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The Modal Logic of Cluster-Decomposable Kripke Interpretations

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Abstract We deal with the modal logic of cluster-decomposable Kripke interpretations, present an axiomatization, and prove some additional results regarding this logic.

1 Introduction

A Kripke interpretation $\mathfrak{M} = \langle U, R, I \rangle$ is called *cluster-decomposable* if its set of possible worlds U can be partitioned into two (disjoint) sets U' and $U'' \neq \varnothing$ such that the accessibility relation R is of the form

$$R = (R \cap (U' \times U')) \cup (U \times U''),$$

where $\langle U'', U'' \times U'', I\,|_{U''} \rangle$ is called the *terminal cluster* of $\mathfrak{M}$. In what follows we denote by $\mathcal{CD}$ the class of all cluster-decomposable Kripke interpretations. Propositional modal logics characterized by subclasses of $\mathcal{CD}$ play a very important role in semantics of nonmonotonic logics. This is because cluster-decomposable Kripke interpretations are tightly connected to *minimal knowledge* and *maximal ignorance* (see [4]; [3], Section 9.3; and [1]). Typical examples of such logics are rather strong logics S5, Sw5, KD45, and S4F (see [1]). Additional lesser-known logics can be found in [5]. In this paper, we describe the modal logic characterized by $\mathcal{CD}$. This logic is denoted by C.[1]

The paper is organized as follows. In the next section we recall the Kripke semantics of modal logic. In Section 3 we list the axioms of C and derive some of their basic consequences. Finally, in Section 4 we prove completeness of C with respect to $\mathcal{CD}$.

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2 Propositional Modal Logic

We start with the language of classical propositional logic that contains propositional variables and only two classical propositional connectives, $\bot$ (a logical constant *falsity*) and $\supset$ (implication). Connectives $\top$ (*truth*), $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and $\equiv$ (equivalence) are defined in a usual manner; for example, $\neg\varphi$ is $\varphi \supset \bot$. The language of propositional modal logic is obtained from the language of classical propositional logic by extending it with a modal connective L (necessary). As usual, the dual connective M (possibly) is defined by $\neg L \neg$.

The weakest *normal* modal logic K results from the classical propositional logic by adding the inference rule,

NEC $\quad \varphi \vdash L\varphi$,

called *necessitation* and the axiom scheme,

k $\quad L(\varphi \supset \psi) \supset (L\varphi \supset L\psi)$.

The *normal* modal logics are obtained by adding to K all instances of some axiom schemes, for example,

t $\quad L\varphi \supset \varphi$,
d $\quad L\varphi \supset M\varphi$,
4 $\quad L\varphi \supset LL\varphi$,
5 $\quad M\varphi \supset LM\varphi$.

Adding ***t*** to K results in T, adding **4** to T results in S4, and so on (see [3], p. 197, or [5] for a more complete description).

For a modal logic S and a set of formulas Γ, called (proper) *axioms*, we write $\Gamma \vdash_{S} \varphi$, if there exists a proof of φ from Γ in S. The unsubscribed $\vdash$ denotes derivability in K.

The Kripke semantics of propositional modal logics is as follows. A *Kripke interpretation* is a triple $\mathfrak{M} = \langle U, R, I \rangle$, where U is a nonempty set of *possible worlds*, $R \subseteq U \times U$ is an *accessibility* relation on U, and I is an assignment to each world in U of a set of propositional variables. We assume that the reader is familiar with the standard definitions of "$(\mathfrak{M}, u)$ *satisfies* a formula φ," denoted $(\mathfrak{M}, u) \models \varphi$, and "$\mathfrak{M}$ *satisfies* φ," or "$\mathfrak{M}$ *is a model of* φ," denoted $\mathfrak{M} \models \varphi$, which appear in [2] or [3].

The Kripke semantics is sound and complete for K. That is, $\Gamma \vdash \varphi$ if and only if φ is satisfied by all Kripke interpretations which satisfy Γ. In particular, a set of formulas is consistent if and only if it has a Kripke model. Kripke interpretations with a reflexive accessibility relation are sound and complete for T, and Kripke interpretations with a reflexive and transitive accessibility relation are sound and complete for S4 (see [3], Corollary 7.51, p. 214).

For the proof of the completeness theorem in Section 3 we shall need the notion of the *canonical* Kripke model (see [2], Section 6, or [3], Sections 7.2 and 7.3, say). In what follows, S and Γ are a modal logic and a set of formulas, respectively.

Definition 2.1 A set of formulas Δ is said to be S,Γ-*consistent* if, for no finite subset Δ' of Δ, $\Gamma \vdash_{S} \neg \bigwedge_{\varphi \in \Delta'} \varphi$.

Definition 2.2 Maximal (with respect to inclusion) S,Γ-consistent sets of formulas are called S,Γ-*maximal*.

Proposition 2.3 ([2], Theorem 6.3, p. 115, or [3], Lemma 7.29, p. 204) *Each* S,Γ-*consistent set of formulas can be extended to an* S,Γ-*maximal set.*

To define the S,Γ-canonical Kripke model we need one more bit of notation. For a set of formulas Δ we define the set of formulas Δ^- by

$$\Delta^- = \{\varphi : L\varphi \in \Delta\}.$$

Definition 2.4 The S,Γ-*canonical* Kripke model $\mathfrak{M}_{\mathrm{S}}^{\Gamma} = \langle U_{\mathrm{S}}^{\Gamma}, R_{\mathrm{S}}^{\Gamma}, I_{\mathrm{S}}^{\Gamma} \rangle$ is defined as follows.

1. U_{S}^{Γ} is the set of all S,Γ-maximal sets of formulas.
2. $R_{\mathrm{S}}^{\Gamma} = \{(u, v) : u, v \in U_{\mathrm{S}}^{\Gamma}, u^- \subseteq v\}$.
3. $I_{\mathrm{S}}^{\Gamma}(u)$ is the set of all propositional variables which belong to u.

Theorem 2.5 ([2], Theorem 6.5, p. 118, or [3], Theorem 7.32, p. 206) *For any formula* φ *and any* $u \in U_S^{\Gamma}$, $(\mathfrak{M}_S^{\Gamma}, u) \models \varphi$ *if and only if* $\varphi \in u$.

3 The Logic C

Let C result in adding to K the following three axiom schemes.

***G*1**	$ML\varphi \supset LM\varphi$.
tL	$L\varphi \supset ML\varphi$.
***M*4**	$ML\varphi \supset MLL\varphi$.

Scheme ***G*1** is well studied in the literature. It belongs to the set ***G′*** consisting of all axiom schemes of the form

$$M^m L^n \varphi \supset L^j M^k \varphi,$$

where $L^0\varphi$ is φ ($M^0\varphi$ is φ) and $L^{i+1}\varphi$ is $LL^i\varphi$ ($M^{i+1}\varphi$ is $MM^i\varphi$) (see [2], p. 182). Modal logics containing ***G*1** are characterized by classes of Kripke interpretations with *convergent* accessibility relation, that is, Kripke interpretations $\langle U, R, I\rangle$ such that R satisfies the following condition ([2], p. 134).

$$\text{If } (u, v'), (u, v'') \in R, \text{ then for some } w \in U, (v', w), (v'', w) \in R.$$[2]

It is easy to verify that C is sound with respect to $\mathcal{CD}$,[3] which together with Theorem 3.1 below implies that C is characterized by $\mathcal{CD}$.

Theorem 3.1 (Completeness) *If each cluster-decomposable Kripke model of* Γ *satisfies* φ, *then* $\Gamma \vdash_{\mathrm{C}} \varphi$.

We postpone the proof of the theorem to Section 4 and first establish a number of properties of C. Some of them, such as the independence of axioms and *nonequivalence of modalities* (see [2], pp. 55–56) are of interest in their own right, and the others are needed for the proof of Theorem 3.1.

In the proofs below we shall use the following two *derived* "modal" rules of inference and two theorems of K. This is in addition to NEC and a number of well-known derived propositional rules.

DR1	$\varphi \supset \psi \vdash L\varphi \supset L\psi$	(cf. a similar rule in [2], p. 30)
DR3	$\varphi \supset \psi \vdash M\varphi \supset M\psi$	(it is dual to DR1, cf. [2], p. 35)
K3	$(L\varphi \wedge L\psi) \supset L(\varphi \wedge \psi)$	(see [2], p. 28)
K6	$M(\varphi \vee \psi) \supset (M\varphi \vee M\psi)$	(see [2], p. 34)

We shall also use the schemes

$\boldsymbol{tL}_\mathrm{d}$	$LM\varphi \supset M\varphi$
$\boldsymbol{M4}_\mathrm{d}$	$LMM\varphi \supset LM\varphi$

which are dual to $\boldsymbol{tL}$ and $\boldsymbol{M4}$, respectively, and the axiom

$\boldsymbol{d1}$	$M\top$

which is equivalent to $\boldsymbol{d}$ (see [2], pp. 43–44).

Below we shall use the following notation. We write

$$\varphi_1 \equiv \varphi_2 \equiv \varphi_3 \equiv \cdots \equiv \varphi_{n-1} \equiv \varphi_n$$

instead of

$$\bigwedge_{i=1}^{n-1} \varphi_i \equiv \varphi_{i+1}.$$

Proposition 3.2 $C \vdash MML\varphi \equiv MLL\varphi \equiv LML\varphi \equiv ML\varphi$.

Proof

1.	$ML\varphi \supset MLL\varphi$	$\boldsymbol{M4}$	
2.	$MML\varphi \supset MMLL\varphi$	DR3	1
3.	$MLL\varphi \supset LML\varphi$	$\boldsymbol{G1}$	
4.	$MMLL\varphi \supset MLML\varphi$	DR3	3
5.	$LML\varphi \supset ML\varphi$	$\boldsymbol{tL}_\mathrm{d}$	
6.	$MLML\varphi \supset MML\varphi$	DR3	5
7.	$MML\varphi \equiv MMLL\varphi \equiv MLML\varphi$	implication cycle	2, 4, 6
8.	$ML\varphi \supset MLL\varphi$	$\boldsymbol{M4}$	
9.	$MLL\varphi \equiv LML\varphi \equiv ML\varphi$	implication cycle	3, 5, 8

Thus, we have

$$MML\varphi \equiv ML(ML\varphi) \equiv LML(ML\varphi) \equiv LMM(L\varphi) \equiv LM(L\varphi) \equiv ML\varphi,$$

where the first equivalence is by 7, the second equivalence is by 9, the third equivalence is by 7 and DR1, the fourth equivalence is dual to 9, and the last equivalence is again by 9. □

Proposition 3.3 $\boldsymbol{tL} \vdash \boldsymbol{d}$.

Proof We shall prove $\boldsymbol{d1}$ instead of $\boldsymbol{d}$.

1.	$L\top$	a theorem of K	
2.	$L\top \supset ML\top$	$\boldsymbol{tL}$	
3.	$ML\top \supset M\top$	a theorem of K	
4.	$M\top$	modus ponens (twice)	1, 2, 3

□

For the proof of independence of $\boldsymbol{G1}$, $\boldsymbol{tL}$, and $\boldsymbol{M4}$ we need the following trivial observation.

Proposition 3.4 *$\boldsymbol{t} \vdash \boldsymbol{tL}$ and $\boldsymbol{4} \vdash \boldsymbol{M4}$.*

Proof Scheme $\boldsymbol{tL}_\mathrm{d}$ is an instance of $\boldsymbol{t}$ for $M\varphi$, and scheme $\boldsymbol{M4}$ is obtained from $\boldsymbol{4}$ by DR3. □

Proposition 3.5

1. $\boldsymbol{G1}, \boldsymbol{M4} \nvdash \boldsymbol{tL}$.
2. $\boldsymbol{tL}, \boldsymbol{M4} \nvdash \boldsymbol{G1}$.
3. $\boldsymbol{G1}, \boldsymbol{tL} \nvdash \boldsymbol{M4}$.

Proof Let $\mathfrak{M}$ be a Kripke interpretation with the empty accessibility relation. Then $\mathfrak{M} \models \boldsymbol{G1}, \boldsymbol{M4}$ and $\mathfrak{M} \not\models \boldsymbol{d1}$. Thus, by Proposition 3.3, $\mathfrak{M} \not\models \boldsymbol{tL}$.

Consider a Kripke interpretation $\mathfrak{M} = \langle U, R, I\rangle$, where $U = \{u, v, w\}$, R is the reflexive and transitive closure of $\{u\} \times \{v, w\}$, and $I(u) = I(v) = \{p\}$, and $I(w) = \varnothing$ (see Figure 1). Then $\mathfrak{M} \models$ S4 and, therefore, $\mathfrak{M} \models \boldsymbol{tL}, \boldsymbol{M4}$. However, $(\mathfrak{M}, u) \not\models MLp \supset LMp$.

Consider a Kripke interpretation $\mathfrak{M} = \langle U, R, I\rangle$, where $U = \{u, v, w\}$, R is the reflexive closure of $\{(u, v), (v, w)\}$, $I(u) = I(v) = \{p\}$, and $I(w) = \varnothing$ (see Figure 2). Then $\mathfrak{M} \models \boldsymbol{G1}$, because R is convergent and, by Proposition 3.4, $\mathfrak{M} \models \boldsymbol{tL}$, because R is reflexive. However, $(\mathfrak{M}, u) \not\models MLp \supset MLLp$. □

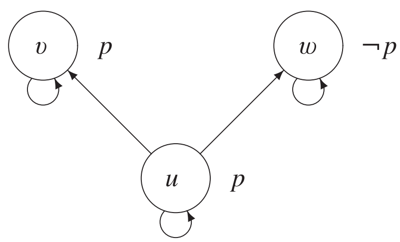

Figure 1 $\boldsymbol{tL}, \boldsymbol{M4} \nvdash MLp \supset LMp$.

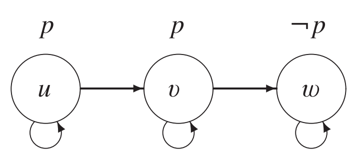

Figure 2 $\boldsymbol{G1}, \boldsymbol{tL} \nvdash MLp \supset MLLp$.

Next we shall classify modalities in C (cf. *nonequivalence of modalities* [2], pp. 55–56).

Theorem 3.6 *In* C *each formula of the form* $M^{i_1}L^{j_1}M^{i_2}L^{j_2}\ldots M^{i_n}L^{j_n}\varphi$, *where* $\sum_{k=1}^{n}(i_k + j_k) > 0$ *is equivalent to one of the following:*

1. $L^i\varphi$, $i = 1, 2, \ldots$,
2. $ML\varphi$,
3. $LM\varphi$, *or*
4. $M^i\varphi$, $i = 1, 2, \ldots$.

Each formula on "level" i *implies in* C *each formula on the level* $i + 1$, $i = 1, 2, 3$. *Neither formula implies a different formula on the same or upper level.*

Proof Note that we prove that some of these modalities are not equivalent even in the presence of $\boldsymbol{t}$.

Nonequivalence of modalities on level 1 First, we show that for $0 \leq k < i$, $\boldsymbol{G1}, \boldsymbol{t}, \boldsymbol{M4} \nvdash L^k\varphi \supset L^i\varphi$ and $\mathrm{C} \nvdash L^i\varphi \supset L^k\varphi$. Let $\mathfrak{M}_i = \langle U_i, R_i, I_i\rangle$ and $\mathfrak{M}'_i = \langle U_i, R'_i, I'_i\rangle$ be the following Kripke interpretations (see Figures 3 and 4, respectively).

$$U_i = \{u_1, u_2, \ldots, u_{i+2}\}.$$
$$R_i = \{(u_j, u_{j+1}) : j = 1, 2, \ldots, i\} \cup U_i \times \{u_{i+2}\},$$
$$\text{and } R'_i \text{ is the reflexive closure of } R_i.$$
$$I_i(u_j) = \{p\} \text{ if } j \leq i, \text{ and } I_i(u_{i+1}) = I_i(u_{i+2}) = \varnothing.$$
$$I'_i(u_j) = \{p\} \text{ if } j \neq i+1, \text{ and } I'_i(u_{i+1}) = \varnothing.$$

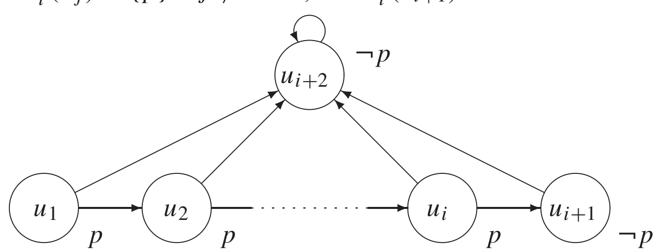

Figure 3 $\mathfrak{M}_i \models \mathrm{C}$, but $\mathfrak{M}_i \not\models L^i\neg p \supset L^k\neg p$ for $k < i$.

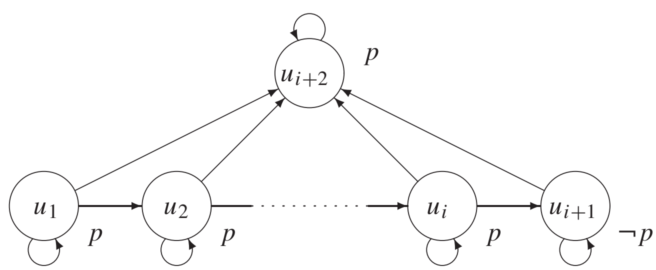

Figure 4 $\mathfrak{M}'_i \models \mathrm{C}, \boldsymbol{t}$, but $\mathfrak{M}'_i \not\models L^k p \supset L^i p$ for $k < i$.

Both $\mathfrak{M}_i$ and $\mathfrak{M}'_i$ are cluster-decomposable, with the cluster $\{u_{i+2}\}$. Thus, by soundness, $\mathfrak{M}_i \models \mathrm{C}$ and $\mathfrak{M}'_i \models \mathrm{C}$, and, since R'_i is reflexive, $\mathfrak{M}'_i \models \boldsymbol{t}$. It is easy to see that $(\mathfrak{M}_i, u_1) \not\models L^i\neg p \supset L^k\neg p$ and $(\mathfrak{M}'_i, u_1) \not\models L^k p \supset L^i p$ for $k = 0, \ldots, i-1$.

Relations between modalities on levels 1 and 2 For each $i = 1, 2, \ldots$, a formula of the form $L^i\varphi \supset ML^i\varphi$ is an instance of $\boldsymbol{tL}$, and, by Proposition 3.2, $\mathrm{C} \vdash ML^i\varphi \supset ML\varphi$. Therefore, $\mathrm{C} \vdash L^i\varphi \supset ML\varphi$.

We proceed to show that for no $i = 0, 1, \ldots$, $\mathrm{C} \vdash ML\varphi \supset L^i\varphi$. Consider a Kripke interpretation $\mathfrak{M}' = \langle U', R', I'\rangle$, where $U' = \{u, v\}$, R' is the reflexive closure of $\{(u, v)\}$, $I'(u) = \varnothing$, and $I'(v) = \{p\}$ (see Figure 5). $\mathfrak{M}'$ is cluster-decomposable; however, $(\mathfrak{M}', u) \not\models MLp \supset L^i p$ for any i.

Relations between modalities on levels 2 and 3 By $\boldsymbol{G1}$, $\mathrm{C} \vdash ML\varphi \supset LM\varphi$, and we need to show that the converse implication is not derivable in C. Consider a Kripke interpretation $\mathfrak{M} = \langle U, R, I\rangle$, where $U = \{u, v, w\}$, R is the reflexive closure of $\{(u, v), (u, w), (v, w), (w, v)\}$, $I(u) = I(v) = \{p\}$, and $I(w) = \varnothing$ (see Figure 6). $\mathfrak{M}$ is cluster-decomposable: its terminal cluster is $\{v, w\}$. Since R is reflexive, $\mathfrak{M} \models \boldsymbol{t}$. However, $(\mathfrak{M}, u) \not\models LMp \supset MLp$.

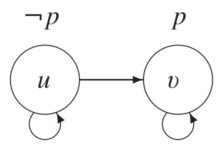

Figure 5 $\boldsymbol{G1}, \boldsymbol{t}, \boldsymbol{M4} \not\vdash MLp \supset L^i p$.

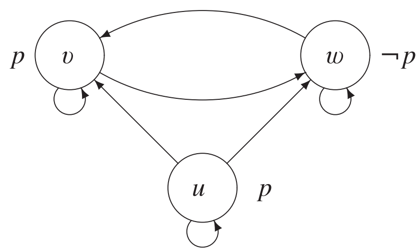

Figure 6 C, $\boldsymbol{t} \not\vdash LMp \supset MLp$.

Relations between modalities on levels 3 and 4, and nonequivalence of modalities on level 4 This follows by duality between levels i and $5 - i$.

In order to complete the proof we note that, by Proposition 3.2, each formula of the form $M^{i_1} L^{j_1} M^{i_2} L^{j_2} \dots M^{i_n} L^{j_n} \varphi$, where $i_n, j_n > 0$, is equivalent in C to $ML\varphi$. The case of the modal prefix ending with M is dual to the above. □

The following proposition is needed for the proof of Theorem 3.1 in Section 4.

Proposition 3.7 C $\vdash (ML\varphi \wedge ML\psi) \supset ML(\varphi \wedge \psi)$.

Proof Note that $(MLL\varphi \wedge LML\psi) \supset MML(\varphi \wedge \psi)$ is a theorem of K. Therefore, $((ML\varphi) \wedge ML\psi) \supset ML(\varphi \wedge \psi)$ propositionally follows from the first formula and Proposition 3.2. □

Now consider the logic S4.2 that contains S4 (***t*** and **4**) and ***G1*** (see [2]). We can easily prove that this logic is equivalent to C with S4.

Proposition 3.8 *The logic* S4.2 *and the logic* C *with* S4 *are equivalent; that is,* S4.2 ⊢ C, S4*, and* C, S4 ⊢ S4.2.

Proof The first part trivially follows from Proposition 3.4, and the second part follows from the fact that C with S4 contains S4.2. □

It is easy to see that S4.2 is sound with respect to $\mathcal{CD}$ for reflexive and transitive Kripke interpretations; that is, if $\Gamma \vdash_{S4.2} \varphi$, then each cluster-decomposable, reflexive, and transitive Kripke model of Γ satisfies φ. This observation, together with Proposition 3.9 below, implies that S4.2 is *characterized* by $\mathcal{CD}$ restricted to reflexive and transitive Kripke interpretations.

Proposition 3.9 ($\mathcal{CD}$ completeness for S4.2) *If each cluster-decomposable, reflexive, and transitive Kripke model of* Γ *satisfies* φ*, then* $\Gamma \vdash_{S4.2} \varphi$.

The proof of Proposition 3.9 is very similar to the proof of Theorem 3.1, and we postpone this proof to Section 4 which contains the proof of Theorem 3.1.

3.1 An alternative axiomatization of C In this section we present a "shorter" axiomatization of C which consists of only two axiom schemes.

Let the logic AC result in adding to K the following axiom schemes.

$\boldsymbol{tL}$ $\quad L\varphi \supset ML\varphi$.

$\boldsymbol{C2}$ $\quad ML\varphi \supset LMLL\varphi$.

We shall also use the scheme $\boldsymbol{C2}_{\mathrm{d}}$ below which is dual to $\boldsymbol{C2}$.

$\boldsymbol{C2}_{\mathrm{d}}$ $\quad MLMM\varphi \supset LM\varphi$.

Note that the axiom scheme $\boldsymbol{C2}$ is obtained by adding an additional modality L to the consequent of the scheme $\boldsymbol{M4}$.

First, we show that AC is derivable in C.

Proposition 3.10 *C* ⊢ *AC*.

Proof $\boldsymbol{tL}$ belongs to C, and $\boldsymbol{C2}$ follows from $\boldsymbol{M4}$ and Proposition 3.2. □

Now we shall show that C is derivable from AC and, therefore, these logics are equivalent.

Proposition 3.11 *AC* ⊢ ***M4***.

Proof

1.	$ML\varphi \supset LMLL\varphi$	$\boldsymbol{C2}$	
2.	$LMLL\varphi \supset MLL\varphi$	$\boldsymbol{tL}_{\mathrm{d}}$	
3.	$ML\varphi \supset MLL\varphi$	syllogism	1, 2

□

Proposition 3.12 *AC* ⊢ ***G1***.

Proof

1.	$MLLL\varphi \supset MLMM\varphi$	follows from $\boldsymbol{d}$	Proposition 3.3
2.	$MLMM\varphi \supset LM\varphi$	$\boldsymbol{C2}_{\mathrm{d}}$	
3.	$MLLL\varphi \supset LM\varphi$	syllogism	1, 2
4.	$MLL\varphi \supset MLLL\varphi$	$\boldsymbol{M4}$	
5.	$ML\varphi \supset MLL\varphi$	$\boldsymbol{M4}$	
6.	$ML\varphi \supset LM\varphi$	a "long" syllogism	5, 4, 3

□

4 Proof of Theorem 3.1

For a Kripke interpretation $\mathfrak{M} = \langle U, R, I\rangle$ and a world $u \in U$ we define the set of formulas $\Delta_{(\mathfrak{M},u)}$ by

$$\Delta_{(\mathfrak{M},u)} = \{\varphi : (\mathfrak{M}, u) \models ML\varphi\}.$$

Lemma 4.1 *Let $\mathfrak{M} = \langle U, R, I\rangle$ be a Kripke interpretation satisfying* C. *Then for each world $u \in U$ and each set of formulas Γ satisfied by $\mathfrak{M}$, the set of formulas $\Delta_{(\mathfrak{M},u)}$ is* C,Γ*-consistent.*

Proof Let Δ' be a finite subset of $\Delta_{(\mathfrak{M},u)}$. Then, by Proposition 3.7, $(\mathfrak{M}, u) \models ML \bigwedge_{\varphi\in\Delta'} \varphi$. That is, for some $v \in U$ such that $(u, v) \in R$, $(\mathfrak{M}, v) \models L \bigwedge_{\varphi\in\Delta'} \varphi$. Since $\mathfrak{M} \models$ C, by Proposition 3.3, $\mathfrak{M} \models M\top$. Thus, there exists a world w such that $(v, w) \in R$, which implies $(\mathfrak{M}, w) \models \bigwedge_{\varphi\in\Delta'} \varphi$. Now, $\Gamma \not\vdash_{\mathrm{C}} \neg \bigwedge_{\varphi\in\Delta'} \varphi$ follows from the fact that $\mathfrak{M} \models$ C, Γ. □

Lemma 4.2 *Let $\mathfrak{M} = \langle U, R, I\rangle$ be a Kripke interpretation satisfying* C *and let the worlds $u, v \in U$ belong to the same connected component of $\mathfrak{M}$. Then $\Delta_{(\mathfrak{M},u)} = \Delta_{(\mathfrak{M},v)}$.*

Proof Assume that $(u, v) \in R$.

Let $\varphi \in \Delta_{(\mathfrak{M},u)}$. Then $(\mathfrak{M}, u) \models ML\varphi$. By Proposition 3.2, $(\mathfrak{M}, u) \models LML\varphi$, implying $(\mathfrak{M}, v) \models ML\varphi$. Thus, $\varphi \in \Delta_{(\mathfrak{M},v)}$.

Similarly, let $\varphi \in \Delta_{(\mathfrak{M},v)}$. Then $(\mathfrak{M}, v) \models ML\varphi$, implying $(\mathfrak{M}, u) \models MML\varphi$. By Proposition 3.2, $(\mathfrak{M}, u) \models ML\varphi$ and, therefore, $\varphi \in \Delta_{(\mathfrak{M},u)}$.

Now the proof follows by induction on the path between u and v. □

Finally, we shall prove a "partial completeness" result.

Lemma 4.3 *Each connected component of $\mathfrak{M}^{\Gamma}_{C}$ belongs to $\mathcal{CD}$.*

Proof Let $\mathfrak{M} = \langle U, R, I\rangle$ be a connected component of $\mathfrak{M}^{\Gamma}_{\mathrm{C}}$, $u \in U$ and let $\Delta = \Delta_{(\mathfrak{M},u)}$. By Lemma 4.2, Δ does not depend on a particular choice of u, and, by Lemma 4.1, it is C,Γ-consistent. Let $U_{\Delta} = \{u \in U^{\Gamma}_{\mathrm{C}} : \Delta \subseteq u\}$. By Proposition 2.3, $U_{\Delta} \neq \varnothing$.

Next we observe that for each $u \in U$ and each $v \in U_{\Delta}$, $u^{-} \subseteq v$; that is, $(u, v) \in R$. Let $\varphi \in u^{-}$. Then $L\varphi \in u$, and, by ***tL***, $ML\varphi \in u$. Therefore, $\varphi \in \Delta$, implying $\varphi \in v$.

Note that the above observation implies $U_{\Delta} \subseteq U$ and $R|_{U_{\Delta}} = U_{\Delta} \times U_{\Delta}$.

To complete the proof, we shall show that for each $u \in U_{\Delta}$ and $v \in U$, if $(u, v) \in R$ then $v \in U_{\Delta}$. Let $u \in U_{\Delta}$, $v \in U$, and $(u, v) \in R$; that is, $u^{-} \subseteq v$. Let $\varphi \in \Delta$. By ***M4***, $L\varphi \in \Delta$ and, therefore, $L\varphi \in u$, implying $\varphi \in v$. □

Now we are ready for the proofs of Theorem 3.1 and Proposition 3.9.

Proof of Theorem 3.1 Assume that $\Gamma \not\vdash_{\mathrm{C}} \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is C,Γ-consistent, and, by Proposition 2.3 and Theorem 2.5, for some $u \in U^{\Gamma}_{\mathrm{C}}$, $(\mathfrak{M}^{\Gamma}_{\mathrm{C}}, u) \models \neg\varphi$. By Lemma 4.3, the connected component of $\mathfrak{M}^{\Gamma}_{\mathrm{C}}$ containing u is cluster-decomposable, and it does not satisfy φ. □

Proof of Proposition 3.9 In the case of S4 $\subseteq$ Γ, each connected component $\mathfrak{M} = \langle U, R, I\rangle$ of $\mathfrak{M}^{\Gamma}_{\mathrm{C}}$ is reflexive and transitive (cf. the proofs of [2], Theorem 6.7, p. 120, and [2], Theorem 6.9, p. 120), and the proof follows from Lemma 4.3. □

Notes

1. It easily follows that C is contained in every logic characterized by a subclass of $\mathcal{CD}$.

2. Cluster-decomposable Kripke interpretations are convergent.

3. That is, if $\Gamma \vdash_{\mathrm{C}} \varphi$, then each cluster-decomposable Kripke model of Γ satisfies φ.

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