(T_{\mathrm{v}}, \mathcal{F})$ by axioms with the following import (initial universal quantifiers are dropped for the sake of legibility; the numbering continues that of [9], Section 4):

(13) $D(p, s, k, x) \leftrightarrow \exists |y| \leq |p^s x|(x = p^s y \vee \|x - p^k y\|_{p^s} < \|x\|_{p^s} = \|y\|_{p^s})$ (for p prime and $1 \leq k \leq s$);

(14) $E(\alpha) \leftrightarrow Z(\alpha)$;

(15) $y = 1 \bmod \alpha \leftrightarrow E(\alpha) \wedge 0 < y \wedge \alpha \leq \|y\| \wedge \forall |z| \leq |y|(0 < z \wedge \alpha \leq \|z\| \rightarrow \|z - y\| < \alpha)$;

(16) $E(p, s, c, x) \leftrightarrow \exists |y| \leq |2x|(y = 1 \bmod \|x\|_{p^s} \wedge \|x - cy\|_{p^s} < \|x\|_{p^s}))$ (for p prime, $s \geq 1$ and $c \in \mathbb{Z}$);

(17) $p(s, s, \alpha) > r \leftrightarrow I_{p,r+1}(\alpha)$ (for p prime, $1 \leq s$ and $r \geq 0$);

(18) $\neg p(s, k, \alpha) > r$ (for p prime, $1 \leq k < s$ and $r \geq 0$);

(19) $y = l \bmod \alpha \leftrightarrow \exists |z| \leq |y|(z = 1 \bmod \alpha \wedge \|y - lz\| < \alpha)$ (when $l \neq 0, 1$);

(20) $y < l_+ \bmod \alpha \leftrightarrow E(\alpha) \wedge (\alpha \leq \|y\| \rightarrow \exists |z| \leq |y|(z = 1 \bmod \alpha \wedge y < l_+ z \wedge \alpha \leq \|y - l_+ z\|))$ (when $0 < l_+$);

(21) $y > l_+ \bmod \alpha \leftrightarrow E(\alpha) \wedge \exists |z| \leq |y|(z = 1 \bmod \alpha \wedge l_+ z < y \wedge \alpha \leq \|l_+ z - y\|)$ (when $0 < l_+$);

(22) $y < l_- \bmod \alpha \leftrightarrow E(\alpha) \wedge \exists |z| \leq |y|(z = 1 \bmod \alpha \wedge y < l_- z \wedge \alpha \leq \|l_- z - y\|)$ (when $l_- < 0$);

(23) $y > l_- \bmod \alpha \leftrightarrow E(\alpha) \wedge (\alpha \leq \|y\| \rightarrow \exists |z| \leq |y|(z = 1 \bmod \alpha \wedge l_- z < y \wedge \alpha \leq \|l_- z - y\|))$ (when $l_- < 0$).

Lemma 3.5 *Suppose $\mathcal{L}_v$ contains no function symbols*[5] *and T_v implies that every element with a successor has an immediate successor. If $n \geq 2$, then every Σ_n- (Π_n-) formula of $\mathcal{L}$ is equivalent modulo* $\mathrm{Lex}(T_\mathrm{v}, \mathcal{F})\hat{}$ *to a Σ_n- (Π_n-) formula of $(\mathcal{L}_\mathrm{v}^2)\hat{}$ without group-sort quantifiers. There is such a Σ_n- (Π_n-) formula which is a positive propositional combination of atomic and negated atomic $\mathcal{L}$-formulas, formulas $\pm D(p, s, k, t)$, $\pm E(p, s, c, t)$, $\pm t Rl$* mod $\|t\|$ *($R \in \{=, <, >\}$, $l \neq 0$), and formulas obtained from Σ_n- (Π_n-) formulas of $\mathcal{L}_v$ through replacement of certain free variables of value sort by terms $\|t\|$ or $\|t\|_{p^s}$.*

Proof Let $\varphi(\overline{x})$ be a Σ_n- (Π_n-) formula of $\mathcal{L}$ and $\widetilde{\mathrm{EXL}}^\# = \widetilde{\mathrm{EXL}} \amalg (\mathcal{L}_\mathrm{v} - \{\leq, 0\})$. The proof of Theorem 4.1 in [9] provides an ELL-formula $\widetilde{\varphi}(\overline{x})$ such that if $\mathcal{A} \models \mathrm{Lex}(T_\mathrm{v}, \mathcal{F})$ and $\widetilde{\mathcal{A}} \models \widetilde{\mathrm{EXT}}_1$ is built from $\mathcal{A}$ as in the proof of Theorem 4.1, then

$$\mathcal{A} \models \varphi[\overline{a}] \text{ iff } \widetilde{\mathcal{A}} \models \widetilde{\varphi}[\overline{a}]$$

for all $\overline{a}$ from $\mathcal{A}$. The definition of $\widetilde{\varphi}(\overline{x})$ makes it a Σ_n- (Π_n-) formula if $\varphi(\overline{x})$ is a Σ_n- (Π_n-) formula. Corollary 3.4 provides a Σ_n- (Π_n-) formula $\widetilde{\varphi}^\diamondsuit(\overline{x})$ of $\widetilde{\mathrm{EXL}}^\#$, without group-sort quantifiers, that is equivalent to $\widetilde{\varphi}(\overline{x})$ modulo $\widetilde{\mathrm{EXT}}_1$; one may assume that $\widetilde{\varphi}^\diamondsuit(\overline{x})$ is a positive propositional combination of atomic ELL-formulas, formulas $\pm D(p, s, k, t)$, $\pm E(p, s, c, t)$, $\pm t R$ mod $A(t)$ ($R \in \{=, <, >\}$, $l \neq 0$), and formulas obtained from Σ_n- (Π_n-) formulas of $\widetilde{\mathrm{EXL}}_\mathrm{s}^\#$ through replacement of certain free variables of subgroup sort by terms $A(t)$ or $F(p, s, t)$. One obtains the desired Σ_n- (Π_n-) formula of $(\mathcal{L}_\mathrm{v}^2)\hat{}$ by translating $\widetilde{\varphi}^\diamondsuit(\overline{x})$ into $(\mathcal{L}_\mathrm{v}^2)\hat{}$ according to the following scheme, which as in the proof of [9], Theorem 4.1, relies on a bijection $\gamma_i \mapsto G_i$ between value-sort variables γ_i and subgroup-sort variables G_i. First one associates with each $\widetilde{\mathrm{EXL}}^\#$-term t an $(\mathcal{L}_\mathrm{v}^2)\hat{}$-term t^* as follows: variables of group sort and the group-sort constant 0 are fixed; * commutes with the function symbols $+, -$ of group sort; each G_i goes to γ_i; the subgroup-sort constant $\varnothing$ goes to the value-sort constant 0, and all other constants of subgroup sort are fixed; $A(t)^*$ is $\|t^*\|$; and $F(p, s, t)^*$ is $\|t^*\|_{p^s}$. When t, u are ELL-terms, $(t = u)^*$ is $t^* = u^*$ and $(t < u)^*$ is $t^* < u^*$;[6] * commutes with every $D(p, s, k, \_\_)$, $E(p, s, c, \_\_)$, $E(\dots)$, $p(s, k, \dots) > r$, $\_\_Rl$ mod $\dots$ ($R \in \{=, <, >\}$ and $l \neq 0$), and relation symbol of $\mathcal{L}_\mathrm{v} - \{\leq, 0\}$; when t is a term of group sort and R, S are terms of subgroup sort, $(R \subset S)^*$ is $R^* < S^*$, $(t \in R)^*$ and $(t = 0 \bmod R)^*$ are $\|t^*\| < R^*$, $(t < 0 \bmod R)^*$ is $t^* < 0 \wedge R^* \leq \|t^*\|$, and $(t > 0 \bmod R)^*$ is $t^* > 0 \wedge R^* \leq \|t^*\|$. Finally, * commutes with all connectives and quantifiers. Certainly ψ^* is Σ_n (Π_n) if ψ is Σ_n (Π_n), and will have the extra structure described in Lemma 3.5 if ψ has the extra structure described in Corollary 3.4. The remarks below will explain why $\varphi(\overline{x})$ is equivalent modulo $\mathrm{Lex}(T_\mathrm{v}, \mathcal{F})\hat{}$ to $\widetilde{\varphi}^{\diamondsuit *}(\overline{x})$.

Suppose $\mathcal{A}\hat{} \models \mathrm{Lex}(T_\mathrm{v}, \mathcal{F})\hat{}$. As noted above, for all $\overline{a}$ from A,

$$\mathcal{A}\hat{} \models \varphi[\overline{a}] \text{ iff } (\mathcal{A}\hat{} \upharpoonright \mathcal{L}_\mathrm{v}^2)\tilde{} \models \widetilde{\varphi}[\overline{a}],$$

and since $(\mathcal{A}\hat{} \upharpoonright \mathcal{L}_\mathrm{v}^2)\tilde{} \models \widetilde{\mathrm{EXT}}_1$,

$$(\mathcal{A}\hat{} \upharpoonright \mathcal{L}_\mathrm{v}^2)\tilde{} \models \widetilde{\varphi}[\overline{a}] \text{ iff } (\mathcal{A}\hat{} \upharpoonright \mathcal{L}_\mathrm{v}^2)\tilde{} \models \widetilde{\varphi}^\diamondsuit[\overline{a}].$$

Because every element of $\|A\|$ with a successor has an immediate successor, the mapping

$$\alpha \in \|A\| \mapsto [0, \alpha) \in \mathcal{I}_o$$

is a surjection of the value domain of $\mathcal{A}\hat{}$ onto the subgroup domain $\mathcal{I} \supseteq \mathcal{I}_s$ of $(\mathcal{A}\hat{} \upharpoonright \mathcal{L}_v^2)\tilde{}$. Because this mapping is order-preserving, it is an order-isomorphism, and but for different choices of symbols and primitives $\mathcal{A}\hat{} \upharpoonright \mathcal{L}_v$ and $(\mathcal{A}\hat{} \upharpoonright \mathcal{L}_v^2)\tilde{}$ $\upharpoonright \widetilde{\mathrm{EXL}}_s^{\#}$ would be isomorphic. In fact, the definitions allow one to show, by induction on logical complexity,[7] that if $\psi(\overline{x}, \overline{Y})$ is any $\widetilde{\mathrm{EXL}}^{\#}$-formula, $\overline{b}$ comes from A, and $\overline{\Gamma}$ comes from $\|A\|$, then

$$(\mathcal{A}\hat{} \upharpoonright \mathcal{L}_v^2)\tilde{} \models \psi[\overline{b}, \overline{[0, \Gamma)}] \text{ iff } \mathcal{A}\hat{} \models \psi^*[\overline{b}, \overline{\Gamma}].$$

So

$$\mathcal{A}\hat{} \models \varphi[\overline{a}] \text{ iff } \mathcal{A}\hat{} \models \widetilde{\varphi}^{\Diamond *}[\overline{a}]$$

for all $\overline{a}$ from A, and the argument is complete. □

One may apply Lemma 3.5 to sentences as follows.

Lemma 3.6 *Suppose $\mathcal{L}_v$ contains no function symbols, T_v implies that elements with successors have immediate successors, and every formula in $\mathcal{F}$ is logically equivalent to an $\mathcal{L}_v$-formula in Σ_m and to an $\mathcal{L}_v$-formula in Π_m. If $n \geq 2$, then every Σ_n- (Π_n-) sentence of $\mathcal{L}$ is equivalent modulo* $\mathrm{Lex}(T_v, \mathcal{F})$ *to a $\Sigma_{n+\max(m-1,0)}$- ($\Pi_{n+\max(m-1,0)}$-) sentence of $\mathcal{L}_v$.*

Proof Let φ be a Σ_n- (Π_n-) sentence of $\mathcal{L}$. Since $n \geq 2$, there is a Σ_n- (Π_n-) sentence ψ of $(\mathcal{L}_v^2)\hat{}$, equivalent to φ modulo $\mathrm{Lex}(T_v, \mathcal{F})\hat{}$, which has all the special properties noted at the end of Lemma 3.5. When t is a closed term of group sort, $\mathrm{Lex}(T_v, \mathcal{F}) \models t = 0$: so in ψ, occurrences of $\|t\|$ or $\|t\|_{p^s}$ may be replaced by 0 ($\in \mathcal{L}_v$), occurrences of $D(p, s, k, t)$ may be replaced by $0 = 0$, occurrences of $t = 1 \bmod \rho$, $E(p, s, c, t)$, $t = l \bmod \rho$ ($l \neq 0, 1$), $t > l_+ \bmod \rho$ ($l_+ > 0$), and $t < l_- \bmod \rho$ ($l_- < 0$) may be replaced by $0 \neq 0$, and occurrences of $t < l_+ \bmod \rho$ ($l_+ > 0$) and of $t > l_- \bmod \rho$ ($l_- < 0$) may be replaced by $E(\rho)$. The new axioms of $\mathrm{Lex}(T_v, \mathcal{F})\hat{}$ also allow one to replace occurrences of $p(s, k, \rho) > r$ by $0 \neq 0$ when $s \neq k$. One thus obtains a (prenex) Σ_n- (Π_n-) sentence θ of $(\mathcal{L}_v^2)\hat{}$, equivalent to φ modulo $\mathrm{Lex}(T_v, \mathcal{F})\hat{}$, whose quantifier-free matrix is a propositional combination of $\mathcal{L}_v$-formulas and formulas $p(s, s, \rho) > r$ and $E(\rho)$ for certain $\mathcal{L}_v$-terms ρ. By hypothesis each formula $p(s, s, \rho) > r$ and $E(\rho)$ is equivalent modulo $\mathrm{Lex}(T_v, \mathcal{F})\hat{}$ to an $\mathcal{L}_v$-formula in Σ_m and to an $\mathcal{L}_v$-formula in Π_m: so θ is equivalent modulo $\mathrm{Lex}(T_v, \mathcal{F})\hat{}$ to a $\Sigma_{n+\max(m-1,0)}$- ($\Pi_{n+\max(m-1,0)}$-) sentence γ of $\mathcal{L}_v$. Because $\mathrm{Lex}(T_v, \mathcal{F})\hat{}$ is a definitional expansion of $\mathrm{Lex}(T_v, \mathcal{F})$, φ is equivalent to γ modulo $\mathrm{Lex}(T_v, \mathcal{F})$. □

Let T_v be the $\mathcal{L}_v = \{\leq, 0\}$-theory of linearly ordered sets, with least element 0, in which elements with successors have immediate successors. In the family $\mathcal{F}$ of $\mathcal{L}_v$-formulas $Z(\alpha)$ and $I_{p,n}(\alpha)$ let $Z(\alpha)$ be $\alpha \neq 0$, $I_{p,0}(\alpha)$ be $\alpha = \alpha$, $I_{p,1}(\alpha)$ be $\alpha \neq 0$, and $I_{p,n}(\alpha)$ be $0 \neq 0$ when $n > 1$.[8] Because every formula of $\mathcal{F}$ belongs to $\Sigma_0 \cap \Pi_0$, Lemma 3.6 says that when $n \geq 2$, every Σ_n- (Π_n-) sentence φ of $\mathcal{L}$ is equivalent modulo $\mathrm{Lex}(T_v, \mathcal{F})$ to a Σ_n- (Π_n-) sentence φ^* of $\mathcal{L}_v$. If $\theta > 0$ is an ordinal, then since $T_\theta = \mathrm{Th}(\theta, \leq, 0) \supseteq T_v$, $\mathrm{Lex}(T_\theta, \mathcal{F}) \supseteq \mathrm{Lex}(T_v, \mathcal{F})$ and φ is equivalent to φ^* modulo $\mathrm{Lex}(T_\theta, \mathcal{F})$, the complete $\mathcal{L}_v^2$-theory of $\overleftarrow{\Pi}_{0<i<\theta}\mathbb{Z}$ ([9], Theorem 4.1). When θ is congruent[9] modulo ω^ω to an ordinal $\leq \omega^\omega$—for example, when $\theta \leq \omega^\omega$—$T_{\theta,\mathcal{F}}$, the set of $\mathcal{L}$-consequences of $\mathrm{Lex}(T_\theta, \mathcal{F})$, admits elimination of unbounded quantifiers ([9], Theorem 5.2, Lemma 5.4). But there may be no limit to the complexity of the bounded formulas, as the following result shows.

Theorem 3.7 *There is no $n \geq 0$ such that every $\mathcal{L}$-formula is equivalent modulo $T_{\omega^\omega,\mathcal{F}}$ to a $b\Sigma_n$-formula.*

Proof Suppose otherwise, and fix a suitable $n \geq 0$. By Corollary 5.3 of [9], $T_{\omega^\omega,\mathcal{F}}$ may be axiomatized by the $\mathcal{L}$-axioms T_o for ordered Abelian groups together with a set T_b of $\mathcal{L}$-sentences

$$\forall x_1 \exists x_2 \beta(x_1, x_2),$$

where $\beta(x_1, x_2)$ is a bounded $\mathcal{L}$-formula. By hypothesis each such $\beta(x_1, x_2)$ is equivalent modulo $T_{\omega^\omega,\mathcal{F}}$ to a formula $\widetilde{\beta}(x_1, x_2)$ in $b\Sigma_n$. So $T_{\omega^\omega,\mathcal{F}}$ may be axiomatized by T_o together with all Π_{n+1}-sentences

$$\forall x_1 \exists x_2 \widetilde{\beta}(x_1, x_2)$$

and all essentially universal sentences

$$\forall x_1 \forall x_2 (\beta \leftrightarrow \widetilde{\beta}),$$

where in both cases $\forall x_1 \exists x_2 \beta(x_1, x_2) \in T_b$. Without loss of generality, $n \geq 1$: so each Π_{n+1}-sentence $\forall x_1 \exists x_2 \widetilde{\beta}(x_1, x_2)$, as above, is equivalent modulo $\mathrm{Lex}(T_v, \mathcal{F})$ to a Π_{n+1}-sentence $(\forall x_1 \exists x_2 \widetilde{\beta}(x_1, x_2))^\star$ of $\mathcal{L}_v$. Since the $\mathcal{L}_v$-theory $T_{\omega^\omega} \subseteq \mathrm{Lex}(T_{\omega^\omega}, \mathcal{F})$ is complete,

$$T_{\omega^\omega} \models (\forall x_1 \exists x_2 \widetilde{\beta}(x_1, x_2))^\star.$$

[4], 4.3.1; [5], Note 1 on p. 60; and [6], 5.2.2, imply that T_{ω^ω} and T_{ω^n} have the same Π_{2n} consequences,[10] and so the same Π_{n+1} consequences ($n \geq 1$): thus

$$T_{\omega^n} \models (\forall x_1 \exists x_2 \widetilde{\beta}(x_1, x_2))^\star$$

and so

$$\mathrm{Lex}(T_{\omega^n}, \mathcal{F}) \models (\forall x_1 \exists x_2 \widetilde{\beta}(x_1, x_2))^\star.$$

Because $\mathrm{Lex}(T_v, \mathcal{F}) \subseteq \mathrm{Lex}(T_{\omega^n}, \mathcal{F})$,

$$\mathrm{Lex}(T_{\omega^n}, \mathcal{F}) \models \forall x_1 \exists x_2 \widetilde{\beta}(x_1, x_2)$$

and

$$T_{\omega^n,\mathcal{F}} \models \forall x_1 \exists x_2 \widetilde{\beta}(x_1, x_2)$$

whenever $\forall x_1 \exists x_2 \beta \in T_b$. Since $\overleftarrow{\Pi}_{0<i<\omega^n}\mathbb{Z}$ is a convex subgroup of $\overleftarrow{\Pi}_{0<i<\omega^\omega}\mathbb{Z}$, in which the essentially universal sentences $\forall x_1 \forall x_2(\beta \leftrightarrow \widetilde{\beta})$ are true, they are true in $\overleftarrow{\Pi}_{0<i<\omega^n}\mathbb{Z}$. So its complete $\mathcal{L}$-theory $T_{\omega^n,\mathcal{F}}$ implies a set of axioms for the complete $\mathcal{L}$-theory $T_{\omega^\omega,\mathcal{F}}$, and

$$T_{\omega^n,\mathcal{F}} = T_{\omega^\omega,\mathcal{F}}.$$

Yet the proof of Lemma 5.1 in [9] provides an $\mathcal{L}$-formula defining $\|x\| \leq \|y\|$ in the $\mathcal{L}_v^2$-structures based on $\overleftarrow{\Pi}_{0<i<\omega^n}\mathbb{Z}$ and on $\overleftarrow{\Pi}_{0<i<\omega^\omega}\mathbb{Z}$ ([9], Section 4). So one may conclude that

$$(\omega^n, \leq, 0) \equiv (\omega^\omega, \leq, 0),$$

contrary to [4], 5.3; [5], Note 1 on p. 60; and [6], 5.2(2,3). $\square$

A similar argument yields the following conclusion.

Theorem 3.8 *If $n \geq 1$, not every $\mathcal{L}$-formula is equivalent modulo $T_{\omega^n,\mathcal{F}}$ to a $b\Sigma_n$-formula.*

Proof When $n \geq 3$, one may repeat the proof of Theorem 3.7 to show that if every $\mathcal{L}$-formula is equivalent modulo $T_{\omega^n,\mathcal{F}}$ to a $b\Sigma_n$-formula, then $(\omega^{n-1}, \leq, 0) \equiv (\omega^n, \leq, 0)$, contrary to [4], 5.3; [5], Note 1 on p. 60; and [6], 5.2(2,3). The assumption that $n \geq 3$ implies that $n + 1 \leq 2(n - 1)$, and so allows one to invoke 4.3.1 of [4] to conclude that T_{ω^n} and $T_{\omega^{n-1}}$ have the same Π_{n+1} consequences.[11]

If $n = 2$, one may use an example like that at the end of Section 2 to reach the desired conclusion. For $i \geq 0$ let

$$H_i = \mathbb{Z} \overleftarrow{\times} \overleftarrow{\prod}_{0<j\leq i} \mathbb{Q} \overleftarrow{\times} \overleftarrow{\prod}_{i<l\leq\omega} \mathbb{Z}$$

and

$$K_i = H_i \overleftarrow{\times} \overleftarrow{\prod}_{0<m<\omega} \mathbb{Z}.$$

$H_i \leq K_i \models T_{\omega^2,\mathcal{F}}$ for all i, and $\{H_i\}_{i\geq 0}$ is a chain whose union

$$H = \mathbb{Z} \overleftarrow{\times} \overleftarrow{\prod}_{0<j<\omega} \mathbb{Q} \overleftarrow{\times} \mathbb{Z}$$

is a convex subgroup of

$$K = \mathbb{Z} \overleftarrow{\times} \overleftarrow{\prod}_{0<j<\omega} \mathbb{Q} \overleftarrow{\times} \mathbb{Z} \overleftarrow{\times} \overleftarrow{\prod}_{0<m<\omega^2} \mathbb{Z} \models T_{\omega^2,\mathcal{F}}.$$

There is an $\mathcal{L}$-formula $\varphi(x)$ defining the set of elements of value ω in any model of $\mathrm{Lex}(T_{\omega^2}, \mathcal{F})$ ([9], Theorem 5.6). The element $(\overline{0}, 1) \in H_0$ obeys $\varphi(x)$ in every K_i but not in K (where $(\overline{0}, 1)$ has value 2): so by Theorem 2.4, $\varphi(x)$ is not equivalent modulo $T_{\omega^2,\mathcal{F}}$ to a $b\Pi_2$-formula. Thus $\neg\varphi(x)$ is not equivalent modulo $T_{\omega^2,\mathcal{F}}$ to a $b\Sigma_2$-formula.

If $n = 1$, the identity map embeds $\overleftarrow{\prod}_{0<i<\omega} \mathbb{Z} \models T_{\omega,\mathcal{F}}$ in $\mathbb{Z} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \overleftarrow{\prod}_{3\leq i<\omega} \mathbb{Z} \models T_{\omega,\mathcal{F}}$. There is an $\mathcal{L}$-formula $\psi(x)$ defining the set of elements of value 2 in any model of $T_{\omega,\mathcal{F}}$ ([9], Theorem 5.6). Since $(0, 1, \overline{0})$ obeys $\psi(x)$ in $\overleftarrow{\prod}_{0<i<\omega} \mathbb{Z} \models T_{\omega,\mathcal{F}}$ but not in $\mathbb{Z} \overleftarrow{\times} \mathbb{Q} \overleftarrow{\times} \overleftarrow{\prod}_{3\leq i<\omega} \mathbb{Z} \models T_{\omega,\mathcal{F}}$—where $(0, 1, \overline{0})$ has value 1—Corollary 2.3 implies that $\psi(x)$ is not equivalent modulo $T_{\omega,\mathcal{F}}$ to a $b\Sigma_1$-formula. $\square$

When $\theta > \omega^\omega$ is not congruent modulo ω^ω to ω^ω, $T_{\theta,\mathcal{F}}$ does not admit elimination of unbounded quantifiers ([9], Section 5, Note 9). In this case there is no $n \geq 0$ such that every bounded $\mathcal{L}$-formula $\beta(\overline{x})$ is equivalent modulo $T_{\theta,\mathcal{F}}$ to a formula $\widetilde{\beta}(\overline{x})$ in $b\Sigma_n$: otherwise, the essentially universal sentences

$$\forall\overline{x}(\beta \leftrightarrow \widetilde{\beta})$$

would follow from $T_{\theta,\mathcal{F}}$ and so hold in the convex subgroup $\overleftarrow{\prod}_{0<i<\omega^\omega} \mathbb{Z}$ of $\overleftarrow{\prod}_{0<i<\theta} \mathbb{Z}$, contrary to Theorem 3.7.

4 Upper Bounds on Bounded-Quantifier Complexity

When $0 < \theta < \omega^\omega$ has Cantor normal form

$$\theta = \omega^r n_r + \cdots + \omega^s n_s$$

and $T_{\theta,\mathcal{F}}$ is the set of $\mathcal{L}$-consequences of $\mathrm{Lex}(T_\theta, \mathcal{F})$ as in [9], Section 5, one may show that every $\mathcal{L}$-formula is equivalent modulo $T_{\theta,\mathcal{F}}$ to a formula in $b\Sigma_{2r+11}$. So an $\mathcal{L}$-formula $\varphi(\overline{x})$ will be equivalent in $\overleftarrow{\prod}_{0<i<\omega^n} \mathbb{Z}$ to a $b\Sigma_{2n+11}$-formula, though by Theorem 3.8 $\varphi(\overline{x})$ need not be equivalent to a $b\Sigma_n$-formula. To obtain such upper bounds for the bounded-quantifier complexity of formulas equivalent to $\varphi(\overline{x})$, one starts by invoking Lemma 3.5 to find an $(\mathcal{L}^2_{\mathrm{v}})\hat{}$-formula $\varphi'(\overline{x})$, equivalent to $\varphi(\overline{x})$ modulo $\mathrm{Lex}(T_\theta, \mathcal{F})\hat{}$, which is a positive propositional combination of atomic and negated atomic $\mathcal{L}$-formulas, formulas $\pm D(p, s, k, t)$, $\pm E(p, s, c, t)$, $\pm t R l \bmod \|t\|$ ($R \in \{=, <, >\}$, $l \neq 0$), and formulas obtained from $\mathcal{L}_v$-formulas through replacement of certain free variables of value sort by terms $\|t\|$ or $\|t\|_{p^s}$. Because Lemma 5.1 of [9] allows one to translate formulas of $(\mathcal{L}_v^2)\hat{}$ into formulas of $\mathcal{L}$, the rest of the argument involves estimating the bounded-quantifer complexity of $\mathcal{L}$-formulas equivalent to $D(p, s, k, t)$, $E(p, s, c, t)$, $t R l \bmod \|t\|$ ($R \in \{=, <, >\}$, $l \neq 0$), and substitution instances (as above) of $\mathcal{L}_v$-formulas. To handle the latter type of formula one exploits the fact that there is an upper bound, determined by r, for the bounded-quantifier complexity of any $\mathcal{L}_v$-formula.

Starting with the $\mathcal{L}_v = \{\leq, 0\}$-theory of well-ordered sets, one may define bounded $\mathcal{L}_v$-formulas as in Section 2 by letting $u \lhd v$ be $u \leq v$. This choice determines classes $b\Sigma_n$, $b\Pi_n$ of $\mathcal{L}_v$-formulas as before. In their study of the $\{\leq\}$-theory W of well-ordered sets, Doner, Mostowski, and Tarski introduce $\{\leq\}$-formulas

$$L_\kappa(x),\ M_{\kappa\lambda}(x, y),\ N_{\kappa\lambda}(x)$$

when $\kappa \geq 0$ and $\lambda \geq 1$ (see [2], pp. 12, 16).[12]

Lemma 4.1

(i) *$L_\kappa(x)$ is equivalent modulo W to a $b\Pi_{2\kappa}$-formula of $\mathcal{L}_v$.*
(ii) *$M_{\kappa\lambda}(x, y)$ is equivalent modulo W to a $b\Sigma_{2\kappa+1}$-formula of $\mathcal{L}_v$.*
(iii) *$N_{\kappa\lambda}(x)$ is equivalent modulo W to a $b\Sigma_{2\kappa+2}$-formula of $\mathcal{L}_v$.*

The proof is a simple induction on κ, based on the definitions in [2], pp. 12, 16.

Doner, Mostowski, and Tarski also define, for each positive $\alpha < \omega^\omega$, a $\{\leq\}$-formula $H_\alpha(x)$ that defines α in $(\theta, \leq)$ whenever $\alpha < \theta$ ([2], p. 42).

Lemma 4.2 *When $0 < \alpha < \omega^\omega$ has Cantor normal form*

$$\alpha = \omega^\nu \lambda_\nu + \cdots + \omega^0 \lambda_0,$$

$H_\alpha(x)$ is equivalent modulo $W \cup \{\forall x(0 \leq x)\}$ to an $\mathcal{L}_v$-formula in $b\Pi_{2\nu+3}$.

Proof By definition $H_\alpha(x)$ is

$$\neg M_{\nu+1,1}(0, x) \wedge \bigwedge_{\kappa \leq \nu} G_\kappa(x),$$

where

$$G_\kappa(x) = \begin{cases} N_{\kappa,\lambda_\kappa}(x) \wedge \neg N_{\kappa,\lambda_\kappa+1}(x) & \text{if } \lambda_\kappa \neq 0 \\ \neg N_{\kappa,1}(x) & \text{if } \lambda_\kappa = 0. \end{cases}$$

By Lemma 4.1, $\neg M_{\nu+1,1}(0, x)$ is equivalent modulo $W \cup \{\forall x(0 \leq x)\}$ to a formula in $b\Pi_{2(\nu+1)+1}$, $N_{\kappa\lambda}(x)$ is equivalent modulo W to a formula in $b\Sigma_{2\kappa+2}$, and $\neg N_{\kappa\lambda}(x)$ is equivalent modulo W to a formula in $b\Pi_{2\kappa+2}$: so the prenexing rules give the result desired. □

T_θ is the $\mathcal{L}_v$-theory of $(\theta, \leq, 0)$. Lemmas 4.1 and 4.2 yield the following conclusion.

Lemma 4.3 *When $0 < \theta < \omega^\omega$ has Cantor normal form*

$$\theta = \omega^r n_r + \cdots + \omega^s n_s,$$

every $\mathcal{L}_v$-formula is equivalent modulo T_θ to a formula in $b\Sigma_{2r+5}$.

Proof The proof of Lemma 5.4 in [9] shows that every $\mathcal{L}_v$-formula is equivalent modulo T_θ to a propositional combination of quantifier-free formulas and formulas

$$L_\kappa(t),\, M_{\kappa\lambda}(t,u),\, N_{\kappa\lambda}(t),\ \text{and}\ \forall y \leq x \neg H_\gamma(y)$$

where $0 < \gamma < \theta$. Corollary 5 and Theorem 25 of [2] allow one to assume that $\kappa \leq r$ in all these formulas. So by Lemmas 4.1 and 4.2 they are equivalent modulo T_θ to formulas in

$$b\Pi_{2r}, b\Sigma_{2r+1}, b\Sigma_{2r+2},\ \text{and}\ b\Pi_{2r+4}.$$

Propositional combinations of such formulas will be equivalent to formulas in $b\Sigma_{2r+5}$. □

Turn now to $\mathcal{L}_v^2$ and the theories $T_{\theta,\mathcal{F}}$ studied in Section 5 of [9]. These correspond to classes $\mathcal{F}$ of $\mathcal{L}_v$-formulas $Z(\alpha)$ and $I_{p,n}(\alpha)$ (p prime, $n \geq 0$) chosen so that

($\star$) There are primes $p_1 < \cdots < p_k$ such that in $(\theta, \leq, 0)$

$$\forall\alpha\forall\beta(0 < \alpha < \beta \wedge \neg L_1(\beta) \to \vee_{j=1}^{k} I_{p_j,1}(\beta)).$$

For such θ and $\mathcal{F}$ one may state the following result.

Lemma 4.4 *The $\mathcal{L}_v^2$-formulas*

$$\|x\| \leq \|y\|,\ \|x\| \leq \|y\|_l,\ \|y\|_l \leq \|x\|,\ \textit{and}\ \|y\|_l \leq \|z\|_m$$

are equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})$ *to $\mathcal{L}$-formulas in*

$$b\Pi_2,\ b\Pi_2,\ b\Sigma_3,\ \textit{and}\ b\Pi_4.$$

Proof See the proof of Lemma 5.1 in [9]. □

Combining the last two lemmas, one reaches the following conclusion.

Lemma 4.5 *Suppose $0 < \theta < \omega^\omega$ has Cantor normal form*

$$\theta = \omega^r n_r + \cdots + \omega^s n_s.$$

If $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ are $m+n$ distinct variables of $\mathcal{L}_v$, $\varphi(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$ is an $\mathcal{L}_v$-formula, and $k_1, \ldots, k_n$ are prime powers greater than one, there is an $\mathcal{L}$-formula $\varphi_{\overline{k}}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in $b\Pi_{2r+9}$ such that

$$\varphi(\|x_1\|, \ldots, \|x_m\|, \|y_1\|_{k_1}, \ldots, \|y_n\|_{k_n}) \leftrightarrow \varphi_{\overline{k}}(\overline{x}, \overline{y})$$

modulo $\mathrm{Lex}(T_\theta, \mathcal{F})$.

Proof By Lemma 4.3 one may assume that φ belongs to $b\Pi_{2r+5}$, that none of the αs or βs is bound in φ, and that φ is a $\{\leq\}$-formula (0 may be inserted later as $\|0\|$). Let

$$\delta \mapsto v_\delta$$

be a bijection, between the variables of $\mathcal{L}_v$ and the variables of $\mathcal{L}$, that sends each α_i to x_i and each β_i to y_i. Suppose φ is

$$\forall\gamma_1 \leq \delta_1 \ldots \exists\gamma_t \leq \delta_t \ldots \theta,$$

where θ is the quantifier-free matrix of φ. For variables δ in φ let

$$w_\delta = \begin{cases} \|v_\delta\| & \text{if } \delta \text{ is bound in } \varphi \\ \|x_i\| & \text{if } \delta \text{ is } \alpha_i \\ \|y_i\|_{k_i} & \text{if } \delta \text{ is } \beta_i. \end{cases}$$

The value domain of any model of $\mathrm{Lex}(T_\theta, \mathcal{F})$ consists of the elements $\|x\|$, and when $\|x\| \le \|y\|$ (or $\|x\| \le \|z\|_l$), then $\|x\| = \|x'\|$ for some x' with $|x'| \le |y|$ (or $\|x\| = \|x''\|$ for some x'' with $|x''| \le |z|$). So $\varphi(\|x_1\|, \ldots, \|x_m\|, \|y_1\|_{k_1}, \ldots, \|y_n\|_{k_n})$ is equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})$ to the $\mathcal{L}^2_{\mathrm{v}}$-formula

$$\forall |v_{\gamma_1}| \le |v_{\delta_1}|(w_{\gamma_1} \le w_{\delta_1} \to \ldots \exists |v_{\gamma_t}| \le |v_{\delta_t}|(w_{\gamma_t} \le w_{\delta_t} \wedge \ldots \theta') \ldots),$$

where θ' results from θ when each variable δ in θ is replaced by w_δ. The prenexing rules allow one to convert the displayed formula to an equivalent one with quantifier prefix

$$\forall |v_{\gamma_1}| \le |v_{\delta_1}| \ldots \exists |v_{\gamma_t}| \le |v_{\delta_t}| \ldots$$

of class $b\Pi_{2r+5}$ (in $\mathcal{L}$) and matrix a propositional combination of formulas

$$\|x\| \le \|y\|, \ \|x\| \le \|y\|_l, \ \|y\|_l \le \|x\|, \text{ and } \|y\|_l \le \|z\|_m.$$

Lemma 4.4 implies that the matrix is equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})$ to an $\mathcal{L}$-formula in $b\Pi_5$; so since $2r+5$ is odd, $\varphi(\|x_1\|, \ldots, \|x_m\|, \|y_1\|_{k_1}, \ldots, \|y_n\|_{k_n})$ is equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})$ to an $\mathcal{L}$-formula in $b\Pi_{2r+9}$. □

As an example of the procedure described in the proof of Lemma 4.5, consider the $\mathcal{L}^2_{\mathrm{v}}$-formula

$$x \ne 0 \wedge \forall \alpha \le \|x\|(\|x\| \not\le \alpha \to \exists \beta \le \|x\|(\beta \not\le \alpha \wedge \|x\| \not\le \beta)),$$

which in the $\mathcal{L}^2_{\mathrm{v}}$-structure $\mathcal{H}$ corresponding to $H = \mathbb{Q} \overleftarrow{\times} \overleftarrow{\prod}_{0<i\le\omega} \mathbb{Z}$ defines the set of elements of value ω. Following the proof of Lemma 4.5, one obtains

$$\begin{aligned} x \ne 0 \wedge \forall |y| \le |x|(&\|y\| \le \|x\| \wedge \|x\| \not\le \|y\| \\ &\to \exists |z| \le |x|(\|z\| \le \|x\| \wedge \|z\| \not\le \|y\| \wedge \|x\| \not\le \|z\|)) \end{aligned}$$

and

$$\begin{aligned} x \ne 0 \wedge \forall |y| \le |x| \exists |z| \le |x|(&\|y\| \le \|x\| \wedge \|x\| \not\le \|y\| \\ &\to \|z\| \le \|x\| \wedge \|z\| \not\le \|y\| \wedge \|x\| \not\le \|z\|). \end{aligned}$$

Since $\|x\| \le \|y\|$ is equivalent in $\mathcal{H}$ to an $\mathcal{L}$-formula in $b\Pi_2$ (Lemma 4.4), the last formula displayed is equivalent in $\mathcal{H}$ to an $\mathcal{L}$-formula in $b\Pi_4$.

One may now combine all these lemmas to state the following theorem.

Theorem 4.6 *Suppose* $0 < \theta < \omega^\omega$ *has Cantor normal form*

$$\theta = \omega^r n_r + \cdots + \omega^s n_s.$$

Every $\mathcal{L}$-formula is equivalent modulo $T_{\theta,\mathcal{F}}$ to an $\mathcal{L}$-formula in $b\Pi_{2r+11}$.

Proof Let $\varphi(\overline{x})$ be an $\mathcal{L}$-formula. Lemma 3.5 says that modulo $\mathrm{Lex}(T_\theta, \mathcal{F})\hat{}$, $\varphi(\overline{x})$ is equivalent to a positive propositional combination of atomic and negated atomic $\mathcal{L}$-formulas, formulas $\pm D(p, s, k, t)$, $\pm E(p, s, c, t)$, $\pm t R l \bmod \|t\|$ ($R \in \{=, <, >\}$, $l \ne 0$), and formulas $\psi(\|t_1\|, \ldots, \|t_m\|, \|u_1\|_{k_1}, \ldots, \|u_n\|_{k_n})$, with $\psi(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$ an $\mathcal{L}_{\mathrm{v}}$-formula. Lemma 4.5 says that each formula $\psi(\|t_1\|, \ldots, \|t_m\|, \|u_1\|_{k_1}, \ldots, \|u_n\|_{k_n})$ is equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})$ to an $\mathcal{L}$-formula in $b\Pi_{2r+9}$. Lemma 4.4 implies that $D(p, s, k, t)$ is equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})\hat{}$ to an $\mathcal{L}$-formula in $b\Sigma_5$, and Lemma 4.5 implies that $t = 1 \bmod \|t\|$ is equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})\hat{}$ to an $\mathcal{L}$-formula in $b\Pi_{2r+9}$ ($E(\|t\|)$—that is, $Z(\|t\|)$—dominates). Thus $E(p, s, c, t)$ is equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})\hat{}$ to an $\mathcal{L}$-formula in $b\Sigma_{2r+10}$ ($y = 1 \bmod \|t\|_{p^s}$ dominates). A similar argument shows that each formula $t = l \bmod \|t\|$ ($l \neq 0, 1$) and each formula $tRl \bmod \|t\|$ ($R \in \{<, >\}$, $l \neq 0$) is equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})\hat{}$ to an $\mathcal{L}$-formula in $b\Sigma_{2r+10}$. So $\varphi(\overline{x})$ is equivalent modulo $\mathrm{Lex}(T_\theta, \mathcal{F})\hat{}$ to a propositional combination of $\mathcal{L}$-formulas in $b\Sigma_{2r+10}$, and so to an $\mathcal{L}$-formula in $b\Pi_{2r+11}$. $\mathrm{Lex}(T_\theta, \mathcal{F})\hat{}$ is a conservative extension of $\mathrm{Lex}(T_\theta, \mathcal{F})$, and $T_{\theta,\mathcal{F}}$ consists of the $\mathcal{L}$-consequences of $\mathrm{Lex}(T_\theta, \mathcal{F})$. So the argument is complete. □

The arguments for Lemma 4.5 and Theorem 4.6 also yield an answer to the question, raised in the Conclusion to [9], about normal forms for formulas modulo theories $T_{\theta,\mathcal{F}}$ which may not admit elimination of unbounded quantifiers (but for which ($\star$) is still assumed). By [2], Theorem 19 and Theorem 24, every $\{\leq\}$-formula is equivalent modulo T_θ to a propositional combination of atomic formulas and formulas

$$L_\kappa(x),\ M_{\kappa\lambda}(x, y),\ M_{\kappa\lambda}(0, y),\ N_{\kappa\lambda}(x),\ \text{and}\ M_{\kappa\lambda}(x, \infty).$$

All these formulas, except possibly $M_{\kappa\lambda}(x, \infty)$, are equivalent to bounded formulas. But one may show that modulo T_θ, $M_{\kappa\lambda}(x, \infty)$ is equivalent to an essentially existential formula: the sentence $L_\kappa(\infty)$ occurring in the definition of $M_{\kappa,1}(x, \infty)$ ([2], p. 16) has fixed truth value in $(\theta, \leq, 0)$ and so may be replaced in $M_{\kappa,1}(x, \infty)$ by an appropriate quantifier-free sentence of $\{\leq, 0\}$; since formulas $L_\kappa(x)$ are bounded, an easy argument by induction on λ shows that each $M_{\kappa\lambda}(x, \infty)$ is equivalent modulo T_θ to an essentially existential formula. The proof of Lemma 4.5 now shows that for each essentially existential $\{\leq, 0\}$-formula $\varphi(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$ and sequence $k_1, \ldots, k_n$ of prime powers greater than one, there is an essentially existential $\mathcal{L}$-formula $\varphi_{\overline{k}}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ for which

$$\varphi(\|x_1\|, \ldots, \|x_m\|, \|y_1\|_{k_1}, \ldots, \|y_n\|_{k_n}) \leftrightarrow \varphi_{\overline{k}}(\overline{x}, \overline{y})$$

modulo $\mathrm{Lex}(T_\theta, \mathcal{F})$. The only change in the argument concerns unbounded existential quantifiers

$$\ldots \exists \gamma_i \ldots$$

which may occur in the quantifier prefix of the (prenex) formula $\varphi(\overline{\alpha}, \overline{\beta})$: such quantifiers are changed to

$$\ldots \exists v_{\gamma_i} \ldots,$$

while occurrences of γ_i in the matrix of φ are changed to w_{γ_i} (the notation here is as in the proof of Lemma 4.5). Combining this new version of Lemma 4.5 with the argument for Theorem 4.6, one concludes that every $\mathcal{L}$-formula is equivalent modulo $T_{\theta,\mathcal{F}}$ to a propositional combination of essentially existential formulas.

5 Conclusion

One may improve Theorem 4.6 when $\theta < \omega$ and $Z(\alpha)$ never holds: that is, when $T_{\theta,\mathcal{F}}$ is the complete $\mathcal{L}$-theory of a polyregular group G of finite rank in which no quotient of successive groups in the polyregular system is a $\mathbb{Z}$-group. For Weispfenning has shown that the theory of such a group admits elimination of quantifiers in the language

$$L(I) = \{0, +, -, \equiv_n, <, S_k, \equiv_n^k\}_{1 \leq n < \omega, 1 \leq k \leq K},$$

where K is the polyregular rank of the group ([1], p. 106), $\equiv_n$ stands for congruence mod n in G, S_k singles out the elements of value at most k in the group, and

$$x \equiv_n^k y \text{ iff } x + S_k \text{ is congruent mod } n \text{ to } y + S_k \text{ in } G/S_k$$

([10],Theorem 2.9). In G,

$$x \equiv_n y \text{ iff } \exists |z| \leq |x-y|(x = y + nz),$$

$$S_k(x) \text{ iff } \forall |y_1| \leq |x| \ldots \forall |y_k| \leq |x|(\|y_1\| < \cdots < \|y_k\| < \|x\| \to y_1 = 0),$$

and

$$x \equiv_n^k y \text{ iff } \exists |z| \leq |x-y|(S_k(z) \wedge \exists |w| \leq |x-y-z|(x = y+z+nw)).$$

So with the help of Lemma 4.4 one may show that $x \equiv_n y$, $S_k(x)$, and $x \equiv_n^k y$ are equivalent in G to $\mathcal{L}$-formulas in $b\Sigma_1$, $b\Pi_2$, and $b\Sigma_3$. Weispfenning's quantifier elimination therefore implies that every $\mathcal{L}$-formula is equivalent in G to an $\mathcal{L}$-formula in $b\Sigma_4$.

Though a version of Weispfenning's quantifier elimination still works when various factor groups S_k/S_{k-1} are $\mathbb{Z}$-groups, this version introduces, for each k with $S_k/S_{k-1} \equiv \mathbb{Z}$, a new constant symbol 1_k corresponding to the least positive element $1_k + S_{k-1}$ of S_k/S_{k-1}, and when $k > 1$ such elements 1_k are not $\mathcal{L}$-definable in G. Yet when $k = 1$, 1_k will be the least positive element of G, and so may be defined by the $\mathcal{L}$-formula

$$0 < x \wedge \forall |y| \leq |x|(0 < y \to y = x).$$

If one supposes that G is a lexicographic extension of $\mathbb{Z}$ by a finite lexicographic product of regular groups that are neither divisible nor $\mathbb{Z}$-groups, then for any $\mathcal{L}$-formula $\varphi(\overline{x})$ the argument sketched above provides an $\mathcal{L}$-formula $\psi(\overline{x}, y)$ of $b\Sigma_4$ such that if 1 is the least positive element of G, $\varphi(\overline{x})$ is equivalent in G to $\psi(\overline{x}, 1)$. Let

$$\gamma = \begin{cases} 0 = 0 & \text{if } G \models \psi[\overline{0}, 1] \\ 0 \neq 0 & \text{otherwise.} \end{cases}$$

Then $\varphi(\overline{x})$ is equivalent in G to

$$(\overline{x} = \overline{0} \wedge \gamma) \vee \bigvee_j [x_j \neq 0 \wedge \exists |y| \leq |x_j|(0 < y \wedge \forall |z| \leq |y|(0 < z \to z = y) \wedge \psi(\overline{x}, y))],$$

which is equivalent in G to an $\mathcal{L}$-formula in $b\Sigma_4$. So in a finite lexicographic product $H_1 \overleftarrow{\times} \cdots \overleftarrow{\times} H_k$ of regular groups H_i that are neither divisible nor $\mathbb{Z}$-groups if $i > 1$, every $\mathcal{L}$-formula is equivalent to an $\mathcal{L}$-formula in $b\Sigma_4$ rather than $b\Sigma_{11}$ as Theorem 4.6 would claim.

To bring the lower and upper bounds of Sections 3 and 4 still closer together, one might try to apply back-and-forth arguments in the style of Fraïssé directly to the ordered Abelian groups $\overleftarrow{\Pi}_{0<i<\theta}\mathbb{Z}$ (and not merely to the ordered sets $(\theta, \leq, 0)$). Developing Ehrenfeucht-Fraïssé games for bounded-quantifier equivalence in this context might also yield a model-theoretic test for $b\Pi_{2n}$-sentences alternative to Theorem 2.5.

Notes

1. See [9], Section 4, for a definition of Hahn products $\overleftarrow{\Pi}_{0<i}G_i$ of ordered Abelian groups $\{G_i\}_{0<i\in I}$ over an ordered index set $(I, <, 0)$ with least element 0; an element $(g_i)_{0<i}$ of $\overleftarrow{\Pi}_{0<i}G_i$ is positive just in case it is not zero and g_j is positive when j is the greatest coordinate at which $(g_i)_{0<i}$ is nonzero.

2. See the Introduction of [9] for a definition of lexicographic products $H \overleftarrow{\times} K$ of ordered Abelian groups H, K; an element (h,k) of $H \overleftarrow{\times} K$ is positive just in case it is not zero and either $0 < k$ or both $0 = k$ and $0 < h$.

3. An essentially universal (existential) formula ([3], p. 31) is a formula built from atomic and negated atomic formulas with the help of conjunction, disjunction, bounded quantifiers, and universal (existential) quantifiers.

4. In a $b\Sigma_{2n-1}$-chain, a tuple from $\mathcal{A}_j$ which satisfies a $b\Sigma_{2n-1}$-formula in $\mathcal{A}_j$ satisfies the same formula in $\mathcal{A}_{j+1}$.

5. Here and in Lemma 3.5 this restriction may be avoided. Because $n > 0$, one may assume that all formulas are unnested ([8], pp. 58–59), and then a slight change in the definition of $\tilde{}$ from [9], Theorem 4.1, will still work.

6. Note that strict inequalities are not atomic $\mathcal{L}$-formulas; thus the need for "negated atomic" in the statement of Lemma 3.5.

7. See Lemma 2.4 of [9] for justification of the bounded quantifiers in the axioms for $D(p,s,k,x)$ and $E(p,s,c,x)$.

8. So $\mathrm{Lex}(T_{\mathrm{v}}, \mathcal{F})$ is true in Hahn products of $\mathbb{Z}$ over ordered sets, with least element, in which elements with successors have immediate successors.

9. According to a slight variant of [2], p. 51, ordinals α and β are congruent modulo ω^ω just in case $\alpha = \beta < \omega^\omega$ or $\alpha, \beta \geq \omega^\omega$ and there is an ordinal δ with $\alpha = \omega^\omega\delta + \beta$ or $\beta = \omega^\omega\delta + \alpha$.

10. Though [6] uses a language without constant symbols, its results apply here because $2n \geq 2$: so when $\theta > 0$ is an ordinal, $\forall\overline{x}_1 \ldots \exists\overline{x}_{2n}\varphi(\overline{x}, 0)$ is equivalent in $(\theta, \leq, 0)$ to $\forall y \forall\overline{x}_1 \ldots \exists\overline{x}_{2n}\exists z(y \leq z \to \varphi(\overline{x}, y))$.

11. Since $n - 1 > 0$, the results of [6] still apply to a language with a constant symbol for zero; see Note 10.

12. To follow the notation of [2], the next few lemmas use lowercase Roman letters for variables in $\mathcal{L}_{\mathrm{v}}$-formulas.

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