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Maximal Three-Valued Clones with the Gupta-Belnap Fixed-Point Property

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Abstract This paper gives a propositional reformulation of the fixed-point problem posed by Gupta and Belnap, using the stipulation logic of Visser. After presenting a solution for clones of three-valued operators that include the constant functions, I determine the maximal three-valued clones with constants that have the fixed-point property, giving different characterizations of them.

1 Introduction

Consider a first-order language $\mathcal{L}$ built with the usual connectives and quantifiers, interpreted by a scheme on a set of truth values E that includes the values 0 (false) and 1 (true). Suppose $\mathcal{L}$ has a monadic predicate T. A ground model for $\mathcal{L}$ is a pair $M = (D, I)$, where D is the domain and I a function that interprets all non-logical symbols of $\mathcal{L}$ except T. Any function g from D to E and any ground model $M = (D, I)$ yield a model $M + g$ of $\mathcal{L}$, using g as the interpretation of the predicate T. We will call val_{M+g} the function that assigns to each sentence of $\mathcal{L}$ its truth value according to the model $M + g$. To make the language self-referential, we will suppose that D includes the sentences of $\mathcal{L}$. We say that T is a truth-predicate for $\mathcal{L}$ in $M + g$ if

$$g(d) = \begin{cases} \text{val}_{M+g}(d), & \text{if } d \text{ is a sentence of } \mathcal{L} \\ 0, & \text{otherwise.} \end{cases}$$

Let us define a function, called the *jump* function and denoted ρ_M, on the set of possible interpretations of T (that is, the set of functions from D to E):

$$\rho_M(g)(d) = \begin{cases} \text{val}_{M+g}(d), & \text{if } d \text{ is a sentence of } \mathcal{L} \\ 0, & \text{otherwise.} \end{cases}$$

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It is obvious that T is a truth predicate for $\mathcal{L}$ in $M + g$ if and only if g is a fixed point of ρ_M. Following Gupta and Belnap's definition in [2] (Def. 2B.11), we say that a scheme has the *fixed-point property* if and only if for every language $\mathcal{L}$ (whose logical connectives are interpreted with that scheme) and ground model M of $\mathcal{L}$ the jump ρ_M has a fixed point. Intuitively, a scheme has the fixed-point property if any language interpreted with that scheme can contain its own truth predicate.

Suppose that $\mathcal{L}$ is a classical first-order language and a is a constant of $\mathcal{L}$ such that $I(a) = \neg Ta$. Then $\neg Ta$ represents in $\mathcal{L}$ the Liar sentence: 'this sentence is false'. If T were a truth-predicate for $\mathcal{L}$, then $\mathrm{val}_{M+g}(Ta) = g(I(a)) = \mathrm{val}_{M+g}(\neg Ta)$. This is impossible if $\neg$ represents classical negation, proving that the classical scheme does not have the fixed-point property. The work of Kripke, Martin and Woodruff, and others[1] showed that the three-valued Kleene schemes (weak and strong) have the fixed-point property. This result is a corollary of the theorem below, which establishes an important sufficient condition for the fixed-point property.

Given a partial order $(E, \leq)$, a set $A \subseteq E$ is *consistent* if every pair of elements of A has an upper bound in $(E, \leq)$. A partial order $(E, \leq)$ is a *ccpo* (coherent complete partial order) if and only if every consistent subset of E has a least upper bound. Then the following fixed-point theorem can be proved.[2]

Theorem 1.1 (Visser) *If $(E, \leq)$ is a ccpo and the logical operators of a scheme are monotonic on that order, then the scheme has the fixed-point property.*

In the Kleene schemes, the set of truth values is $E_3 = \{0, 1, 2\}$, 0 being the value 'false', 1 the value 'true' and the value 2 being assigned to sentences that lack a classical truth-value (paradoxes and other pathological sentences). The relevant ccpo is the order induced by the degree of information that the values give: $2 \leq 0$, $2 \leq 1$. This is called the order of knowledge on E_3 and will be denoted as E_3^k. Although Kleene languages represent a nice generalization of classical connectives, there is an important difference between the classical scheme and the Kleene schemes. It is well known that the classical scheme is functionally complete, but the Kleene schemes are not. That is, classical negation and conjunction suffice to define any Boolean operator, but there are operators on E_3 that cannot be expressed with the Kleene connectives. One of those operators is the unary connective $\downarrow p$ such that $\downarrow 0 = 0$, $\downarrow 1 = 0$, and $\downarrow 2 = 1$. This connective can be interpreted as 'p lacks (classical) truth value' and reflects syntactically the semantic fact that there are sentences lacking classical truth value. Gupta and Belnap showed that the weak Kleene scheme expanded with this connective has the fixed-point property, although the operators are nonmonotonic on the order of knowledge and Theorem 1.1 cannot be applied. On the other hand, the strong Kleene scheme expanded with $\downarrow$ does not have the fixed-point property. This poses the question of how many operators can be added to the Kleene schemes without losing the fixed-point property. Generalizing we arrive at the *(Gupta-Belnap) general fixed-point problem*.

Problem 1.2 Given a set E of truth values, characterize the class of truth-functional first-order schemes on E that satisfy the fixed-point property.[3]

The next section introduces a propositional version of this problem, using the stipulation logic of Visser;[4] then a simple characterization of the fixed-point property in the three-valued propositional case is provided. In the remaining sections of the paper I will determine explicitly the maximal three-valued propositional schemes that have the fixed-point property and study some of their properties.

2 Propositional Version of the Fixed-Point Problem

2.1 Stipulation logic Let us consider, given a fixed set $P = \{p_0, p_1, \ldots\}$ of atomic sentences and a set Σ of symbols for logical connectives, a sentential language $\mathcal{L}_\Sigma(P)$. A (truth-functional) *propositional scheme* ρ is a function that assigns to each connective σ of Σ a function $\rho(\sigma)$ of the corresponding arity on some set E of truth values. An *interpreted language* (on E) is a pair $(\mathcal{L}_\Sigma(P), \rho)$. Given an interpreted language $(\mathcal{L}_\Sigma(P), \rho)$ and $\varphi \in \mathcal{L}_\Sigma(P)$, $(\varphi)_\rho$ represents the function that the scheme ρ assigns to the sentential formula φ. As usual, we use $\varphi(p_{i_0}, \ldots, p_{i_{n-1}})$ to indicate that the only sentences appearing in the formula φ are $p_{i_0}, \ldots, p_{i_{n-1}}$. A *valuation* is a function $v : P \to E$. It can be extended in the canonical way to a function $v^* : \mathcal{L}_\Sigma(P) \to E$ using the propositional scheme: if $\varphi(p_{i_0}, \ldots, p_{i_{n-1}}) \in \mathcal{L}_\Sigma(P)$, $v^*(\varphi) = (\varphi)_\rho(v(p_{i_0}), \ldots, v(p_{i_{n-1}}))$.

Given $P' \subseteq P$, a *stipulation* is a map $s : P' \to \mathcal{L}_\Sigma(P)$. Stipulations are intended to express self-referential sentences: each atomic sentence $p \in P'$ refers to the sentence $s(p)$. For example, let us consider the liar sentence: 'this sentence is false'. If we call it ℓ, then ℓ says ℓ *is false* or, by Convention T, ℓ says $\neg\ell$, so the liar sentence is expressed by the stipulation $s(\ell) = \neg\ell$. Given a stipulation $s : P' \to \mathcal{L}_\Sigma(P)$, a valuation v is *s-consistent* if for all $p \in P'$, $v(p) = v^*(s(p))$. Let us say that an interpreted language has the *fixed-point property* if for every stipulation s there is an s-consistent valuation. The Gupta-Belnap general fixed-point problem can be specialized as follows.

Problem 2.1 Given a set E of truth values, characterize the interpreted languages on E that have the fixed-point property.

The solution to this propositional version of the fixed-point problem is a necessary condition for the solution to the general fixed-point problem, in the sense that if a first-order scheme has the fixed-point property, then its underlying propositional interpreted language has the fixed-point property.

2.2 Clones of functions Let us present some definitions from the theory of algebras of functions that will be used to give a semantic reformulation of the fixed-point property.

Let E be a set, $\mathcal{O}_E^{(n)}$ the set of n-ary functions on E, and $\mathcal{O}_E = \bigcup_{n\geq 1} \mathcal{O}_E^{(n)}$ the set of all finitary functions on E. E_k will denote the set $\{0, 1, \ldots, k-1\}$. We write $\mathcal{O}_k$ instead of $\mathcal{O}_{E_k}$. A *clone* (of functions on E) is a set of functions of $\mathcal{O}_E$ which contains the projections (i.e., the functions $e_i^n(x_0, \ldots, x_{n-1}) = x_i$ for all $n \geq 1$ and $0 \leq i \leq n-1$) and is closed under composition of functions. A clone is a *clone with constants* if it contains the constant functions: $c_a^n(x_0, \ldots, x_{n-1}) = a$, for all $n \geq 1$ and all $a \in E$. Let $X \subseteq \mathcal{O}_E$; then $\langle X \rangle$ represents the clone with constants generated by X (that is, the least clone that contains X and the constant functions). If F is a clone, $F^{(n)}$ represents the set of functions of F with n variables. The set $F^{(1)}$ is closed under composition and is called the *transformation monoid* of the clone F.

In order to characterize the clones of functions we need the notion of a function preserving a relation. Let E be a finite set and $\mathcal{R}_E$ the set of all finitary relations on E. As a convenient pictorial device, an n-ary relation will be represented as a matrix, each column of the matrix being an element of the relation. Two matrices represent the same relation if they have the same columns, irrespective of their order. Given $f \in \mathcal{O}_E^{(n)}$ and an m-ary relation $R \in \mathcal{R}_E$, *f preserves R*, or R is *invariant for f*, when for every matrix $(a_{ij})_{m\times n}$ of elements of E, if

$$(a_{00},\ldots,a_{(m-1)0}),\ldots,(a_{0(n-1)},\ldots,a_{(m-1)(n-1)})\in R,$$

then

$$(f(a_{00},\ldots,a_{0(n-1)}),\ldots,f(a_{(m-1)0},\ldots,a_{(m-1)(n-1)}))\in R$$

(that is, if the columns of the matrix are elements of R, then the column made by the function applied to the rows is also an element of R).

Given $Q\subseteq\mathcal{R}_E$, Pol Q is the set of functions that preserve all the relations in Q (called the polymorphisms of Q). We write Pol R instead of Pol$\{R\}$. Polymorphisms provide an important characterization of clones:[5] $F\subseteq\mathcal{O}_E$ is a clone of functions if and only if there is $Q\subseteq\mathcal{R}_E$ such that $F=\text{Pol } Q$.

2.3 Semantic version of the fixed-point problem Given a clone $F\subseteq\mathcal{O}_E$, an interpreted language $(\mathcal{L}_\Sigma(P),\rho)$ is *adequate* for F if $F=\langle\{\rho(\sigma):\sigma\in\Sigma\}\rangle$. It is obvious that all interpreted languages adequate for a clone are equivalent with respect to the fixed-point property. Thus we say that a clone has the *fixed-point property* if some adequate interpreted language for it has the fixed-point property. We will be interested mainly in languages that have connectives that are interpreted as the unary constants. Constants are added to propositional languages to allow for the expression of empirical sentences. For example, the sentence 'this sentence is false or snow is white' can be expressed by the stipulation $s(p)=\neg p\vee q$, $s(q)=c_1$, where p stands for the whole sentence, q stands for 'snow is white' and c_1 is the constant 'truth'. Every interpreted language $(\mathcal{L}_\Sigma(P),\rho)$ generates the clone with constants $\langle\{\rho(\sigma):\sigma\in\Sigma\}\rangle$. Now we can give a propositional semantic version of the fixed-point problem.

Problem 2.2 Given some set E of truth values, characterize the set of clones with constants of $\mathcal{O}_E$ that have the fixed-point property.

We will prove one general lemma that simplifies the solution.

Let us consider an interpreted language $(\mathcal{L}_\Sigma(P),\rho)$ and a stipulation $s:Q\to\mathcal{L}_\Sigma(P)$, $Q\subseteq P$. A *substipulation* s' is the restriction of s to some subset $P'\subseteq Q$ (i.e., $s':P'\to\mathcal{L}_\Sigma(P)$ such that $s'(p)=s(p)$ for all $p\in P'$).

Lemma 2.3 *Let $(\mathcal{L}_\Sigma(P),\rho)$ be an interpreted language defined on a finite set of truth values and $s:P\to\mathcal{L}_\Sigma(P)$ be a stipulation. If for any finite substipulation s' there is an s'-consistent valuation, then there is an s-consistent valuation.*

Proof Let s^n denote the substipulation that restricts the stipulation s to the set $\{p_0,\ldots,p_{n-1}\}$. Let us use E_k as the set of truth values. We build a tree using the s^n-consistent valuations. The nodes of the tree will be sequences $(a_0,\ldots,a_{n-1})$, with $n\in\omega$, $a_n\in E_k$. The first node is the empty sequence (). Given one node $(a_0,\ldots,a_{n-1})$, its successors are the nodes $(a_0,\ldots,a_n)$ such that the partial valuation $v(p_i)=a_i$, for $0\le i\le n$ can be extended to an s^{n+1}-consistent valuation (if any). Since there is an s^n-consistent valuation, for every $n\in\omega$, the tree is infinite. It is also finitely generated, since every node has at most k successors. Applying König's Lemma, the tree has an infinite branch, with nodes $a^n=(a^n_0,\ldots,a^n_{n-1})$, for $n\in\omega$. Then the valuation such that $v(p_n)=a^{n+1}_n$ for all n is an s-consistent valuation. □

3 A Solution to the Three-Valued Case

From now on we will follow some conventions to simplify notation: we will use ambiguously the same symbol for a function of a clone and for its name in an adequate language for the clone, and use ordinary variables $x_i, y_i, \ldots$ instead of sentence letters $p_0, p_1, \ldots$. We will use $\overline{x}$ to denote an n-tuple $(x_0, \ldots, x_{n-1})$ where the value n can be determined by the context (the same convention applies to $\overline{y}$, $\overline{a}$, etc.). Given a clone $F \subseteq \mathcal{O}_3$, a stipulation s is then a system of equations $x_i = f_i(x_{i_1}, \ldots, x_{i_{r_i}})$ $(i = 0, 1, \ldots)$, with $f_i \in F$ and $i_1, \ldots, i_{r_i}, r_i \in \omega$, and an s-consistent valuation is a solution of the system of equations.

The solution to Problem 2.2 in the three-valued case is based on the transformation monoids, which offer a nice classification of the clones. We say that a transformation monoid is *stable* (or that it is a *stable monoid*) if all its functions have a fixed point. When a clone has a stable monoid, we say that the clone has the *unary fixed-point property*. We say that a clone F has the *uniform fixed-point property* if for every finite stipulation $x_i = f_i(\overline{x}, \overline{y})$, $f_i \in F^{(n+k)}$ $(i = 0, \ldots, n-1)$, there are functions $g_i \in F^{(k)}$ such that $g_i(\overline{y}) = f_i(\overline{g}(\overline{y}), \overline{y})$.[6] We say that F has the *uniform unary fixed-point property* if for every $f \in F^{(n+1)}$ there is $g \in F^{(n)}$ such that $g(\overline{y}) = f(g(\overline{y}), \overline{y})$.

In order to prove the theorem we need a lemma, due to Smullyan.

Lemma 3.1 (Smullyan) *If $F \subseteq \mathcal{O}_E$ has the uniform unary fixed-point property, then F has the uniform fixed-point property.*

Proof The proof is by induction on the number of stipulations. The case of the stipulation of one variable is immediate. Consider the stipulation of $n+1$ variables given by the system of equations,

$$x = f(x, \overline{y}, \overline{z}),$$
$$y_i = g_i(x, \overline{y}, \overline{z}),$$

for $i = 0, \ldots, n-1$. By the uniform unary fixed-point property, we have a function h such that $h(\overline{y}, \overline{z}) = f(h(\overline{y}, \overline{z}), \overline{y}, \overline{z})$. Take $k_i(\overline{y}, \overline{z}) := g_i(h(\overline{y}, \overline{z}), \overline{y}, \overline{z})$. Consider the stipulation determined by $y_i = k_i(\overline{y}, \overline{z})$ $(i = 0, \ldots, n-1)$. Let the functions u_i provide a uniform solution of this last system (that exists by induction hypothesis). Let $v(\overline{z}) := h(u_0(\overline{z}), \ldots, u_{n-1}(\overline{z}), \overline{z})$. Then v and the u_i constitute a uniform solution for our original system. □

The solution to the fixed-point problem is given by the following theorem:[7] (for the notation of unary three-valued functions, see the Appendix (Section 9)).

Theorem 3.2 (Visser) *A clone with constants $F \subseteq \mathcal{O}_3$ has the fixed-point property if and only if it has the unary fixed-point property.*

Proof Let F be a clone with constants in $\mathcal{O}_3$. Suppose that $F^{(1)}$ is a stable monoid. We want to show that F has the uniform fixed-point property. By Lemma 3.1, it is sufficient to show that F has the uniform unary fixed-point property.

We define $\sharp F$ as follows. If $\neg_i \in F$ $(i \in E_3)$, then $\sharp F := i$. If no function $\neg_i$ belongs to F, $\sharp F := 0$. Since the composition of two different functions $\neg_i$ is a function without fixed points, $\sharp F$ is well defined.

We want to show that for any $f \in F^{(1)}$, $f^2(\sharp F) = f(f^2(\sharp F))$. If $\sharp F$ is a fixed point of $f \in F^{(1)}$, we are done. If it is not, $f(\sharp F) = a \neq \sharp F$. If $f(a) = a$, then

$f(f^2(\sharp F)) = a = f^2(\sharp F)$. Otherwise, there are only two possible cases: either $f(a) = \sharp F$ or $f(a) = b$, with b such that $\{0, 1, 2\} = \{\sharp F, a, b\}$ and $f(b) = b$. The first case is not possible because f would be $\neg_b$, contradicting the definition of $\sharp F$, and the second case satisfies the property $f(f^2(\sharp F)) = b = f^2(\sharp F)$. Therefore, $f^2(\sharp F) = f(f^2(\sharp F))$. Since this equation holds for any $f \in F$ and since F has constants, it follows that, for any $n+1$-ary g, $g(g(\sharp F, \overline{y}), \overline{y}) = g(g(g(\sharp F, \overline{y}), \overline{y}), \overline{y})$. Hence, F has uniform fixed points. This implies that any finite stipulation has a consistent valuation; by Lemma 2.3, F has the fixed-point property. □

To give a philosophical interpretation of this theorem, notice that a unary function without a fixed point can be considered as a generalization of classical negation (as far as fixed-point properties are involved). Then the theorem states that a three-valued propositional scheme has the fixed-point property if and only if it does not express (a generalized) negation. Notice that Kleene negation $\neg_2$ is not a generalized negation in this sense, unlike β_2, usually called "strong negation" or "exclusion negation."

4 Definition of the Principal Clones

The aim of the remaining sections of this paper is to give different characterizations of the maximal three-valued clones with constants having the fixed-point property. The clones will be defined using certain conditions that can be checked very easily from the truth table of the logical operators. Before we state the definitions, we need to introduce some new notation. Let $f \in \mathcal{O}_3$. The derived set of f, denoted der f, is the set of all functions which can be obtained from f with some (all, none) of its variables replaced by constants. I_{01} is the clone Pol(01), that is, the set of all functions that preserve the set $\{0, 1\}$. If $f \in I_{01}$, then the restriction of f, denoted re f, is the function re $f : E_2 \to E_3$ defined as re $f(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1})$, for all $x_0, \ldots, x_{n-1} \in E_2$. E_2^t (E_3^t) will denote the order of truth on E_2 (respectively, E_3), determined by $0 \leq 1$ (respectively, $0 \leq 2 \leq 1$). We recall from Section 1 that E_3^k is the order of knowledge on E_3, determined by $2 \leq 0$, $2 \leq 1$. We will also use the concept of inner automorphism: let σ be a permutation of a set E and let us define the mapping $(-)^\sigma : \mathcal{O}_E \to \mathcal{O}_E$ such that, for every $f \in \mathcal{O}_E^{(n)}$, $(f)^\sigma(x_0, \ldots, x_{n-1}) = \sigma^{-1} f(\sigma x_0, \ldots, \sigma x_{n-1})$. This mapping is called an inner automorphism. If $F \subseteq \mathcal{O}_E$, let us define $F^\sigma = \{(f)^\sigma : f \in F\}$.

Definition 4.1 We will call principal clones the following twelve clones with constants in $\mathcal{O}_3$.

1. M_2 is the clone of the monotonic functions on the order E_3^t.
2. K_2 is the clone of the monotonic functions on the order E_3^k.
3. H_2 is the clone of all functions $f \in \mathcal{O}_3$ such that, for every $g \in$ der f, if $g \neq c_2$, then $g \in I_{01}$ and re g is monotonic on the order E_2^t.
4. G_2 is the clone of all functions $f \in \mathcal{O}_3$ that satisfy the following conditions:
   (a) For every $g \in$ der f, if $g \neq c_2$, then $g \in I_{01}$.
   (b) If $f(a_0, \ldots, a_{n-1}) \neq 2$, for some $a_i \in E_3$ and $a_{i_0} = \cdots = a_{i_{j-1}} = 2$, for $0 \leq j \leq n-1$ and $0 \leq i_0 \leq \cdots \leq i_{j-1} \leq n-1$, then the function

$$\text{re}\, f(a_0, \ldots, a_{i_0-1}, x_0, a_{i_0+1}, \ldots, a_{i_{j-1}-1}, x_{j-1}, a_{i_{j-1}+1}, \ldots, a_{n-1})$$

is constant.

5. For each $F \in \{M, K, H, G\}$, $F_0 = (F_2)^{\neg_1}$ and $F_1 = (F_2)^{\neg_0}$.

As an illustration of the definition, let us consider the following functions:

f_1	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

f_2	0	1	2
0	1	1	1
1	0	1	2
2	2	1	2

f_3	0	1	2
0	0	1	0
1	1	1	0
2	0	0	2

f_4	0	1	2
0	0	0	1
1	0	0	0
2	1	0	2

If we restrict our attention to the clones M_2, K_2, H_2, and G_2, the function f_1 belongs only to M_2, f_2 (the conditional of the strong Kleene scheme) belongs only to K_2, f_3 belongs only to H_2, and f_4 belongs only to G_2. For example, $f_3(2, 0) \neq 2$, but the function re $f_3(x, 0)$ is the identity function on E_2, so $f_3 \notin G_2$.

The following proposition determines the transformation monoids of the principal clones and shows that they are precisely the maximal stable monoids.[8]

Proposition 4.2 *$\mathcal{O}_3^{(1)}$ has the following maximal stable monoids (all are supposed to include e_0^1 and the constant unary functions):*

$$M_0^{(1)} = \{\alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_4, \beta_5\} \qquad M_1^{(1)} = \{\alpha_0, \alpha_1, \alpha_3, \gamma_2, \gamma_4, \gamma_5\}$$
$$K_0^{(1)} = \{\neg_0, \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_2, \gamma_4\} \qquad K_1^{(1)} = \{\neg_1, \alpha_2, \alpha_3, \beta_0, \beta_5, \gamma_2, \gamma_3\}$$
$$H_0^{(1)} = \{\alpha_3, \beta_5, \gamma_2, \gamma_3, \gamma_4, \gamma_5\} \qquad H_1^{(1)} = \{\alpha_1, \gamma_4, \beta_0, \beta_1, \beta_4, \beta_5\}$$
$$G_0^{(1)} = \{\neg_0, \gamma_2, \gamma_3, \gamma_4, \gamma_5\} \qquad G_1^{(1)} = \{\neg_1, \beta_0, \beta_1, \beta_4, \beta_5\}$$
$$M_2^{(1)} = \{\beta_0, \beta_1, \beta_5, \gamma_2, \gamma_3, \gamma_4\}$$
$$K_2^{(1)} = \{\neg_2, \alpha_1, \alpha_3, \beta_4, \beta_5, \gamma_4, \gamma_5\}$$
$$H_2^{(1)} = \{\beta_0, \gamma_2, \alpha_0, \alpha_1, \alpha_2, \alpha_3\}$$
$$G_2^{(1)} = \{\neg_2, \alpha_0, \alpha_1, \alpha_2, \alpha_3\}$$

The proof of this result is given in the Appendix (Section 9).

The next three sections will analyze more deeply the structure and properties of the principal clones. In these sections we will

(a) prove that the principal clones are all the clones with constants maximal for the fixed-point property and characterize the principal clones as the clones of functions satisfying a certain relation,
(b) give a finite generator system of operators for the principal clones,
(c) locate the principal clones in the lattice of three-valued clones. They are maximal, submaximal, or subsubmaximal elements in the lattice. In a sense, this shows that the clones with constants maximal for the fixed-point property are "big" clones, hosting a great variety of operators.

5 Maximality of the Principal Clones

It is easily shown that, given any set E, for any transformation monoid $M \subseteq \mathcal{O}_E^{(1)}$ the set of clones F such that $F^{(1)} = M$ is a complete lattice. Consider the function Φ that assigns to each transformation monoid M the greatest element of that lattice (i.e., the unique clone that contains any clone with transformation monoid M). In general, the function Φ is not monotonic: consider $N := K_2^{(1)} \backslash \{c_0, c_1\} \subset \mathcal{O}_3^{(1)}$ and the clone $I_2 := \mathrm{Pol}(2) \subset \mathcal{O}_3$. I_2 is a clone with the fixed-point property (the valuation that assigns 2 to all variables is s-consistent, for all stipulations s) and it is easy to see that

$\Phi(N) = I_2$ (use Theorem 7.1 below). Lemma 5.2 yields that $\Phi(K_2^{(1)}) = K_2$. Then $N \subset K_2^{(1)}$, but $\Phi(N) \nsubseteq \Phi(K_2^{(1)})$ (since $c_0 \notin I_2$). Proposition 5.3 will show that Φ is monotonic in some special cases: consider $M \subseteq \mathcal{O}_3^{(1)}$ to be a maximal stable monoid and J the principal clone corresponding to M. Then for every transformation monoid $N \subseteq \mathcal{O}_3^{(1)}$, if $N \subseteq M$, then $\Phi(N) \subseteq \Phi(M) = J$. This proposition will provide the key step to prove that the principal clones are all the clones with constants maximal for the fixed-point property.

Lemma 5.1 *Let $(E, \leq)$ be a finite partial order and $F \subseteq \mathcal{O}_E$ be the clone of all monotonic functions on $\leq$, and let us consider a clone $G \subseteq \mathcal{O}_E$ such that $G^{(1)} \subseteq F^{(1)}$. Then $G \subseteq F$.*

Proof Let E, F, and G be as given in the hypothesis of the lemma. Consider a function $f \in G^{(n)}$ and elements $\overline{a}, \overline{b} \in E^n$ such that $a_i \leq b_i$ for $0 \leq i \leq n-1$. Let us consider the functions $h_i(x) = f(b_0, \ldots, b_{i-1}, x, a_{i+1}, \ldots, a_{n-1})$, for $0 \leq i \leq n-1$. By hypothesis, all h_i are monotonic on $\leq$. Then $h_0(a_0) \leq h_0(b_0) = h_1(a_1) \leq h_1(b_1) = h_2(a_2) \leq \cdots \leq h_{n-2}(b_{n-2}) = h_{n-1}(a_{n-1}) \leq h_{n-1}(b_{n-1})$, showing that $f(\overline{a}) \leq f(\overline{b})$. □

Lemma 5.2 *Given $M \in \left\{M_2^{(1)}, K_2^{(1)}, H_2^{(1)}, G_2^{(1)}\right\}$, let F be a clone with constants such that $F^{(1)} \subseteq M$, and let J be the principal clone corresponding to M. Then $F \subseteq J$.*

Proof (Cases of $M_2^{(1)}$ and $K_2^{(1)}$) By Lemma 5.1, given the definitions of M_2 and K_2.

Case of $H_2^{(1)}$ Let us consider a clone F and $f \in F$ such that $f \notin H_2$. By the definition of H_2, there is a function $g \in \operatorname{der} f$ such that $g \neq c_2$ and either $g \notin I_{01}$ or re g is nonmonotonic on the order E_2^t. Let us consider both possibilities in turn.

(1) There are $\overline{a} \in E_2^n$ such that $g(\overline{a}) = 2$. Since $g \neq c_2$, there are $\overline{b} \in E_3^n$ such that $g(\overline{b}) \neq 2$. Let us consider the collection of functions $h_i(x) := g(b_0, \ldots, b_{i-1}, x, a_{i+1}, \ldots, a_{n-1})$, for $0 \leq i \leq n-1$. Call k the least i such that, for all j such that $i \leq j \leq n-1$, $h_j(x) \neq c_2$ (k exists because $h_{n-1}(b_{n-1}) \neq 2$) and suppose that $h_k \in I_{01}$. If $k > 0$, then $h_{k-1}(b_{k-1}) = h_k(a_k) \in E_2$; that is, $h_{k-1}(x) \neq c_2$, contradicting the minimality of k. If $k = 0$, $h_0 \in I_{01}$ contradicts that $h_0(a_0) = g(\overline{a}) = 2$. Therefore, $h_k \notin I_{01}$ and $h_k \neq c_2$. By the definition of $H_2^{(1)}$, $h_k \notin H_2^{(1)}$.

(2) There are $\overline{a}, \overline{b} \in E_2^n$ such that $a_i \leq b_i$ for $0 \leq i \leq n-1$ and $g(\overline{a}) = 1$, $g(\overline{b}) = 0$. Let us consider the functions $h_i(x)$ defined as in case (1). Call k the least i such that, for all j such that $i \leq j \leq n-1$, $h_j(b_j) = 0$. If either $a_k = b_k$ or $h_k(a_k) = 0$, then $h_k(a_k) = h_k(b_k)$. If $k > 0$, then $h_{k-1}(b_{k-1}) = h_k(a_k) = h_k(b_k) = 0$, contradicting the minimality of k. If $k = 0$, $h_0(a_0) = h_0(b_0) = 0$ contradicts that $h_0(a_0) = g(\overline{a}) = 1$. So $a_k = 0$, $b_k = 1$, and $h_k(a_k) = 1$, and by the definition of $H_2^{(1)}$, $h_k \notin H_2^{(1)}$.

In either case there is $h_k \in F^{(1)}$ such that $h_k \notin H_2^{(1)}$, contradicting that $F^{(1)} \subseteq H_2^{(1)}$.

Case of $G_2^{(1)}$ Consider a clone F and $f \in F$ such that $f \notin G_2$. Let us distinguish two cases.

(1) There is a function $g \in \operatorname{der} f$ such that $g \neq c_2$ and $g \notin I_{01}$. An argument analogous to (1) in the case of $H_2^{(1)}$ proves that there is a function in $F^{(1)}$ that does not belong to $G_2^{(1)}$.

(2) Without losing generality, we can suppose that for certain numbers m, l such that $m + l = n$ there are values $a_0, \ldots, a_{l-1} \in E_2$ such that $f(a_0, \ldots, a_{l-1}, 2, \ldots, 2) \neq 2$ and the function $\operatorname{re} f(a_0, \ldots, a_{l-1}, x_0, \ldots, x_{m-1})$ is not constant. We define $g(x_0, \ldots, x_{m-1}) = f(a_0, \ldots, a_{l-1}, x_0, \ldots, x_{m-1})$. We can suppose that g belongs to I_{01} (otherwise, case (1) applies). Then $g(2, \ldots, 2) \neq 2$ and there are $b_0, \ldots, b_{m-1}$, $c_0, \ldots, c_{m-1} \in E_2$ such that $g(b_0, \ldots, b_{m-1}) = 1$ and $g(c_0, \ldots, c_{m-1}) = 0$. Let us consider the collection of functions of the form $h(x) := g(d_0, \ldots, d_{i-1}, x, d_{i+1}, \ldots, d_{m-1})$, for some $d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{m-1} \in E_3$. We will deduce a contradiction assuming the following property: (*) for any function h of that form, if $h(2) \neq 2$, then $h(0) = h(1) \in E_2$. Consider the functions $q_i(x_0, \ldots, x_i) = \operatorname{re} g(x_0, \ldots, x_i, 2, \ldots, 2)$, for all $0 \leq i \leq m-1$ and the functions defined like q_i but with a permutation of the variables in $\operatorname{re} g$ (let us say that these functions are similar to q_i). We will prove by induction on the number of variables that all the functions similar to q_i are constant functions with value in E_2. Since $g(2, \ldots, 2) \neq 2$, by the property (*) $\operatorname{re} g(x_0, 2, \ldots, 2)$, $\operatorname{re} g(2, x_1, 2, \ldots, 2), \ldots, \operatorname{re} g(2, \ldots, 2, x_{m-1})$ are constant functions with value in E_2. This proves the case $i = 0$. Now suppose that all the functions similar to q_{i-1} are constant with value in E_2, and let us take arbitrary elements $d_0, \ldots, d_i, e_0, \ldots, e_i \in E_2$. By induction hypothesis, $q_i(2, d_1, \ldots, d_i) \neq 2$, and by the property (*),

$$q_i(0, d_1, \ldots, d_i) = q_i(1, d_1, \ldots, d_i);$$

hence,

$$q_i(d_0, \ldots, d_i) = q_i(e_0, d_1, \ldots, d_i).$$

Let us suppose now that

$$q_i(d_0, \ldots, d_i) = q_i(e_0, \ldots, e_{j-1}, d_j, \ldots, d_i)$$

for some $1 \leq j \leq i$; then by induction hypothesis again,

$$q_i(e_0, \ldots, e_{j-1}, 2, d_{j+1}, \ldots, d_i) \neq 2,$$

and by the property (*),

$$q_i(e_0, \ldots, e_{j-1}, 0, d_{j+1}, \ldots, d_i) = q_i(e_0, \ldots, e_{j-1}, 1, d_{j+1}, \ldots, d_i);$$

hence,

$$q_i(e_0, \ldots, e_{j-1}, d_j, \ldots, d_i) = q_i(e_0, \ldots, e_j, d_{j+1}, \ldots, d_i).$$

These identities show that

$$q_i(d_0, \ldots, d_i) = q_i(e_0, \ldots, e_j, d_{j+1}, \ldots, d_i).$$

We have proved by induction that

$$q_i(d_0, \ldots, d_i) = q_i(e_0, \ldots, e_i).$$

This shows that q_i is a constant function. The argument obviously generalizes to all functions similar to q_i. As a particular case of the induction, we obtain that q_{m-1} is a constant function, contradicting that $g(b_0, \ldots, b_{m-1}) \neq 0$ and $g(c_0, \ldots, c_{m-1}) \neq 1$.

Since (*) implies a contradiction, there are elements $d_0, \ldots, d_{m-1} \in E_3$ such that the function $h(x) = g(d_0, \ldots, d_{i-1}, x, d_{i+1}, \ldots, d_{m-1})$ satisfies $h(2) \neq 2$ and re h is nonconstant. By the definition of $G_2^{(1)}$, $h \notin G_2^{(1)}$. □

The following proposition extends Lemma 5.2 to all maximal stable monoids.

Proposition 5.3 *Let $M \subseteq \mathcal{O}_3^{(1)}$ be a maximal stable monoid, J the principal clone corresponding to M, and let $F \subseteq \mathcal{O}_3$ be a clone with constants such that $F^{(1)} \subseteq M$. Then $F \subseteq J$.*

Proof Consider M and J as given in the proposition. We know from Definition 4.1 that there is a permutation σ of E_3 and a clone $G \in \{M_2, K_2, H_2, G_2\}$ such that $J = G^\sigma$. Consider $N := G^{(1)}$. Using Proposition 4.2 it is easy to check that $M = N^\sigma$. Let F be a clone such that $F^{(1)} \subseteq M = N^\sigma$. Then it is easy to prove that $(F^{\sigma^{-1}})^{(1)} = (F^{(1)})^{\sigma^{-1}} \subseteq N$. By Lemma 5.2, $F^{\sigma^{-1}} \subseteq G$. Hence $F \subseteq G^\sigma = J$. □

A consequence of this proposition is the characterization of the clones with constants maximal for the fixed-point property.

Theorem 5.4 *The principal clones are all the clones with constants maximal for the fixed-point property.*

Proof That the principal clones have the fixed-point property is a consequence of Theorem 3.2 and the fact that their transformation monoids are stable. Since no maximal stable monoid is included in another, it follows that no principal clone is included in another principal clone. To prove completeness, consider any set $A \subseteq \mathcal{O}_3$ such that for every principal clone J there is $f \in A$ such that $f \notin J$. Our aim is to prove that $\langle A \rangle$ is a clone without the fixed-point property. Suppose there is a principal clone J such that $\langle A \rangle^{(1)} \subseteq J^{(1)}$. By Proposition 5.3, $\langle A \rangle \subseteq J$, contradicting the definition of A. Hence, for all maximal stable monoids M, $\langle A \rangle^{(1)} \nsubseteq M$. Since any stable monoid is included in a maximal stable monoid (due to the finiteness of the lattice of three-valued transformation monoids), it follows that $\langle A \rangle^{(1)}$ is not a stable monoid. Therefore, $\langle A \rangle$ does not have the fixed-point property. □

We will now state as a corollary of Proposition 5.3 another characterization of the principal clones in terms of polymorphisms of certain relations associated with transformation monoids. Consider a transformation monoid $M = \{f_0, \ldots, f_{n-1}\} \subseteq \mathcal{O}_k^{(1)}$, $k \geq 2$. Then let us consider the relation,

$$\Gamma(M) = \begin{pmatrix} f_0(0) & \ldots & f_{n-1}(0) \\ \vdots & & \vdots \\ f_0(k-1) & \ldots & f_{n-1}(k-1) \end{pmatrix}.$$

It is easy to prove the following two facts about the relation $\Gamma(M)$.[9]

Lemma 5.5

1. *For any clone $F \subseteq \mathcal{O}_k$, $F \subseteq \operatorname{Pol}\Gamma(F^{(1)})$.*
2. *For any transformation monoid $M \subseteq \mathcal{O}_k^{(1)}$, $(\operatorname{Pol}\Gamma(M))^{(1)} = M^{(1)}$.*

Lemma 5.5 gives a determination of the function Φ: for every transformation monoid $M \subseteq \mathcal{O}_k^{(1)}$, $\Phi(M) = \operatorname{Pol}\Gamma(M)$. It also shows that $J \subseteq \operatorname{Pol}\Gamma(J^{(1)})$, for every principal clone J. Proposition 5.3 implies that the principal clone J contains every clone with transformation monoid $J^{(1)}$. In particular, by Lemma 5.5, $\mathrm{Pol}\,\Gamma(J^{(1)}) \subseteq J$. Therefore, for every principal clone J, $J = \mathrm{Pol}\,\Gamma(J^{(1)})$.[10]

6 Generator Systems for the Principal Clones

A finite generator system will be given for the principal clones with index 2. Generators for the rest can then be found analogously. This property has a special interest, because it gives a basis for the maximal propositional scheme that has the fixed-point property and contains the nonmonotonic scheme of Gupta and Belnap (notice that the clone generated by the weak Kleene operators plus $\downarrow$ is strictly included in G_2).

It is well known that the clone K_2 is the clone generated by the strong Kleene scheme with constants, so it is finitely generated. The following theorem shows that the clones M_2, H_2, and G_2 are also finitely generated. We will use the following special binary functions:

$\wedge_w$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\wedge_s$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\wedge_o$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\odot$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$\boxdot$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	2

Considering the values 0, 1, 2 as representing the truth values ‘false’, ‘true’, and ‘neither true nor false’, respectively, some of these functions have interesting interpretations. The functions $\wedge_s$ and $\wedge_w$ are the conjunction operators of the strong and weak Kleene schemes, respectively. The operator $\wedge_o$ can be read, as a mnemonic rule, as a conjunction operator that incorporates an “overlooking” policy toward pathological sentences: when one of the conjuncts is a pathological sentence, it just returns the value of the other sentence.

For each $\wedge_i$, $i \in \{w, s, o\}$, we define as usual $x \vee_i y := \neg_2(\neg_2 x \wedge_i \neg_2 y)$. Let $f(x_0, \ldots, x_{n-1}) \in \mathcal{O}_3^{(n)}$. We say that the variable x_i is a *contaminant variable* if, for every $a_0, \ldots, a_{n-1} \in E_3$, $f(a_0, \ldots, a_{n-1}) = 2$ whenever $a_i = 2$.

Generator systems for the principal clones are given by the following theorem.

Theorem 6.1

1. $M_2 = \langle \wedge_s, \vee_s, \gamma_2, \beta_1 \rangle$.
2. $K_2 = \langle \neg_2, \wedge_s \rangle$.
3. $H_2 = \langle \wedge_w, \vee_w, \wedge_o, \vee_o \rangle$.
4. $G_2 = \langle \neg_2, \wedge_w, \odot \rangle$.

Proof (M_2) Given $f \in M_2^{(n)}$ ($f \notin \{c_0, c_1, c_2\}$) and $\overline{a} \in E_3^n$, let us define the following functions:

$$g_{\overline{a},i}(\overline{x}) := \begin{cases} c_1 & \text{if } a_i = 0 \\ \beta_0(x_i) & \text{if } a_i = 1 \\ \gamma_2(x_i) & \text{if } a_i = 2 \end{cases}$$

$$h_{\overline{a},i}(\overline{x}) := \begin{cases} c_1 & \text{if } a_i = 0 \\ \beta_1(x_i) & \text{if } a_i = 1 \\ \gamma_4(x_i) & \text{if } a_i = 2 \end{cases}$$

$g_{\overline{a}}(\overline{x}) := g_{\overline{a},0}(\overline{x}) \wedge_s \cdots \wedge_s g_{\overline{a},n-1}(\overline{x})$.

If some $a_i \neq 0$, then $h_{\overline{a}}(\overline{x}) := h_{\overline{a},0}(\overline{x}) \wedge_s \cdots \wedge_s h_{\overline{a},n-1}(\overline{x})$; otherwise, $h_{\overline{0}}(\overline{x}) := c_2$. $m(\overline{x}) := \bigvee_{\overline{a}: f(\overline{a})=1} g_{\overline{a}}(\overline{x}) \vee_s \bigvee_{\overline{a}: f(\overline{a})=2} h_{\overline{a}}(\overline{x})$. By construction, $m \in \langle \wedge_s, \vee_s, \gamma_2, \beta_1 \rangle$ (notice that $\beta_0 = \gamma_2 \circ \beta_1$ and $\gamma_4 = \beta_1 \circ \gamma_2$). We claim that $f = m$. Consider $\overline{b} \in E_3^n$.

Case 1: $f(\overline{b}) = 0$. Suppose $f(\overline{a}) = 1$. If $a_i \leq b_i$ for all i, then, by definition of M_2, $f(\overline{b}) = 1$, contradicting the hypothesis. Thus there is an i, $1 \leq i \leq n$, such that $a_i \nleq b_i$. Three pairs of values are possible for a_i and b_i:

1. if $a_i = 2$ and $b_i = 0$, then $g_{\overline{a},i}(\overline{b}) = \gamma_2(0) = 0$;
2. if $a_i = 1$ and $b_i = 2$, then $g_{\overline{a},i}(\overline{b}) = \beta_0(2) = 0$;
3. if $a_i = 1$ and $b_i = 0$, then $g_{\overline{a},i}(\overline{b}) = \beta_0(0) = 0$.

Therefore, $g_{\overline{a},i}(\overline{b}) = 0$ and this implies $g_{\overline{a}}(\overline{b}) = 0$. It follows that $\bigvee_{\overline{a}: f(\overline{a})=1} g_{\overline{a}}(\overline{b}) = 0$. Now suppose $f(\overline{a}) = 2$. Applying a reasoning analogous to the case $f(\overline{a}) = 1$ we find that there is i, $0 \leq i \leq n-1$, such that $h_{\overline{a},i}(\overline{b}) = 0$. This implies $h_{\overline{a}}(\overline{b}) = 0$ and $\bigvee_{\overline{a}: f(\overline{a})=1} h_{\overline{a}}(\overline{b}) = 0$. Therefore, $m(\overline{b}) = 0$.

Case 2: $f(\overline{b}) = 1$. If $b_i = 1$, then $g_{\overline{b},i}(\overline{b}) = \beta_0(1) = 1$; if $b_i = 2$, then $g_{\overline{b},i}(\overline{b}) = \gamma_2(2) = 1$. Therefore, $g_{\overline{b},i}(\overline{b}) = 1$ and, by the definition of the strong Kleene disjunction, $m(\overline{b}) = 1$.

Case 3: $f(\overline{b}) = 2$. With an argument analogous to the one used in case 1 it is easy to prove that if $f(\overline{a}) = 1$, then $g_{\overline{a}}(\overline{b}) = 0$, and so $\bigvee_{\overline{a}: f(\overline{a})=1} g_{\overline{a}}(\overline{b}) = 0$. If $f(\overline{a}) = 2$, then by definition of h the value of $h_{\overline{a}}(\overline{b})$ has to be either 0 or 2. But $h_{\overline{b}}(\overline{b}) = 2$. Therefore, $\bigvee_{\overline{a}: f(\overline{a})=2} h_{\overline{a}}(\overline{b}) = 2$ and $m(\overline{b}) = 2$.

(K_2) See [1], Section 4.1.

(H_2) The proof is by induction on the number of variables of $f \in H_2$. It is easy to verify that $H_2^{(1)} = \langle \wedge_w, \vee_w, \wedge_o, \vee_o \rangle^{(1)}$. We will use the following auxiliary functions:

σ_0	0	1	2
0	0	0	0
1	0	0	0
2	0	1	2

σ_1	0	1	2
0	0	0	0
1	0	1	0
2	0	0	0

σ_2	0	1	2
0	0	1	0
1	1	1	0
2	0	0	0

All belong to H_2, as it is shown by the definitions,

$$\sigma_0(x, y) = \alpha_1(x) \wedge_o y,$$
$$\sigma_1(x, y) = \beta_0(x) \wedge_w \beta_0(y),$$
$$\sigma_2(x, y) = (\alpha_2(x) \wedge_w \beta_0(y)) \vee_w (\beta_0(x) \wedge_w \alpha_2(y)).$$

Let $f \in H_2^{(n)}$, $f \neq c_2$. By the definition of H_2, $f \in I_{01}$ and re f is monotonic on E_2^t. Since $f \in I_{01}$, let us consider the function re f as a function re $f \in \mathcal{O}_2$. The theorem of characterization of all two-valued clones, due to Post,[11] implies that re $f \in \langle \wedge_c, \vee_c \rangle$, where $\wedge_c$ and $\vee_c$ are the usual classical operators of conjunction and disjunction. Therefore, there is a construction of the function re f using projections, the constant functions, and the functions $\wedge_c$ and $\vee_c$. Let us define recursively the function $(\text{re } f)^* \in \mathcal{O}_3$ as

$(c_0)^* := c_0$, $(c_1)^* := \alpha_2$,

$$(e_i^n)^*(\overline{x}) := \alpha_2(x_0)\wedge_w \cdots \wedge_w \alpha_2(x_{i-1})\wedge_w \beta_0(x_i)\wedge_w \alpha_2(x_{i+1})\wedge_w \cdots \wedge_w \alpha_2(x_{n-1});$$

if $h = h_1 \wedge_c h_2$, then $h^* := \sigma_1(h_1^*, h_2^*)$, and if $h = h_1 \vee_c h_2$, then $h^* := \sigma_2(h_1^*, h_2^*)$.

It is easy to prove by induction that $\text{re}(\text{re } f)^* = \text{re } f$ and that $(\text{re } f)^*(\overline{a}) = 0$, if some $a_i = 2$. Moreover, $(\text{re } f)^* \in H_2$. Let us define $m \in \mathcal{O}_3^{(n)}$:

$$m(\overline{x}) := (\text{re } f)^* \vee_w \sigma_0(x_0, f(2, x_1, \ldots, x_{n-1})) \vee_w \cdots \vee_w \sigma_0(x_{n-1}, f(x_0, \ldots, x_{n-2}, 2)).$$

By induction hypothesis, $m \in H_2$. Consider $\overline{a} \in E_3^n$. By construction, if $a_i = 2$, then $\sigma_0(a_i, f(a_0, \ldots, a_{i-1}, 2, a_{i+1}, \ldots, a_{n-1})) = f(\overline{a})$, and if $a_i \in E_2$, then $\sigma_0(a_i, f(a_0, \ldots, a_{i-1}, 2, a_{i+1}, \ldots, a_{n-1})) = 0$. Therefore, if some $a_i = 2$, $(\text{re } f)^*(\overline{a}) = 0$ and $m(\overline{a}) = f(\overline{a})$. If all $a_i \in E_2$, then $m(\overline{a}) = (\text{re } f)^*(\overline{a}) = f(\overline{a})$.

(G_2) By induction on the number of variables. It is easy to check that $G_2^{(1)} = \langle \neg_2, \wedge_w, \odot \rangle^{(1)}$. Let $f \in G_2^{(n)}$. Suppose that f has a contaminant variable, say x_0; then we define $g \in \mathcal{O}_3^{(n)}$:

$$g(\overline{x}) := (x_0 \wedge_w f(1, x_1, \ldots, x_{n-1})) \vee_w (\neg_2 x_0 \wedge_w f(0, x_1, \ldots, x_{n-1})).$$

Consider $\overline{a} \in E_3^n$. If $a_0 = 2$, then $f(2, a_1, \ldots, a_{n-1}) = 2$, because x_0 is a contaminant variable, and $g(\overline{a}) = 2$. If $a_0 \in E_2$, $f(0, a_1, \ldots, a_{n-1}) \neq 2$ and $f(1, a_1, \ldots, a_{n-1}) \neq 2$, then, by the definition of g, $g(\overline{a}) = f(\overline{a})$. If either $f(1, a_1, \ldots, a_{n-1}) = 2$ or $f(0, a_1, \ldots, a_{n-1}) = 2$, then $f(x, a_1, \ldots, a_{n-1}) \notin I_{01}$, and by the definition of G_2, $f(x, a_1, \ldots, a_{n-1}) = c_2$. In particular, $f(\overline{a}) = 2$. Hence $f(\overline{a}) = g(\overline{a}) = 2$.

If f has no contaminant variable, the result is a consequence of these three claims.

Claim 1: For every i, $0 \leq i \leq n-1$, and every element $a_0, \ldots, a_{n-2} \in E_2$, $f(a_0, \ldots, a_{i-1}, 2, a_i, \ldots, a_{n-2}) \in E_2$.

Proof: If $f(a_0, \ldots, a_{i-1}, 2, a_i, \ldots, a_{n-2}) = 2$, with $a_0, \ldots, a_{n-2} \in E_2$, then the function $f(x_0, \ldots, x_{i-1}, 2, x_i, \ldots, x_{n-2}) \notin I_{01}$ and, by the definition of G_2, $f(x_0, \ldots, x_{i-1}, 2, x_i, \ldots, x_{n-2}) = c_2$; that is, the variable x_i is contaminant, contradicting the hypothesis.

Claim 2: Either re $f = c_0$ or re $f = c_1$.

Proof: Consider elements $\overline{a}, \overline{b} \in E_2^n$. Let us prove by induction that $f(\overline{a}) = f(\overline{b})$. Consider the functions $h_i(x) := f(b_0, \ldots, b_{i-1}, x, a_{i+1}, \ldots, a_{n-1})$. Suppose that for some i, $0 \leq i \leq n-1$, $f(\overline{a}) = h_i(a_i)$. By hypothesis, $f \neq c_2$, and by definition of G_2, $f \in I_{01}$; therefore, $h_i(a_i) \in E_2$. By Claim 1, $h_i(2) \in E_2$. By definition of G_2, re $h_i(x)$ is a constant function, and then $f(\overline{a}) = h_i(a_i) = h_i(b_i) = h_{i+1}(a_{i+1})$.

Claim 3: Let us define the functions $g_i, g, h \in \mathcal{O}_3^{(n)}$:

$$g_i(\overline{x}) := (x_i \wedge_w \neg_2(x_i)) \odot f(x_0, \ldots, x_{i-1}, 2, x_{i+1}, \ldots, x_{n-1}),$$
$$g(\overline{x}) := g_0(\overline{x}) \vee_w \cdots \vee_w g_{n-1}(\overline{x}),$$
$$h(\overline{x}) := g(\overline{x}) \vee_w (\alpha_2(x_0) \wedge_w \cdots \wedge_w \alpha_2(x_{n-1})).$$

If re $f = c_0$, then $f = g$ and if re $f = c_1$, then $f = h$.

Proof: Consider $\overline{a} \in E_3^n$. By the definitions of the functions, if $a_i \in E_2$, then $g_i(\overline{a}) = 0$ and if $a_i = 2$, then $g_i(\overline{a}) = f(\overline{a})$. If some $a_i = 2$, then $g(\overline{a}) = f(\overline{a})$ and if all $a_i \in E_2$, then $g(\overline{a}) = 0$. Therefore, if re $f = c_0$, then $f = g$. In a similar way it can be shown that if re $f = c_1$, then $f = h$. □

7 The Principal Clones in the Lattice of Three-Valued Clones

The aim of this section is to determine the position of the principal clones in the lattice of three-valued clones. Although the lattice of two-valued clones is denumerable and was completely determined by Post, a complete description of the lattice of clones on a set of k elements, $k > 2$, is not to be expected, because the cardinality of the set of k-valued clones is $2^{\aleph_0}$ when $2 < k < \omega$. One of the important problems concerning the structure of the lattice of clones that has been solved is the determination of all the maximal clones. The description was given by Yablonskiĭ for the three-valued clones and was generalized by Rosenberg for all k-valued clones.[12] Lau determined the set of all submaximal clones of $\mathcal{O}_3$, that is, the set of all clones that are maximal in the maximal three-valued clones. Let us state the results which will be used later.

Theorem 7.1 (Yablonskiĭ) *Let $\{i, j, k\} = \{0, 1, 2\}$. $\mathcal{O}_3$ has exactly the following 18 maximal clones:*

$$
\begin{array}{llcl}
\textit{1–3} & I_i & = & \mathrm{Pol}(i) \\
\textit{4–6} & I_{ij} & = & \mathrm{Pol}(ij) \\
\textit{7–9} & M_i & = & \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & j & j & i \\ 0 & 1 & 2 & i & k & k \end{pmatrix} \\
\textit{10–12} & U_i & = & \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & j & k \\ 0 & 1 & 2 & k & j \end{pmatrix} \\
\textit{13–15} & C_i & = & \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & i & j & i & k \\ 0 & 1 & 2 & j & i & k & i \end{pmatrix} \\
\textit{16} & T & = & \mathrm{Pol}(\{(a,b,c) \in E_3^3 : \mathrm{card}(\{a,b,c\}) \leq 2\}) \\
\textit{17} & L & = & \mathrm{Pol}(\{(a,b,c,d) \in E_3^4 : a+b = c+d \mod 3)\}) \\
\textit{18} & S & = & \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}.
\end{array}
$$

This theorem classifies the clones M_i as maximal clones in $\mathcal{O}_3$. One of Lau's theorems[13] classifies each clone K_i as maximal in the clone C_i, hence as submaximal of $\mathcal{O}_3$. In order to classify the clones H_i and G_i we will use another theorem of Lau, which we quote here.[14]

Theorem 7.2 (Lau) *Let $\{i, j, k\} = \{0, 1, 2\}$. U_i has exactly the following 13 maximal clones:*

$$L_i^1 = U_2 \cap I_i$$
$$L_i^2 = U_2 \cap I_{jk}$$
$$L_i^3 = U_2 \cap I_{ij}$$
$$L_i^4 = U_2 \cap I_{ik}$$
$$L_i^5 = U_2 \cap \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & j & k & j & i \\ 0 & 1 & 2 & k & j & i & j \end{pmatrix}$$
$$L_i^6 = U_2 \cap \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & k & j & k & i \\ 0 & 1 & 2 & j & k & i & k \end{pmatrix}$$
$$L_i^7 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & j \\ 0 & 1 & 2 & k \end{pmatrix}$$
$$L_i^8 = \mathrm{Pol}\begin{pmatrix} j & i & k & i \\ i & j & i & k \end{pmatrix}$$
$$L_i^9 = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & j & j & k & k \\ 0 & 1 & 2 & k & i & j & i \end{pmatrix}$$
$$L_i^{10} = \mathrm{Pol}\begin{pmatrix} 0 & 1 & 2 & j & j & k & k & i & i & j & k \\ 0 & 1 & 2 & j & j & k & k & i & i & k & j \\ 0 & 1 & 2 & k & i & j & i & j & k & i & i \end{pmatrix}$$
$$L_i^{11} = \mathrm{Pol}\begin{pmatrix} j & j & j & j & k & k & k & k & j & k & i & i & i \\ j & j & k & k & j & j & k & k & j & k & i & i & i \\ j & k & j & k & j & k & j & k & i & i & j & k & i \end{pmatrix}$$
$$L_i^{12} = \mathrm{Pol}\begin{pmatrix} j & j & j & k & k & j & k & k & i \\ j & j & k & k & j & k & j & k & i \\ j & k & j & j & k & k & j & k & i \\ j & k & k & j & j & j & k & k & i \end{pmatrix}$$
$$L_i^{13} = \mathrm{Pol}\, E_2^4 \cup$$
$$\begin{pmatrix} j & j & k & k & j & j & k & k & j & j & k & k & i & i & i & i & i & i & i & i & i & i & i & i & i \\ j & k & j & k & i & i & i & i & i & i & i & i & j & j & k & k & j & j & k & k & i & i & i & i & i \\ i & i & i & i & j & k & j & k & i & i & i & i & j & k & j & k & i & i & i & i & j & j & k & k & i \\ i & i & i & i & i & i & i & i & j & k & j & k & i & i & i & i & j & k & j & k & j & k & j & k & i \end{pmatrix}$$

Before giving the characterizations of the clones H_i and G_i we need to characterize the functions belonging to the clones L_2^7 and L_2^9.

Lemma 7.3 *Let $f \in \mathcal{O}_3$. Then $f \in L_2^7$ if and only if for all $g \in \operatorname{der} f$, if $\operatorname{re} g \neq c_2$, then $g \in I_{01}$ and $\operatorname{re} g$ is monotonic on the order E_2^t.*

Proof (⇒) Consider $f \in L_2^7 = \mathrm{Pol}\left(\begin{smallmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{smallmatrix}\right) =: \mathrm{Pol}\, R_7$ such that $\operatorname{re} f \neq c_2$; that is, there are $\overline{a} \in E_2^n$ such that $f(\overline{a}) \in E_2$. Consider $\overline{b} \in E_2^n$. Without losing generality, we can suppose that $\overline{a} = (0101)$ and $\overline{b} = (0110)$. Considering $\overline{d} = (0111)$, we see that $\binom{\overline{a}}{\overline{d}} \in R_7$ and $\binom{\overline{b}}{\overline{d}} \in R_7$. That implies that $\binom{f(\overline{a})}{f(\overline{d})} \in R_7$ and $\binom{f(\overline{b})}{f(\overline{d})} \in R_7$. Since $f(\overline{a}) \in E_2$, it follows that $f(\overline{d}) \in E_2$ and $f(\overline{b}) \in E_2$, proving that $f \in I_{01}$. The monotonicity of $\operatorname{re} f$ is trivial.

($\Leftarrow$) Let us consider $f \in \mathcal{O}_3$ satisfying the condition and elements $\overline{a} \in E_3^n$ and $\overline{b} \in E_3^n$ such that $\binom{\overline{a}}{\overline{b}} \in R_7$. Substituting c_2 for variables, we can guarantee that $\overline{a}$, $\overline{b} \in E_2$. If re $f = c_2$, then $\binom{f(\overline{a})}{f(\overline{b})} = \binom{2}{2} \in R_7$. If $f \in I_{01}$ and re f is monotonic on the order E_2^t, then $f(\overline{a}) \leq f(\overline{b})$ and $f(\overline{a}), f(\overline{b}) \in E_2$; that is, $\binom{f(\overline{a})}{f(\overline{b})} \in R_7$. □

Lemma 7.4 $L_2^7 = \langle \wedge_w, \vee_w, \wedge_o, \vee_o, \alpha_4 \rangle$.

Proof Consider $f \in L_2^7$, $f \neq c_2$. The proof is similar to the one used in Theorem 6.1, in the case of H_2, but using the following functions instead:

$$g(\overline{x}) := \begin{cases} (\text{re } f)^* & \text{if } f \in I_{01} \text{ and re } f \text{ is monotonic on } E_2^t \\ \alpha_4(x_0) \wedge_o \cdots \wedge_o \alpha_4(x_{n-1}) & \text{if re } f = c_2. \end{cases}$$

$$m(\overline{x}) := g(\overline{x}) \vee_w \sigma_0(x_0, f(2, x_1, \ldots, x_{n-1})) \vee_w \ldots \vee_w \sigma_0(x_{n-1}, f(x_0, \ldots, x_{n-2}, 2)).$$

Notice that g satisfies re g = re f and that $g(\overline{a}) = 0$, if some $a_i = 2$. Consider $\overline{a} \in E_3^n$. If re $f = c_2$, then if all $a_i \in E_2$, then $g(\overline{a}) = 2$. Hence $m(\overline{a}) = 2 = f(\overline{a})$. If re $f = c_2$ and some $a_i = 2$, then $g(\overline{a}) = 0$ and $\sigma_0(a_i, f(a_0, \ldots, a_{i-1}, 2, a_{i+1}, \ldots, a_{n-1})) = f(\overline{a})$; hence $m(\overline{a}) = f(\overline{a})$. If re $f \neq c_2$, then, by Lemma 7.3, $f \in I_{01}$ and re f is monotonic on the order E_2^t. Therefore, if some $a_i = 2$, $(\text{re } f)^*(\overline{a}) = 0$ and $m(\overline{a}) = f(\overline{a})$. If all $a_i \in E_2$, then all $\sigma_0(a_i, f(a_0, \ldots, a_{i-1}, 2, a_{i+1}, \ldots, a_{n-1})) = 0$ and $m(\overline{a}) = (\text{re } f)^*(\overline{a}) = f(\overline{a})$. □

Lemma 7.5 *Let $f \in \mathcal{O}_3$. Then $f \in L_2^9$ if and only if for all $g \in$ der f, if $g \neq c_2$, then $g \in I_{01}$.*

Proof Similar to Lemma 7.3. □

Lemma 7.6 $L_2^9 = \langle \neg_2, \wedge_w, \boxdot, \gamma_0 \rangle$.

Proof Consider $f \in L_2^9$, $f \neq c_2$. The proof is similar to the one used in Theorem 6.1, in the case of H_2. Let us define the following functions:

$$g_{\overline{a}}(\overline{x}) := \alpha_1(x_{i_0}) \boxdot \cdots \boxdot \alpha_1(x_{i_{r-1}}), \quad \text{if } a_{i_0} = \cdots = a_{i_{r-1}} = 2.$$

If $f \neq c_0$,

$$h_{\overline{a},i}(\overline{x}) := \begin{cases} \gamma_0(x_i) & \text{if } a_i = 0 \\ \beta_0(x_i) & \text{if } a_i = 1 \\ \alpha_0(x_i) & \text{if } a_i = 2 \end{cases}$$

$$h_{\overline{a}}(\overline{x}) := h_{\overline{a},0}(\overline{x}) \wedge_w \cdots \wedge_w h_{\overline{a},n-1}(\overline{x})$$

$$m(\overline{x}) := \bigvee_{\overline{a}: f(\overline{a})=2} g_{\overline{a}}(\overline{x}) \vee_w \bigvee_{\overline{a}: f(\overline{a})=1} h_{\overline{a}}(\overline{x}).$$

We claim that $f = m$. Consider $\overline{a} \in E_3^n$. If $f(\overline{a}) = 2$, then $g_{\overline{a}}(\overline{a}) = 2$ and $m(\overline{a}) = 2$ (notice that $g_{\overline{a}}$ is defined, since otherwise $\overline{a} \in E_2^n$, contradicting that $f \in L_2^9$ and $f \neq c_2$). If $f(\overline{a}) = 1$, then let us consider any $\overline{b} \in E_3^n$ such that $f(\overline{b}) = 2$. Suppose that $a_i = 2$ whenever $b_i = 2$. Then, without losing generality, $\overline{a} = (a_0, \ldots, a_{k-1}, 2, \ldots, 2)$ and $\overline{b} = (b_0, \ldots, b_{k-1}, 2, \ldots, 2)$, with $a_0, \ldots, a_{k-1}, b_0, \ldots, b_{k-1} \in E_2$. Since $f(\overline{a}) = 1$, $f(x_0, \ldots, x_{k-1}, 2, \ldots, 2) \neq c_2$, and applying Lemma 7.5 it follows that $f(x_0, \ldots, x_{k-1}, 2, \ldots, 2) \in I_{01}$, contradicting that $f(\overline{b}) = 2$. Therefore, there is $b_i = 2$ such that $a_i \in E_2$. This implies $g_{\overline{b}}(\overline{a}) = 0$. Moreover, $h_{\overline{a}}(\overline{a}) = 1$ and $h_{\overline{b}}(\overline{a}) \in E_2$ by definition of h; hence $m(\overline{a}) = 1$.

If $f(\overline{a}) = 0$, a similar argument to the case $f(\overline{a}) = 1$ shows that $g_{\overline{b}}(\overline{a}) = 0$ when $f(\overline{b}) = 2$. Consider now any $\overline{b}$ such that $f(\overline{b}) = 1$. Since $f(\overline{a}) = 0$, $\overline{a} \neq \overline{b}$; that is, there is $a_i \neq b_i$ and then $h_{\overline{b},i}(a_i) = 0$. Therefore, $h_{\overline{b}}(\overline{a}) = 0$ and finally $m(\overline{a}) = 0$. □

The characterization of the clones H_i and G_i is given by the following propositions.

Proposition 7.7 *The clones H_i $(i \in E_3)$ are the intersection of two maximal clones of U_i, namely,*

$$H_i = L_i^7 \cap L_i^9,$$

and they are maximal in those maximal clones.

Proof In order to simplify notation, let us suppose $i = 2$. The other cases can be characterized similarly. It is trivial to prove that $H_2 = L_2^7 \cap L_2^9$, given the characterizations of the clones involved.

To prove that H_2 is maximal in L_2^7, we must prove that for every $f \in L_2^7 \backslash H_2$, $\langle H_2 \cup \{f\} \rangle = L_2^7$. Since $f \in L_2^7$ and $f \notin H_2$, using the characterization lemmas we find that there is $g \in \operatorname{der} f$ such that $g \neq c_2$ and $\operatorname{re} g = c_2$. Without losing generality, we can suppose that there are elements $\overline{a} \in E_3^n$ such that $a_0 = \cdots = a_{i-1} = 2, a_i, \ldots, a_{n-1} \in E_2, 0 \leq i \leq n-1$, and $g(\overline{a}) \neq 2$. Consider the function $h(x) := g(x, \ldots, x, a_i, \ldots, a_{n-1})$. Since $\operatorname{re} g = c_2$, $h \in \{\alpha_4, \alpha_5\}$ and then (by Theorem 6.1 and Lemma 7.4) $\langle H_2 \cup \{f\} \rangle = \langle H_2 \cup \{h\} \rangle = L_2^7$.

To prove that H_2 is maximal in L_2^9, consider $f \in L_2^9 \backslash H_2$. Since $f \in L_2^9$ and $f \notin H_2$, by the definitions of the clones we find that there is $g \in \operatorname{der} f$ such that $g \neq c_2$, $g \in I_{01}$ and $\operatorname{re} g$ is not monotonic on E_2^t. That means that there are elements $\overline{a}, \overline{b} \in E_2^n$ such that $a_i \leq b_i$ and $g(\overline{a}) = 1$ and $g(\overline{b}) = 0$. Without losing generality, we can suppose that $a_0 = \cdots = a_{i-1} = 0$ and $b_0 = \cdots = b_{i-1} = 1$ and $a_i = b_i, \ldots, a_{n-1} = b_{n-1}$, $0 \leq i \leq n-1$. Consider the function $h(x) = g(x, \ldots, x, a_i, \ldots, a_{n-1})$. Then $h \in \{\neg_2, \gamma_0, \beta_2\}$. Given that $\neg_2(x) = \gamma_0(x) \wedge_w (c_1 \vee_w x) = \beta_2(x) \wedge_w (c_1 \vee_w x)$, $x \boxdot y = (\neg_2(x) \wedge_o y) \wedge_o (x \wedge_o \neg_2(y))$ and $\gamma_0(x) = \neg_2(x \wedge_o c_1)$, it follows that $\langle H_2 \cup \{f\} \rangle = \langle H_2 \cup \{\neg_2\} \rangle = L_2^9$. □

Proposition 7.8 *The clones G_i $(i \in E_3)$ are maximal in L_i^9 and they are not included in any other submaximal of $\mathcal{O}_3$.*

Proof Let us fix $i = 2$, the other cases requiring an analogous treatment. It is trivial to prove that $G_2 \subseteq L_2^9$, considering the characterizations of the clones.

Let us show that G_2 is not included either in any other maximal clone of $\mathcal{O}_3$ distinct from U_2 or in any maximal clone of U_2 distinct from L_2^9. The clones I_i, I_{ij}, and S cannot contain G_2 because they do not contain all constant functions. The function $\neg_2$, which belongs to G_2, is a counterexample to the inclusion in the clones M_i, U_0, U_1, C_0, and C_1. The function α_0 is a counterexample to the clones C_2 and L (the clone L can be characterized as the clone of all functions that have a representation as a lineal polynomial in the field $(\mathbb{Z}_3, +, \cdot)$, and the expression of α_0 as a polynomial of $\mathbb{Z}_3$ of minimal degree is $2x^2 + x$) and $\wedge_w$ is a counterexample to T.

With respect to the other maximal clones of U_2, the clones L_2^1, L_2^2, L_2^3, L_2^4, and L_2^8 do not have all constant functions. The function $\neg_2$ is a counterexample to the clones L_2^5, L_2^6, and L_2^7. The function $\odot$ belongs to G_2, but not to the clone L_{10}. Finally, $\wedge_w$ is a counterexample to the clones L_2^{11}, L_2^{12}, and L_2^{13}. Note that the inclusion of G_2 in L_2^9 is strict, because the function β_0 belongs to L_2^9, but not to G_2.

In order to prove the maximality of G_2 in L_2^9, consider $f \in (L_2^9)^{(n)}$, $f \notin G_2$. Without losing generality, f satisfies that there are $\overline{a} \in E_2^{n-i}$ such that $f(2, \ldots, 2, \overline{a}) \neq 2$ and re $f(x_0, \ldots, x_{i-1}, \overline{a}) \notin \{c_0, c_1\}$. Therefore, there are $\overline{b}, \overline{d} \in E_2^i$ such that $f(\overline{b}, \overline{a}) = 0$ and $f(\overline{d}, \overline{a}) = 1$. Let us consider the functions $g_j(x) := f(d_0, \ldots, d_{j-1}, x, b_{j+1}, \ldots, b_{i-1}, \overline{a})$, $0 \leq j \leq i-1$. For at least one of those functions, say g_k, it is true that $g_k(b_k) \neq g_k(d_k)$, because otherwise $f(\overline{b}, \overline{a}) = f(\overline{d}, \overline{a})$. Moreover, $g_k(2) \in E_2$, because if $g_k(2) = 2$, then the function $f(x_0, \ldots, x_{j-1}, 2, x_{j+1}, \ldots, x_{n-1}) \notin I_{01}$ and, by Lemma 7.5, this implies $f(x_0, \ldots, x_{j-1}, 2, x_{j+1}, \ldots, x_{n-1}) = c_2$, contradicting the fact that $f(2, \ldots, 2, \overline{a}) \neq 2$. We have found a function $g_k \in \{\gamma_0, \gamma_2, \beta_0, \beta_2\}$. Since each of those functions generates γ_0 ($\gamma_0 = \neg_2 \circ \gamma_2 = \beta_0 \circ \neg_2 = \neg_2 \circ \beta_2 \circ \neg_2$), it follows that $\gamma_0 \in \langle G_2 \cup \{f\}\rangle$. □

8 Generalizations and Open Problems

Let us consider another theorem by Visser that gives a propositional version of Theorem 1.1.[15]

Theorem 8.1 (Visser) *If $(E, \leq)$ is a ccpo and all the functions in a clone $F \subseteq \mathcal{O}_E$ are monotonic on that order, then F has the fixed-point property.*

This result can be generalized to partial orders that are not ccpos using a theorem proved by Roddy. Given a partial order $(E, \leq)$, Pol $\leq$ is the clone of all functions monotonic on $\leq$. Let us say that a partial order $(E, \leq)$ is *stable* if all monotonic functions from E to E have a fixed point, that is, if $(\text{Pol} \leq)^{(1)}$ is a stable monoid.[16] Given two partial orders $(E, \leq)$ and $(E', \leq')$ the Cartesian product is given by the pointwise order on the Cartesian product of the sets and will be represented as $(E \times E', \leq \times \leq')$.

Theorem 8.2 (Roddy[17]) *Let $(E, \leq)$ and $(E', \leq')$ be two finite stable partial orders. Then the Cartesian product $(E \times E', \leq \times \leq')$ is a stable partial order.*

Now we can give the generalization of Theorem 8.1.

Theorem 8.3 *Let $(E_k, \leq)$ $(k \geq 2)$ be a stable partial order and $F \subseteq \mathcal{O}_k$ a clone such that $F^{(1)} \subseteq (\text{Pol} \leq)^{(1)}$. Then F has the fixed-point property.*

Proof Consider a partial order $(E_k, \leq)$ $(k \geq 2)$ such that $(\text{Pol} \leq)^{(1)}$ is stable and a clone $F \subseteq \mathcal{O}_k$ such that $F^{(1)} \subseteq (\text{Pol} \leq)^{(1)}$. By Lemma 5.1, $F \subseteq \text{Pol} \leq$. We will prove that Pol $\leq$ has the fixed-point property and a fortiori that F has the fixed-point property. Let us consider a finite stipulation $s : x_i = f_i(\overline{x})$ $(i = 0, \ldots, n-1)$ $(f_i \in (\text{Pol} \leq)^{(n)})$. Consider the function $\rho_s : E_k^n \to E_k^n$ defined as

$$\rho_s(\overline{a}) = (f_0(\overline{a}), \ldots, f_{n-1}(\overline{a}))$$

(the *jump* function). It is obvious that there is an s-consistent valuation if and only if ρ_s has a fixed point. By Theorem 8.2, the partial order $(E_k^n, \leq^n) := (E_k \times \cdots \times E_k, \leq \times \cdots \times \leq)$ is stable. It is easy to check that ρ_s is monotonic on the order $(E_k^n, \leq^n)$. Hence ρ_s has a fixed point and there is an s-consistent valuation. By Lemma 2.3, Pol $\leq$ has the fixed-point property. □

Another simple application of Theorem 8.1 gives a solution to the fixed-point problem in the two-valued case.

Theorem 8.4 *Let $F \subseteq \mathcal{O}_2$ be a clone with constants. Then the following three conditions are equivalent:*

1. *F has the fixed-point property;*
2. *F has the unary fixed-point property;*
3. *All the functions in F are monotonic on the order of truth.*

Proof It is easy to prove that if a function $f \in \mathcal{O}_2$ is not monotonic on the order of truth, then classical negation belongs to $\langle f \rangle$. This shows that (2) implies (3). That (3) implies (1) is a consequence of Theorem 8.1. □

It is natural to ask whether the characterization given by Theorem 3.2 is valid for sets of truth values of cardinality different from two or three. We do not know if the characterization is valid for other finite sets of truth values, but the following counterexample shows that it is not valid when the set of truth values is infinite. Let us consider the clone with constants generated by the following functions on the natural numbers: for all $n, m \in \omega$, $f_n(m) = n$, if $m \leq n$ and $f_n(m) = m$, if $m > n$. It is obvious that the transformation monoid of the clone is stable, yet it does not have the fixed-point property. Consider the stipulation s given by the system $x_n = f_n(x_{n+1})$. Suppose that there is an s-consistent valuation v and consider $l := v(x_0) + 1$. As v is s-consistent, $v(x_n) = f_n(v(x_{n+1}))$, for all $n \in \omega$. Then $v(x_0) = f_0(f_1(\ldots f_l(v(x_{l+1}))\ldots))$. By the definition of f_n, it is true that for all $n, m \in \omega$, $f_n(m) \geq n$ and $f_{n-1} \circ f_n = f_n$. Therefore, $v(x_0) = f_l(v(x_{l+1})) \geq l$, contradicting the definition of l.

Finally we want to explore this situation: if for some set of k truth values a generalization of Theorem 3.2 holds, what can be said about the clones with constants in $\mathcal{O}_k$ maximal for the fixed-point property? Consider the following two hypotheses.

Hypothesis 1: For every clone with constants $F \subseteq \mathcal{O}_k$, F has the fixed-point property if and only if $F^{(1)}$ is a stable monoid.

Hypothesis 2: Let $M \subseteq \mathcal{O}_k^{(1)}$ be a maximal stable monoid and let $F \subseteq \mathcal{O}_k$ be a clone with constants such that $F^{(1)} \subseteq M$. Then $F \subseteq \mathrm{Pol}\,\Gamma(M)$.[18]

With the help of Lemma 5.5, it is trivial to prove the following proposition, which gives a strategy for the classification of the clones with constants maximal for the fixed-point property.

Proposition 8.5

1. *If Hypothesis 1 is true, then*
   (a) *if $M \subseteq \mathcal{O}_k^{(1)}$ is a maximal stable monoid, then* $\mathrm{Pol}\,\Gamma(M)$ *is a clone maximal for the fixed-point property;*

(b) *if $J \subseteq \mathcal{O}_k$ is a clone with constants maximal for the fixed-point property, then there is a stable monoid $M \subseteq \mathcal{O}_k^{(1)}$ such that $J = \operatorname{Pol}\Gamma(M)$.*

2. *If both Hypotheses 1 and 2 are true, then if $J \subseteq \mathcal{O}_k$ is a clone with constants maximal for the fixed-point property, then there is a maximal stable monoid $M \subseteq \mathcal{O}_k^{(1)}$ such that $J = \operatorname{Pol}\Gamma(M)$.*

9 Appendix

We use the following notation for unary three-valued functions:

Permutations

	e_0^1	$\sim_1$	$\sim_2$	$\neg_0$	$\neg_1$	$\neg_2$
0	0	1	2	0	2	1
1	1	2	0	2	1	0
2	2	0	1	1	0	2

Non-bijective functions

	α_0	α_1	α_2	α_3	α_4	α_5
0	0	0	1	1	2	2
1	0	0	1	1	2	2
2	1	2	0	2	0	1

	β_0	β_1	β_2	β_3	β_4	β_5
0	0	0	1	1	2	2
1	1	2	0	2	0	1
2	0	0	1	1	2	2

	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5
0	1	2	0	2	0	1
1	0	0	1	1	2	2
2	0	0	1	1	2	2

	c_0	c_1	c_2
0	0	1	2
1	0	1	2
2	0	1	2

The following matrix represents the elements of the twelve maximal stable monoids, as defined in Proposition 4.2. The element of row F and column f is 1 when $f \in F$ and is 0 when $f \notin F$.

	α_0	α_1	α_2	α_3	β_0	β_1	β_4	β_5	γ_2	γ_3	γ_4	γ_5	$\neg_0$	$\neg_1$	$\neg_2$
$M_0^{(1)}$	0	1	1	1	1	0	1	1	0	0	0	0	0	0	0
$M_1^{(1)}$	1	1	0	1	0	0	0	0	1	0	1	1	0	0	0
$M_2^{(1)}$	0	0	0	0	1	1	0	1	1	1	1	0	0	0	0
$K_0^{(1)}$	1	1	0	0	1	1	0	0	1	0	1	0	1	0	0
$K_1^{(1)}$	0	0	1	1	1	0	0	1	1	1	0	0	0	1	0
$K_2^{(1)}$	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1
$H_0^{(1)}$	0	0	0	1	0	0	0	1	1	1	1	1	0	0	0
$H_1^{(1)}$	0	1	0	0	1	1	1	1	0	0	1	0	0	0	0
$H_2^{(1)}$	1	1	1	1	1	0	0	0	1	0	0	0	0	0	0
$G_0^{(1)}$	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0
$G_1^{(1)}$	0	0	0	0	1	1	1	1	0	0	0	0	0	1	0
$G_2^{(1)}$	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1

Now we can give a proof of Proposition 4.2, which we state again.

Proposition 9.1 *$\mathcal{O}_3^{(1)}$ has exactly the twelve maximal stable monoids $M_i^{(1)}$, $K_i^{(1)}$, $H_i^{(1)}$, $G_i^{(1)}$ $(i \in E_3)$, with the elements defined in the previous matrix (plus e_0^1 and the constant functions).*

Proof It can be checked easily that all these sets are stable monoids and that they are all different. Proving that they are maximal stable monoids and that they are the only maximal stable monoids reduces to proving the following claim: every set of functions $J \subseteq \mathcal{O}_3^{(1)}$ such that, for every transformation monoid $M \in \left\{M_i^{(1)}, K_i^{(1)}, H_i^{(1)}, G_i^{(1)}\right\}$ $(i \in E_3)$ there is $h \in J$ such that $h \notin M$, generates a function without fixed points. We will determine the sets J with this property that are minimal for inclusion. Notice that the composition of two functions $\neg_i, \neg_j$ $(i \neq j)$ is a function without fixed points. Hence we can suppose that J has at most one function $\neg_i$.

We will consider first the case in which $\neg_i \notin J$ for all $i \in E_3$. Looking at the row of $G_0^{(1)}$ in the matrix it is evident that all functions in J cannot be of the type γ_i. Let us suppose that J only contains functions from β_i and γ_i. Then considering the row of $M_2^{(1)}$ we see that either $\gamma_5 \in J$ or $\beta_4 \in J$. If $\gamma_5 \in J$ and $\beta_4 \notin J$, attending to the rows of $K_2^{(1)}$ and $H_0^{(1)}$ we see that J should include one of the sets, $\{\beta_0, \gamma_5\}$, $\{\beta_1, \gamma_5\}$, which are minimal and generate a function without fixed points. If $\beta_4 \in J$ and $\gamma_5 \notin J$, considering the rows of $K_2^{(1)}$ and $H_1^{(1)}$ we discover two new minimal sets, $\{\beta_4, \gamma_2\}$, $\{\beta_4, \gamma_3\}$. If $\gamma_5 \in J$ and $\beta_4 \in J$, we need to add some other function that does not belong to $K_2^{(1)}$, and so these sets are never minimal. The same type of argument can be used to find the minimal sets J only with functions α_i and β_i and the ones with functions α_i and γ_i. They are $\{\alpha_0, \gamma_3\}$, $\{\alpha_1, \gamma_3\}$, $\{\alpha_2, \gamma_4\}$, $\{\alpha_2, \gamma_5\}$, $\{\alpha_2, \beta_1\}$, $\{\alpha_3, \beta_1\}$, $\{\alpha_0, \beta_4\}$, $\{\alpha_0, \beta_5\}$.

Every set J that contains at least one function of each type (α_i, β_i, and γ_i) cannot be included in any stable monoid $M_i^{(1)}$, $G_i^{(1)}$. To determine the minimal sets in this case, let us classify the functions according to their range of values ($\{0, 1\}$, $\{0, 2\}$, $\{1, 2\}$) (see the matrix at the end of the proof). All the functions in J cannot have values in $\{0, 1\}$ (in this case $J \subseteq H_2^{(1)}$, as it is obvious from the matrix). Let us consider the case in which all functions in J have values in $\{0, 1\}$ or $\{0, 2\}$. Looking at the $K_0^{(1)}$-row we see that either $\alpha_2 \in J$ or $\beta_4 \in J$. If $\alpha_2 \in J$, then, since we are supposing that some function $\gamma_i \in J$, it is necessary that $\gamma_4 \in J$, but $\{\alpha_2, \gamma_4\}$ is a minimal set, so we cannot find any new minimal set. If $\beta_4 \in J$, then $\gamma_2 \in J$, but $\{\gamma_2, \beta_4\}$ is a minimal set. By an analogous argument we can check that no new minimal sets can be found unless in J there is at least one function with values in $\{0, 1\}$, another one with values in $\{0, 2\}$, a third one with values in $\{1, 2\}$ and at least one function of each type (α_i, β_i, and γ_i). The three-element sets J that satisfy this condition and do not include any of the two-element minimals are $\{\alpha_1, \beta_5, \gamma_2\}$ and $\{\alpha_3, \beta_0, \gamma_4\}$. A bit of calculation allows to show that every four-element set J with the previous condition includes some already determined two- or three-element minimal set. This completes the list of all minimal sets J without functions $\neg_i$: $\{\beta_0, \gamma_5\}$, $\{\beta_1, \gamma_5\}$, $\{\beta_4, \gamma_2\}$, $\{\beta_4, \gamma_3\}$, $\{\alpha_0, \gamma_3\}$, $\{\alpha_1, \gamma_3\}$, $\{\alpha_2, \gamma_4\}$, $\{\alpha_2, \gamma_5\}$, $\{\alpha_2, \beta_1\}$, $\{\alpha_3, \beta_1\}$, $\{\alpha_0, \beta_4\}$, $\{\alpha_0, \beta_5\}$, $\{\alpha_1, \beta_5, \gamma_2\}$, $\{\alpha_3, \beta_0, \gamma_4\}$.

Let us suppose now that $\neg_0 \in J$. Considering that $\neg_0$ only belongs to the transformation monoids K_0^1 and G_0^1, J must include one of these subsets: $\{\beta_0, \gamma_3\}$, $\{\beta_1, \gamma_3\}$, $\{\alpha_0, \gamma_3\}$, $\{\alpha_1, \gamma_3\}$, $\{\beta_0, \gamma_5\}$, $\{\beta_1, \gamma_5\}$, $\{\alpha_0, \gamma_5\}$, $\{\alpha_1, \gamma_5\}$, $\{\beta_4\}$, $\{\beta_5\}$, $\{\alpha_2\}$, $\{\alpha_3\}$. Some of them are minimal sets J on their own, so the new minimal sets are $\{\neg_0, \beta_4\}$, $\{\neg_0, \beta_5\}$, $\{\neg_0, \alpha_2\}$, $\{\neg_0, \alpha_3\}$, $\{\neg_0, \beta_0, \gamma_3\}$, $\{\neg_0, \beta_1, \gamma_3\}$, $\{\neg_0, \alpha_0, \gamma_5\}$, $\{\neg_0, \alpha_1, \gamma_5\}$. In the same way it can be shown that the minimal sets that include the permutation $\neg_1$ are $\{\neg_1, \gamma_4\}$, $\{\neg_1, \gamma_5\}$, $\{\neg_1, \alpha_0\}$, $\{\neg_1, \alpha_1\}$, $\{\neg_1, \beta_1, \gamma_2\}$, $\{\neg_1, \beta_1, \gamma_3\}$, $\{\neg_1, \alpha_2, \beta_4\}$, $\{\neg_1, \alpha_3, \beta_4\}$. And the minimal sets with $\neg_2$ are $\{\neg_2, \gamma_2\}$, $\{\neg_2, \gamma_3\}$, $\{\neg_2, \beta_0\}$, $\{\neg_2, \beta_1\}$, $\{\neg_2, \alpha_0, \gamma_4, \}$, $\{\neg_2, \alpha_0, \gamma_5, \}$, $\{\neg_2, \alpha_2, \beta_4\}$, $\{\neg_2, \alpha_2, \beta_5\}$. It is easy to check that all the minimal sets generate a function without fixed points. $\square$

Below is the matrix of elements of the maximal stable monoids $K_i^{(1)}$ and $H_i^{(1)}$, with the functions grouped according to their range. The element of row F and column f is 1 when $f \in F$ and is 0 when $f \notin F$.

	α_0	α_2	β_0	γ_2	α_1	β_1	β_4	γ_4	α_3	β_5	γ_3	γ_5
$K_0^{(1)}$	1	0	1	1	1	1	0	1	0	0	0	0
$K_1^{(1)}$	0	1	1	1	0	0	0	0	1	1	1	0
$K_2^{(1)}$	0	0	0	0	1	0	1	1	1	1	0	1
$H_0^{(1)}$	0	0	0	1	0	0	0	1	1	1	1	1
$H_1^{(1)}$	0	0	1	0	1	1	1	1	0	1	0	0
$H_2^{(1)}$	1	1	1	1	1	0	0	0	1	0	0	0

Notes

1. See [3], [5].

2. See [14] and [2], Ch. 2.

3. [2], Problem 2B15. The original problem was restricted to three and four-valued schemes, because the most important many-valued logics proposed as solutions to paradoxes have three or four truth values.

4. See [14].

5. See [7], Section 1.1 or [11]. Rosenberg generalized this characterization for E infinite using infinitary relations. See [13], Chapter 1.

6. Notice that the strict form of these stipulations is not a real restriction of generality, since clones contain the projection functions.

7. My original proof of this theorem was very long, since I was not aware of Smullyan's lemma. The strategy of that proof was presented in [6]. I am greatly indebted to Visser who, acting as a referee for this paper, discovered this proof and communicated it to me.

8. We do not know whether in general the transformation monoids of the k-valued clones with constants maximal for the fixed-point property are always maximal stable monoids. More about this in Section 8.

9. The relation $\Gamma(F^{(1)})$ is a particular application of the operator Γ_F defined in [7], 1.1.16. For more properties of this relation, see [7], 1.1.16–20. A generalization for infinite sets, due to Rosenberg, can be found in [13], Ch. 1.

10. It is not difficult to give a direct proof of this result in the style of Lemma 7.3.

11. See [8]. The theorem can be found in modern notation in [12] and in [13], pp. 36–39.

12. See [10]. The theorems are stated without proof in [7], 4.3 and [13], pp. 29–30.

13. In [4], p. 230, Proposition 6, the clone (5) is K_i.

14. [4], Proposition 7, p. 230.

15. See [14].

16. This property is called the fixed-point property in the literature on orders. We have changed the name to prevent confusion with the (Gupta-Belnap) fixed-point property.

17. See [9].

18. Notice that Hypothesis 2 is equivalent to this property of monotonicity: Let $M, N \subseteq \mathcal{O}_k$ be stable monoids such that N is maximal stable and $M \subseteq N$. Then $\mathrm{Pol}\,\Gamma(M) \subseteq \mathrm{Pol}\,\Gamma(N)$.

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