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# Rumely Domains with Atomic Constructible Boolean Algebra. An Effective Viewpoint

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**Abstract** The archetypal Rumely domain is the ring $\widetilde{\mathbb{Z}}$ of algebraic integers. Its constructible Boolean algebra is atomless. We study here the opposite situation: Rumely domains whose constructible Boolean algebra is atomic. Recursive models (which are rings of algebraic numbers) are proposed; effective model-completeness and decidability of the corresponding theory are proved.

## 1 Introduction

The notion of Rumely domain was introduced by Macintyre and van den Dries [14] in order to axiomatize the theory of $\widetilde{\mathbb{Z}}$, the ring of algebraic integers; an axiomatization formulated in slightly different terms, but also based on Rumely's local-global principle [10], was proposed by Prestel and Schmid [9].

**Definition 1.1 ([14])** A domain $R$ with fraction field $K$ is a Rumely domain if it has the following properties.

**Ru.1**: Its fraction field $K$ is algebraically closed.

**Ru.2**: Every finitely generated ideal of $R$ is principal.

**Ru.3**: (Local-global principle) If $C \subseteq {}^m K$ is a smooth, irreducible, closed curve, $f \in K[X_1, \ldots, X_m]$ and $C_f = \{x \in C : f(x) \neq 0\}$ has points in $(1/a)\,{}^m R$ and in $(1/b)\,{}^m R$ where $a, b \in R \setminus \{0\}$ are relatively prime, then $C_f$ has a point in ${}^m R$.

$R$ is a good Rumely domain if it satisfies, moreover, the following properties.

**Ru.4**: (Good factorization) For all $a, b \in R \setminus \{0\}$, there are $a_1, a_2$ in $R$ such that $a = a_1 a_2$, $a_1$ and $b$ are relatively prime and $b$ belongs to the Jacobson radical of $a_2$.

**Ru.5**: Every nonzero nonunit is the product of two relatively prime nonunits.

**Ru.6**: Its Jacobson radical is equal to $\{0\}$ and $R \neq K$.

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All these properties are first-order expressible in the language of rings $\mathcal{L}_{\text{ring}} = \{0, 1, +, -, \cdot\}$, and we treat them as axioms. All localizations of $\widetilde{\mathbb{Z}}$ satisfy **Ru.1** – **Ru.4**; the ring $\widetilde{\mathbb{Z}}$ satisfies, moreover, **Ru.5** and **Ru.6** [14].

We recall the definition of the constructible Boolean algebra.

**Definition 1.2** Let $R$ be a ring.

(1) $\mathrm{Max}(R)$ denotes the set of maximal ideals of $R$. It is endowed with the Zariski topology: the basic open sets are of the form $D_R(a) = \{\mathfrak{M} \in \mathrm{Max}(R) : a \notin \mathfrak{M}\}$, for $a \in R$. The basic closed sets are the sets $V_R(b) = \{\mathfrak{M} \in \mathrm{Max}(R) : b \in \mathfrak{M}\}$, for $b \in R$.
(2) The constructible Boolean algebra $B(R)$ associated with $R$ is the algebra generated by the basic open sets. Its elements are called constructible.

Properties **Ru.4** and **Ru.5** determine the structure of the constructible algebra: if a Bezout domain $R \neq \mathrm{Frac}(R)$ satisfies **Ru.4**, then one can check that

$$\textbf{Ru.5} \text{ holds in } R \quad \text{iff} \quad B(R) \text{ is atomless.}$$

To obtain atomic constructible algebras, in opposition to **Ru.5**, we shall consider the following definition.

**Definition 1.3**

(1) Let **Atomic.5** be the following property of a ring: given any nonzero nonunit $a$, there is a nonunit $b$ dividing $a$ such that $b$ is not the product of two relatively prime nonunits.
(2) The theory $T^{\text{atomic}}_{\text{ring}}$ is the theory **Ru.1** – **Ru.4** + **Atomic.5** + **Ru.6**.

As expected, for any Bezout domain $R \neq \mathrm{Frac}(R)$ satisfying **Ru.4**, one has the equivalence,

$$\textbf{Atomic.5} \text{ holds in } R \quad \text{iff} \quad B(R) \text{ is atomic.}$$

We propose in this article a study of the theory $T^{\text{atomic}}_{\text{ring}}$, stressing the effective aspects. Basic definitions are introduced in Section 1. Section 2 deals with models of $T^{\text{atomic}}_{\text{ring}}$. As $\widetilde{\mathbb{Z}}$ is a canonical recursive good Rumely domain, we propose some "natural" recursive models of $T^{\text{atomic}}_{\text{ring}} + \mathrm{char} = 0$ (with recursive axiomatizations). Models of $T^{\text{atomic}}_{\text{ring}} + \mathrm{char} = p$, for $p > 0$, are also proposed. Section 3 is devoted to model completeness issues.

We introduce the languages which allow (partial) quantifier elimination.

**Definition 1.4**

(1) Let $\underline{\mathrm{rad}}$ and $\mathrm{size}_{=1}$ be, respectively, binary and unary relation symbols whose interpretations in a ring $R$ are the following:

$$\begin{array}{llll} R \models a \,\underline{\mathrm{rad}}\, b & \text{iff} & a \in \mathrm{rad}_R(b) & \text{(the Jacobson radical of } b \text{ in } R) \\ & \text{iff} & a \in \cap V_R(b). & \\ R \models \mathrm{size}_{=1}(u) & \text{iff} & \multicolumn{2}{l}{u \text{ nonzero nonunit is not the product of two}} \\ & & \multicolumn{2}{l}{\text{relatively prime nonunits.}} \end{array}$$

(2) In any Bezout domain, let $(x : y)$ denote a generator of the principal ideal $(x) : (y)$. For each $n, k, l < \omega$, let $S_{n,k,l}$ be the $2k + 2l$ relation symbol

defined in any Bezout domain as

$$S_{n,k,l}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) \leftrightarrow \exists (u_r)_{r<n}\big[\bigwedge_{r<n}\mathrm{size}_{=1}(u_r) \wedge \bigwedge_{r\neq r'}\gcd(u_r,u_{r'}) = 1 \wedge \bigwedge_{r<n}(\gcd_{j<l}(z_j : t_j))\,\underline{\mathrm{rad}}\,u_r \wedge \bigwedge_{r<n}\gcd(u_r, \prod_{i<k}(x_i : y_i)) = 1\big].$$

In any model $R$ of **Ru.2** + **Ru.4** + **Ru.6**, the meanings of the two predicates $\mathrm{size}_{=1}$ and $S_{n,k,l}$ are the following ones: for $u \in R$, $\mathbf{a},\mathbf{b} \in {}^kR$, $\mathbf{c},\mathbf{d} \in {}^lR$,

$$R \models \mathrm{size}_{=1}(u) \quad \text{iff} \quad V_R(u) \text{ is an atom in } B(R),$$

$$R \models S_{n,k,l}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \quad \text{iff} \quad \left(\begin{array}{l}\text{in } B(R)\text{, the constructible set}\\ D_R(\prod_{i<k}(a_i : b_i)) \cap V_R(\gcd_{j<l}(c_j : d_j))\\ \text{is above at least } n \text{ distinct atoms.}\end{array}\right.$$

Let us note that the predicates $\mathrm{rad}_{k,l}$, for $k, l < \omega$, introduced in [13] and [14] and defined as $\mathrm{rad}_{k,l}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) \leftrightarrow \prod_{i<k}(x_i : y_i))\,\underline{\mathrm{rad}}\,\gcd_{j<l}(z_j : t_j)$, can be recovered from the $S_{n,k,l}$s:

$$T_{\mathrm{ring}}^{\mathrm{atomic}} \vdash \mathrm{rad}_{k,l}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}) \leftrightarrow \neg S_{1,k,l}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}).$$

We shall prove that $T_{\mathrm{ring}}^{\mathrm{atomic}}$ is effectively model complete with respect to the language $\mathcal{L}_{\mathrm{ring}} \cup \{\,\underline{\mathrm{rad}}\,, \mathrm{size}_{=1}\}$ and that it admits a strong form of model completeness relative to the language $\mathcal{L}_{\mathrm{ring}} \cup \{S_{n,k,l} : n,k,l < \omega\}$ (this is very reminiscent of [14] which showed model completeness of the theory of good Rumely domains with respect to $\mathcal{L}_{\mathrm{ring}} \cup \{\,\underline{\mathrm{rad}}\,\}$ and proposed a strong form of model completeness relative to $\mathcal{L}_{\mathrm{ring}} \cup \{\mathrm{rad}_{k,l} : k,l < \omega\}$). The method of proof consists in connecting truth in the model of $T_{\mathrm{ring}}^{\mathrm{atomic}}$ and truth in its constructible Boolean algebra via a Feferman-Vaught type result and in applying a result of Tarski ([11], [12]) concerning (effective) quantifier elimination in the theory of atomic Boolean algebras relative to an adequate language.

Decidability of $T_{\mathrm{ring}}^{\mathrm{atomic}}$ is then easily deduced, in Section 5, from the strong form of model completeness. We also present recursive axiomatizations of some models constructed in Section 2.

We must mention here the work of Darnière [3] and Ershov [5] who proposed a high level of generalization of the theory of Rumely domains. (Noneffective) model completeness and decidability of the theory $T_{\mathrm{ring}}^{\mathrm{atomic}}$ can be obtained by their methods (modulo some argumentation and a version of Tarski's result for relatively complemented lattices). Contrary to these more abstract articles which pursue different goals, our scope is more reduced and this allows some informative effective study (in the spirit and manner of [13] and [14]).

## 2 Basic Notions and Definitions

Let us first recall a few basic notions from the domain of Boolean algebras.

**Definition 2.1** Let $\mathfrak{B} = \langle B, 0, 1, +, \,, \cdot, ^{-}\rangle$ be a Boolean algebra, and let $\mathcal{L}_{\mathrm{boole}}$ be the language $\{0, 1, +, \cdot, ^{-}\}$.

(a) (i) An atom $a \in B$ is a nonzero element such that for any $b \in B$, $0 \le b \le a$ implies $b = 0$ or $b = a$ ($x \le y$ iff $x \cdot y = x$).

(ii) $\mathfrak{B}$ is atomless if it has no atoms.

(iii) $\mathfrak{B}$ is atomic if for any $b \in B \setminus \{0\}$, there exists an atom $a \in B$ such that $a \le b$.

(b) For each $n > 0$, let $R_n$ be the unary relation symbol whose interpretation is

$$R_n(x) \text{ holds} \quad \text{iff} \quad \text{there exist at least } n \text{ distinct atoms} \leq x.$$

We shall be dealing here with atomic Boolean algebras; hence let us set the following definition.

**Definition 2.2**

1. $T_{\text{boole}}^{\text{atomic}}$ is the theory of atomic Boolean algebras, and
2. $\mathcal{L}_{\text{boole}}^{\text{atomic}}$ is the language $\mathcal{L}_{\text{boole}} \cup \{R_n : n < \omega\}$.

The following result is due to Tarski (see [6], p. 73).[1] (See also [12], [4], and [1], chapter 5.5)

**Theorem 2.3 ([11])** *The theory $T_{\text{boole}}^{\text{atomic}}$ admits effective quantifier elimination in $\mathcal{L}_{\text{boole}}^{\text{atomic}}$.*

Let us restrict the discussion now to constructible Boolean algebras of Rumely domains. Since they are fields of sets, we shall also use the set theoretical notation $\cup, \cap, \subseteq, \varnothing, \text{Max}(R)$ for $+, \cdot, \leq, 0, 1$. The role of axioms **Ru.2** and **Ru.4** is essential.

**Lemma 2.4 ([14], 2.12)** *In any Bezout domain $R$ with good factorization, every constructible set in $B(R)$ is a basic open set or a basic closed set.*

Let us first state without proof the following easy but useful fact.

**Fact 2.5** Let $R \neq \text{Frac}(R)$ satisfy **Ru.2** + **Ru.4**.

(a) If $a \in R$ and $\mathfrak{M}$ belongs to $D_R(a)$, then there exists $b \in R \setminus \{0\}$ such that $\mathfrak{M} \in V_R(b) \subseteq D_R(a)$.
(b) Any atom in $B(R)$ is of the form $V_R(b)$, with $|V_R(b)| = 1$, for some $b \in R \setminus \{0\}$.

We can now link properties **Ru.5**, **Atomic.5** to the presence or absence of atoms in the constructible Boolean algebra.

**Lemma 2.6** *Let $R$ ($\neq \text{Frac}(R)$) satisfy* **Ru.2** + **Ru.4** *and let $a \in R \setminus \{0\}$ be a nonunit.*

(a) *$V_R(a)$ is not an atom iff $a$ is the product of two relatively prime nonunits.*
(b) **Ru.5** *holds in $R$ iff $B(R)$ is atomless.*
(c) **Atomic.5** *holds in $R$ iff $B(R)$ is atomic.*

**Proof** Let $R \neq \text{Frac}(R)$ satisfy **Ru.2** + **Ru.4**.

(a) Let $a$ be a nonzero nonunit. We suppose $V_R(a)$ is not an atom. Hence by Fact 2.5(a), there is $b \in R \setminus \{0\}$ with $\varnothing \subsetneq V_R(b) \subsetneq V_R(a)$. By good factorization, we obtain $a_0, a_1$ such that

(i) $a = a_0 a_1$,
(ii) $a_0$ and $b$ are relatively prime,
(iii) $V_R(a_1) \subseteq V_R(b)$.

Hence $V_R(a_1) = V_R(b) \neq \varnothing$, $V_R(a_0) = V_R(a) \setminus V_R(b) \neq \varnothing$, and $a$ is the product of two relatively prime nonunits. The opposite implication is immediate.

(b) We suppose that **Ru.5** holds in $R$. It follows from (a) that no $V_R(b)$, for $b \in R \setminus \{0\}$ can be an atom. By Fact 2.5(b), $B(R)$ is atomless. Conversely let $B(R)$ be atomless. Then for all nonzero nonunit $a \in R$, $V_R(a)$ is not an atom, and we conclude by (a) that **Ru.5** holds in $R$.

(c) Let **Atomic.5** hold in $R$, and let $X$ be a nonempty constructible set. Then by Fact 2.5(a), there is $b \in R \setminus \{0\}$ such that $\varnothing \neq V_R(b) \subseteq X$. By **Atomic.5**, there is $d$ dividing $b$ such that $d$ is not the product of two relatively prime nonunits. Hence by (a), $V_R(d)$ is an atom such that $V_R(d) \subseteq V_R(b) \subseteq X$. Therefore, $B(R)$ is atomic.

Conversely let $B(R)$ be atomic. Then, given a nonzero nonunit $a$ in $R$, there is $b \in R \setminus \{0\}$ such that $V_R(b) \subseteq V_R(a)$ and $V_R(b)$ is an atom. By the same argument as in (a), we can assume that $b$ divides $a$. Since $V_R(b)$ is an atom, $b$ cannot be the product of two relatively prime nonunits. Hence $R$ satisfies **Atomic.5**. $\square$

Let us state some definitions which were (partially) proposed in the introduction.

**Definition 2.7** $\quad$ $\mathrm{size}_{=1}$, $S_n$, and $S_{n,k,l}$, for $n, k, l < \omega$, are the predicates defined as

1. $\mathrm{size}_{=1}(x)$ if and only if $x$ nonzero nonunit is not the product of two relatively prime nonunits.
2. $S_n(x)$ if and only if

$$\exists (x_i)_{i<n}\Big[\bigwedge_{i<n} \mathrm{size}_{=1}(x_i) \wedge \bigwedge_{i \neq j}(\gcd(x_i, x_j) = 1) \wedge \bigwedge_{i<n} x \;\underline{\mathrm{rad}}\; (x_i)\Big].$$

3. $S_{n,k,l}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})$ if and only if

$$\exists u\Big[S_n(u) \wedge (\gcd_{j<l}(z_j : t_j)) \;\underline{\mathrm{rad}}\; u \wedge \gcd(u, \textstyle\prod_{i<k}(x_i : y_i)) = 1\Big].$$

The previous lemma implies that these predicates have their expected meaning.

**Claim 2.8** $\quad$ *Let $R$ be a Bezout domain satisfying* **Ru.4** + **Ru.6**. *Then for any $u$ in $R$, $\mathbf{a}, \mathbf{b} \in {}^kR$, $\mathbf{c}, \mathbf{d} \in {}^lR$, $n, k, l < \omega$,*

1. $R \models \mathrm{size}_{=1}(u) \Leftrightarrow V_R(u)$ *is an atom,*
2. $R \models S_n(u) \Leftrightarrow V_R(u)$ *is above at least n atoms,*
3. $R \models S_{n,k,l}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \Leftrightarrow \left( \begin{array}{l} D_R(\prod_{i<k}(a_i : b_i)) \cap V_R(\gcd_{j<k}(c_j : d_j)) \\ \textit{is above at least n atoms.} \end{array} \right.$

We note the following fact.

**Fact 2.9** $\quad$ In any model $R$ of $T_{\mathrm{ring}}^{\mathrm{atomic}}$, for $n \geq 1$, $u \in R$,

$$R \models S_n(u) \quad \text{iff} \quad |V_R(u)| \geq n.$$

**Proof** $\quad$ We show the implication from right to left for $n \geq 2$. Let us assume $\{\mathfrak{M}_0, \ldots, \mathfrak{M}_{n-1}\} \subseteq V_R(u)$, with all $\mathfrak{M}_i$s distinct. For all $i \neq j$, let $\delta_{i,j} \in \mathfrak{M}_i \setminus \mathfrak{M}_j$. Since $R$ is Bezout, for each $i < n$, let $\delta_i := \gcd\{\delta_{i,j} : j \neq i\}$. Then, for any $i < n$, $\delta_i \in \mathfrak{M}_i \setminus (\bigcup_{j \neq i} \mathfrak{M}_j)$. This gives $\mathfrak{M}_i \in V_R(\delta_i) \setminus (\bigcup_{j \neq i} V_R(\delta_j))$.

All the constructible sets $\big(V_R(\delta_i) \setminus (\bigcup_{j \neq i} V_R(\delta_j))\big) \cap V_R(u)$, for $i < n$, are nonempty and pairwise disjoint. Since $B(R)$ is atomic, each one contains an atom. $\square$

A more algebraic definition of $S_n$—valid only in models of $T_{\mathrm{ring}}^{\mathrm{atomic}}$—could have been $S_n(x)$ if and only if $x$ is the product of $n$ pairwise relatively prime nonunits. Let us set the following definition.

**Definition 2.10** Let $\mathcal{L}_{\text{ring}}^{\text{atomic}} := \mathcal{L}_{\text{ring}} \cup \{\, \underline{\text{rad}}\,, \text{size}_{=1}\}$ and let $\mathcal{L}' := \mathcal{L}_{\text{ring}} \cup \{S_{n,k,l} : n, k, l < \omega\}$.

**Notation 2.11** "$\gcd(x, y)$" or "$(x : y)$" is defined in a Bezout domain, up to multiplication by a unit. This functional notation is convenient, but we shall also use the relational "$(x, y) = (z)$" meaning by $(x, y)$ (respectively, $(z)$) the ideal generated by $x$ and $y$ (respectively, $z$).

## 3 Models of $T_{\text{ring}}^{\text{atomic}}$

The canonical good Rumely domain $\widetilde{\mathbb{Z}}$ can be equipped with the recursive structure defined by Rumely in [10], p. 32 (any $\alpha \in \widetilde{\mathbb{Z}}$ is represented by a pair $(P(x), a + bi)$ where $P(x)$ is a monic irreducible polynomial over $\mathbb{Z}$, $\alpha$ is a root of $P$, and $a + bi$ is a sufficiently good decimal approximation of $\alpha$). It is thus natural to wonder whether one can construct a recursive model of $T_{\text{ring}}^{\text{atomic}} + \text{char} = 0$.

We shall do so by considering localizations of $\widetilde{\mathbb{Z}}$, turning appropriate $V_{\widetilde{\mathbb{Z}}}(u)$, $u \in \widetilde{\mathbb{Z}}$, into atoms by introducing inverses which will kill all maximal ideals in $V_{\widetilde{\mathbb{Z}}}(u)$ except one. It is thus interesting to obtain the following effective decomposition of basic closed sets.

**Lemma 3.1** *There is an effective uniform procedure which produces for each nonzero nonunit $u \in \widetilde{\mathbb{Z}}$ two sequences of algebraic integers $\langle a_n : n < \omega\rangle$, $\langle b_n : n < \omega\rangle$ such that*

1. $\mathfrak{M}_u := \bigcup_{n<\omega} a_n\widetilde{\mathbb{Z}}$ *is a maximal ideal of* $\widetilde{\mathbb{Z}}$,
2. *the* $V_{\widetilde{\mathbb{Z}}}(b_n)$*s, for* $n < \omega$, *are nonempty and pairwise disjoint,*
3. $V_{\widetilde{\mathbb{Z}}}(u) = \{\mathfrak{M}_u\} \mathbin{\mathring{\cup}} \mathring{\bigcup}_{n<\omega} V_{\widetilde{\mathbb{Z}}}(b_n)$, ($\mathring{\cup}$ *meaning disjoint union*).

One can derive the following.

**Corollary 3.2**

(a) *Given a nonunit* $u \in \widetilde{\mathbb{Z}} \setminus \{0\}$, *one can effectively construct a multiplicative set* $S_u = \{s_n : n < \omega\} \subseteq \widetilde{\mathbb{Z}}$ *such that*
  (i) *for all* $\mathfrak{M} \in V_{\widetilde{\mathbb{Z}}}(u)$ *except one,* $\mathfrak{M} \cap S_u \neq \varnothing$,
  (ii) *for all* $\mathfrak{M} \in D_{\widetilde{\mathbb{Z}}}(u)$, $\mathfrak{M} \cap S_u = \varnothing$.

(b) *Let $u$ and $S_u$ be as in* (a). *If* $R := (S_u)^{-1} \cdot \widetilde{\mathbb{Z}}$, *then* $V_R(u)$ *is an atom and* $D_R(u) = \{\mathfrak{M}R : \mathfrak{M} \in D_{\widetilde{\mathbb{Z}}}(u)\}$.

**Proof of Corollary 3.2** (a) Let $u \in \widetilde{\mathbb{Z}}$ be a nonzero nonunit, and let $\mathfrak{M}_u$, $\langle b_n : n < \omega\rangle$ be obtained from Lemma 3.1. One takes for $S_u$ the multiplicative set generated by $1 \cup \{b_n : n < \omega\}$.

(b) Let $R := (S_u)^{-1} \cdot \widetilde{\mathbb{Z}}$. (b) follows from (a) since $\text{Max}(R)$ is the set $\{\mathfrak{M}R : \mathfrak{M} \in \text{Max}(\widetilde{\mathbb{Z}}), \mathfrak{M} \cap S_u \neq \varnothing\}$. □

Let us defer the (somewhat lengthy) proof of Lemma 3.1 and to motivate it, let us propose constructions of recursive models of $T_{\text{ring}}^{\text{atomic}}$ based on this lemma. We present first an "almost" canonical example: a model where every nonzero element belongs to finitely many maximal ideals. A recursive axiomatization of its theory is presented. We then describe more briefly a second example where each prime integer belongs to infinitely many maximal ideals. We finish up with a consistency statement, to be used later, which gives the existence of (possibly nonrecursive) models $R$ of $T_{\text{ring}}^{\text{atomic}}$ whose algebraic part (i.e., $R \cap \widetilde{\mathbb{Q}}$) is $\widetilde{\mathbb{Z}}$ or $\widetilde{\mathbb{Q}}$.

**Example 3.3** We construct a recursive model $R$ of "$T_{\text{ring}}^{\text{atomic}} + \text{char} = 0$" such that $\text{Max}(R)$ has the structure of $\text{Max}(\mathbb{Z})$. The method is to turn all $V_{\widetilde{\mathbb{Z}}}(p)$s, for $p$ prime (rational) number, into atoms.

**Definition 3.4 (Definition of $R$)** By Lemma 3.1, for each prime $p$, one constructs effectively

1. a maximal ideal $\mathfrak{M}_p$ (more exactly, a defining sequence of $\mathfrak{M}_p$),
2. a sequence $\langle b_{p,n} : n < \omega\rangle$ of algebraic integers such that

$$V_{\widetilde{\mathbb{Z}}}(p) = \{\mathfrak{M}_p\} \mathbin{\mathring{\cup}} \mathring{\bigcup}_{n<\omega} V_{\widetilde{\mathbb{Z}}}(b_{p,n}). \tag{1}$$

Let $S$ be the multiplicative set generated by $\{1\} \cup \{b_{p,n} : p \text{ prime}, n < \omega\}$, and let $R := S^{-1}\widetilde{\mathbb{Z}}$.

We check that $R$ is the expected model.

**Ru.1 – Ru.4**: By [14], 2.10, 3.5, any localization of $\widetilde{\mathbb{Z}}$ satisfies **Ru.1 – Ru.4**.

**Atomic.5**: Since $\text{Max}(R) = \{\mathfrak{M}R : \mathfrak{M} \in \text{Max}(\widetilde{\mathbb{Z}}), \mathfrak{M} \cap S = \varnothing\}$, all maximal ideals in $\text{Max}(\widetilde{\mathbb{Z}}) \setminus \{\mathfrak{M}_p : p \text{ prime}\}$ are "killed" in the transition from $\widetilde{\mathbb{Z}}$ to $R$. Let us check that all $\mathfrak{M}_p$s are "preserved." We assume for a contradiction that, for $p$ prime and $s \in S, s \in \mathfrak{M}_p$. Since $\mathfrak{M}_p$ is prime, by definition of $S$, there must exist $b_{q,n}, q$ prime, $n < \omega$, such that $b_{q,n} \in \mathfrak{M}_p$.

(a) If $q = p$, then we would get $\mathfrak{M}_p \in V_{\widetilde{\mathbb{Z}}}(b_{p,n})$ and a contradiction.
(b) If $q \neq p$, since $V_{\widetilde{\mathbb{Z}}}(q) \supseteq V_{\widetilde{\mathbb{Z}}}(b_{q,n})$, we obtain $q \in \mathfrak{M}_p$ which would give $(1) \subseteq (p, q) \subseteq \mathfrak{M}_p$ and also a contradiction.

Hence, for each prime $p$, $V_R(p) = \{\mathfrak{M}_p R\}$ is an atom. Since $\text{Max}(R)$ is the disjoint union $\mathring{\bigcup}\{V_R(p) : p \text{ prime}\}$, $R$ satisfy **Atomic.5**.

**Ru.6**: Let $v \in \widetilde{\mathbb{Z}} \setminus \{0\}$. We check that $v$ does not belong to the radical of $R$. There is a finite set of prime numbers $P$ such that $V_{\widetilde{\mathbb{Z}}}(v) \subseteq \bigcup_{p\in P} V_{\widetilde{\mathbb{Z}}}(p)$. If $q$ is a prime number not in $P$, then necessarily $V_{\widetilde{\mathbb{Z}}}(v) \cap V_{\widetilde{\mathbb{Z}}}(q) = \varnothing$, and hence $V_R(v) \cap V_R(q) = \varnothing$. Since $V_R(q)$ is nonempty, $v$ cannot belong to the radical of $R$. Hence $R$ is a model of $T_{\text{ring}}^{\text{atomic}}$.

Let us present the following result we shall prove in Section 5.

**Proposition 3.5** *The theory of R is recursively axiomatized as*

1. $T_{\text{ring}}^{\text{atomic}} + \mathit{char} = 0$,
2. *the quantifier-free diagram of* $\widetilde{\mathbb{Z}}$,
3. *for each prime p, the axiom* "$\text{size}_{=1}(p)$" *(formally its $\mathcal{L}_{\text{ring}}$ equivalent),*
4. *for each prime p, each $n < \omega$, the axiom "$b_{p,n}$ unit".*

**Example 3.6** We sketch the construction. If instead of turning all $V_{\widetilde{\mathbb{Z}}}(b_{p,n})$s, $p$ prime, $n < \omega$, into empty sets, one transforms them into atoms, then one obtains a situation where if $R'$ is the final model, then every $V_{R'}(p)$ contains infinitely many atoms. More precisely, each $V_{R'}(p)$ is equal to some union $\{\mathfrak{M}_p R'\} \cup \{\mathfrak{M}_{p,n} R' : n < \omega\}$ where the $\{\mathfrak{M}_{p,n} R'\}$s are the atoms $V_{R'}(b_{p,n})$, but $\{\mathfrak{M}_p\, R'\}$ is not an atom, and hence not a constructible set. Since $R \models \neg S_2(p)$ and $R' \models S_2(p)$, for any prime $p$, $R$ and $R'$ are not elementarily equivalent.

Let us present now some consistency results also based on Lemma 3.1 we shall need in Section 5.

**Lemma 3.7** *Let* $\mathbf{t} := \langle t_i : i < k\rangle$ *be a sequence of nonzero nonunits which are pairwise relatively prime in* $\widetilde{\mathbb{Z}}$, *and let* $\mathbf{m} := \langle m_i : i < k\rangle \in {}^k\omega$. *Then there is a localization* $R$ *of* $\widetilde{\mathbb{Z}}$ *satisfying* $T^{\text{atomic}}_{\text{ring}}$ *and such that for each* $i < k$, $V_R(t_i)$ *contains exactly* $m_i$ *atoms.*

**Proof** Let $\mathbf{t}, \mathbf{m}, k$ be as in the lemma. We set $I := \{i < k : m_i \neq 0\}$. Then our strategy is as follows:

(a) for each $i \in I$, to split $V_{\widetilde{\mathbb{Z}}}(t_i)$ into $m_i$ nonempty disjoint basic closed sets $V_{\widetilde{\mathbb{Z}}}(t_{i,s})$, $s < m_i$,
(b) for every $i \in I$, $s < m_i$, to transform $V_{\widetilde{\mathbb{Z}}}(t_{i,s})$ into an atom,
(c) to turn $\operatorname{Max}(\widetilde{\mathbb{Z}}) \setminus (\bigcup_{i<k} V_{\widetilde{\mathbb{Z}}}(t_i)) = D_{\widetilde{\mathbb{Z}}}(\prod_{i<k} t_i)$ into a union of atoms,
(d) for each $i \in (k \setminus I)$, to turn $t_i$ into a unit.

Let us successively take care of all these steps.

(a) We use **Ru.5** to find the appropriate $t_{i,s}$, for $i \in I$, $s < m_i$.

(b) By Corollary 3.2, for each $i \in I$, $s < m_i$, one defines sets $S_{i,s} \subseteq \widetilde{\mathbb{Z}}$ such that

(i) for all $\mathfrak{N} \in V_{\widetilde{\mathbb{Z}}}(t_{i,s})$ except one, $S_{i,s} \cap \mathfrak{N} \neq \varnothing$,
(ii) for all $\mathfrak{N} \in D_{\widetilde{\mathbb{Z}}}(t_{i,s})$, $S_{i,s} \cap \mathfrak{N} = \varnothing$.

(c) Let $u = \prod_{i<k} t_i$. Since $\operatorname{Max}(\widetilde{\mathbb{Z}}) = \mathring{\bigcup}\{V_{\widetilde{\mathbb{Z}}}(p) : p \text{ prime}\}$,

$$D_{\widetilde{\mathbb{Z}}}(u) = \mathring{\bigcup}\{D_{\widetilde{\mathbb{Z}}}(u) \cap V_{\widetilde{\mathbb{Z}}}(p) : p \text{ prime}\}.$$

Let $P$ be the set of primes $p$ such that $D_{\widetilde{\mathbb{Z}}}(u) \cap V_{\widetilde{\mathbb{Z}}}(p)$ is nonempty. Since there is a finite set $T$ of primes such that $V_{\widetilde{\mathbb{Z}}}(u) \subseteq \bigcup_{p\in T} V_{\widetilde{\mathbb{Z}}}(p)$, $P$ must be infinite. By good factorization, for each $p \in P$, there is $u_p \in \widetilde{\mathbb{Z}} \setminus \{0\}$ such that $D_{\widetilde{\mathbb{Z}}}(u) \cap V_{\widetilde{\mathbb{Z}}}(p) = V_{\widetilde{\mathbb{Z}}}(u_p)$. Then $D_{\widetilde{\mathbb{Z}}}(u) = \mathring{\bigcup}\{V_{\widetilde{\mathbb{Z}}}(u_p) : p \in P\}$.

Again, by Corollary 3.2, for each $p \in P$, one can define $S_p \subseteq \widetilde{\mathbb{Z}}$ such that

(i) for all $\mathfrak{N} \in V_{\widetilde{\mathbb{Z}}}(u_p)$ except one, $S_p \cap \mathfrak{N} \neq \varnothing$,
(ii) for all $\mathfrak{N} \in D_{\widetilde{\mathbb{Z}}}(u_p)$, $S_p \cap \mathfrak{N} = \varnothing$.

(d) We simply put $\{t_i : i \in k \setminus I\}$ in the final set: let $S$ be the multiplicative set generated by $\{1\} \cup \{t_i : i \in k \setminus I\} \cup \bigcup\{S_{i,s} : i \in I, s < m_i\} \cup \bigcup\{S_p : p \in P\}$.

Let $R := S^{-1} \cdot \widetilde{\mathbb{Z}}$. We have

$$\operatorname{Max}(R) = \mathring{\bigcup}\{V_R(t_{i,s}) : i \in I, s < m_i\} \mathbin{\mathring{\cup}} \mathring{\bigcup}\{V_R(u_p) : p \in P\}.$$

Hence $\operatorname{Max}(R)$ is a disjoint union of atoms, and **Atomic.5** holds in $R$.

Also since there is an infinite set $P$ of primes $p$ such that $V_R(p)$ is nonempty, by the same argument as in Example 3.3, one checks that **Ru.6** holds in $R$. $\square$

One can obtain models $R$ of $T^{\text{atomic}}_{\text{ring}} + \text{char} = 0$ with any prescribed algebraic part (i.e., $R \cap \widetilde{\mathbb{Q}}$). Let us only consider the following.

**Lemma 3.8** *There exist models $A_0, A_1$ of $T_{\text{ring}}^{\text{atomic}}$ such that $A_0 \cap \widetilde{\mathbb{Q}} = \widetilde{\mathbb{Z}}$ and $A_1 \cap \widetilde{\mathbb{Q}} = \widetilde{\mathbb{Q}}$.*

(We shall see later that, up to elementary equivalence, these models are unique.)

**Proof of Lemma 3.8** We start with $A_0$ and $\widetilde{\mathbb{Z}}$. Let $\Theta$ be the $\mathcal{L}_{\text{ring}}(\widetilde{\mathbb{Z}})$ theory defined as

$$\Theta := T_{\text{ring}}^{\text{atomic}} + \text{ the quantifier-free diagram of } \widetilde{\mathbb{Z}} + \{\text{“}z \text{ nonunit”} : z \text{ nonunit in } \widetilde{\mathbb{Z}}\}.$$

We check that any finite subset of $\Theta$ admits a model. Let $\Theta_{\text{fin}} \subseteq T_{\text{ring}}^{\text{atomic}} + \text{diagram}(\widetilde{\mathbb{Z}}) + \{z_i \text{ nonunit} : i < k\}$, where the $z_i$s are nonzero nonunits of $\widetilde{\mathbb{Z}}$. For each nonempty $I \subseteq k$, by good factorization in $\widetilde{\mathbb{Z}}$, let $z_I \in \widetilde{\mathbb{Z}} \setminus \{0\}$ be such that

$$\begin{aligned} V_{\widetilde{\mathbb{Z}}}(z_I) &:= \left(\bigcap_{i \in I} V_{\widetilde{\mathbb{Z}}}(z_i)\right) \cap \left(\bigcap_{i \notin I} (V_{\widetilde{\mathbb{Z}}}(z_i))^c\right) \\ &= \bigcap_{i \in I} \left(V_{\widetilde{\mathbb{Z}}}(z_i) \setminus V_{\widetilde{\mathbb{Z}}}(\textstyle\prod_{i \notin I} z_i)\right). \end{aligned}$$

For every $i < k$, since $V_{\widetilde{\mathbb{Z}}}(z_i) = \mathring{\bigcup}_{I \ni i} V_{\widetilde{\mathbb{Z}}}(z_I)$, we note that there exists at least one $I$ such that $i \in I$ and $z_I$ is a nonunit in $\widetilde{\mathbb{Z}}$.

**Claim 3.9** *For any ring $R \supseteq \widetilde{\mathbb{Z}}$ and any $u, v, w \in \widetilde{\mathbb{Z}}$,*

$$\left(V_{\widetilde{\mathbb{Z}}}(w) = V_{\widetilde{\mathbb{Z}}}(u) \setminus V_{\widetilde{\mathbb{Z}}}(v)\right) \text{ implies } \left(V_R(w) = V_R(u) \setminus V_R(v)\right).$$

**Proof** We have the implications,

$$\begin{aligned} V_{\widetilde{\mathbb{Z}}}(w) = V_{\widetilde{\mathbb{Z}}}(u) \setminus V_{\widetilde{\mathbb{Z}}}(v) &\implies \begin{cases} \cdot\ (v, w) = (1) \text{ in } \widetilde{\mathbb{Z}}, \\ \cdot\ V_{\widetilde{\mathbb{Z}}}(uv) = V_{\widetilde{\mathbb{Z}}}(wv), \end{cases} \\ &\implies \begin{cases} \cdot\ (v, w) = (1) \text{ in } \widetilde{\mathbb{Z}}, \\ \cdot\ \exists\, m, n < \omega,\ \exists\, \lambda, \mu \in \widetilde{\mathbb{Z}} \text{ such that} \\ \quad (uv)^m = \lambda(wv) \text{ and } (wv)^n = \mu(uv), \end{cases} \\ &\implies \begin{cases} \cdot\ V_R(v) \cap V_R(w) = \varnothing, \\ \cdot\ V_R(u) \cup V_R(v) = V_R(w) \cup V_R(v), \end{cases} \\ &\implies \quad V_R(w) = V_R(u) \setminus V_R(v). \end{aligned}$$

□

Let $\mathbf{t} := \langle t_j : j < m \rangle$ be an enumeration of the set $\{z_I \text{ nonunit} : \varnothing \neq I \subseteq k\}$. $\mathbf{t}$ is a sequence of nonzero nonunits which are relatively prime in $\widetilde{\mathbb{Z}}$. We can thus apply Lemma 3.7: there is a model $R \supseteq \widetilde{\mathbb{Z}}$ of $T_{\text{ring}}^{\text{atomic}}$ such that each $V_R(t_j)$, $j < m$, contains an atom. By Claim 3.9, and the fact that a gcd in $\widetilde{\mathbb{Z}}$ remains a gcd in $R$, we deduce that

$$V_R(z_I) = \left(\bigcap_{i \in I} V_R(z_i)\right) \cap \left(\bigcap_{i \notin I} (V_R(z_i))^c\right) \text{ and } V_R(z_i) = \mathring{\bigcup}_{i \in I} V_R(z_I).$$

We noticed above that, for each $i < k$, there is $I \subseteq k$ with $i \in I$ and $z_I$ nonunit. We derive that there must exist $j < m$ such that $V_R(t_j) \subseteq V_R(z_i)$. Therefore, all $z_i$s are nonunits in $R$. Hence $\Theta_{\text{fin}}$ is consistent. Therefore, $\Theta$ admits a model, and any model $R$ of $\Theta$ satisfies $R \cap \widetilde{\mathbb{Q}} = \widetilde{\mathbb{Z}}$.

To deal with $\widetilde{\mathbb{Q}}$, we consider the theory,

$$T_{\text{ring}}^{\text{atomic}} + \text{ quantifier-free diagram of } \widetilde{\mathbb{Z}} + \{\text{“}z \text{ unit”} : z \in \widetilde{\mathbb{Z}} \text{ and } z \text{ nonzero nonunit}\},$$

and simply replace the requirement "$V_R(t_j)$ contains exactly one atom" by the condition "$V_R(t_j)$ contains no atom". □

It remains to prove Lemma 3.1.

**Proof of Lemma 3.1** By arguments of [10], p. 32 (and, for example, [8], Theorem 2.12.23, about computation of primitive elements), it is possible to construct a "recursive" sequence of number fields $K_n = \mathbb{Q}(\alpha_n), n < \omega$ such that $\alpha_n \in \widetilde{\mathbb{Z}}$, $K_n \subseteq K_{n+1}$, and $\bigcup_{n<\omega} K_n = \widetilde{\mathbb{Q}}$ (one defines inductively the two sequences $\langle P_n(x) : n < \omega\rangle$, $\langle a_n + ib_n : n < \omega\rangle$ as in [10] representation of $\widetilde{\mathbb{Z}}$). Given $u \in \widetilde{\mathbb{Z}}$, it is then possible to obtain effectively the least $n < \omega$ such that $u \in K_n$. For $n < \omega$, we denote by $\mathcal{O}_n$ the ring of integers of $K_n$.

We shall make effective the following (nonconstructive) argument: let $u$ be a nonzero nonunit of $\mathcal{O}_{n_0}$, for $n_0 < \omega$, and let $u\mathcal{O}_{n_0} = (\mathfrak{M}_{0,0})^{e_0} \cdots (\mathfrak{M}_{0,k_0-1})^{e_{k_0-1}}$, for $k_0 \geq 1$, be the factorization of the principal ideal $u\mathcal{O}_{n_0}$ into prime ideals of $\mathcal{O}_{n_0}$. By the "finiteness of the class number" argument ([7], p. 38, or [14], 2.4), there is $n_1 \geq n_0$ such that all ideals $\mathfrak{M}_{0,i}$ become principal in $\mathcal{O}_{n_1}$: for $i < k_0$, let $a_{0,i} \in \mathcal{O}_{n_1}$ be such that $\mathfrak{M}_{0,i}\mathcal{O}_{n_1} = a_{0,i}\mathcal{O}_{n_1}$.

Let us set $\boldsymbol{a_0} := \langle a_{0,i} : i < k_0\rangle$ and $u_1 := a_{0,0}$. We then repeat the procedure with $u_1$ and $\mathcal{O}_{n_1}$, defining $\boldsymbol{a_1} := \langle a_{1,i} : i < k_1\rangle$ and $u_2 := a_{1,0} \ldots$. This way, one builds a sequence (of finite sequences) $\langle \boldsymbol{a_n} : n < \omega\rangle$. If we set $\langle a_n : n < \omega\rangle = \langle a_{n,0} : n < \omega\rangle$ and $\langle b_n : n < \omega\rangle = \langle \prod_{1\leq j<k_n} a_{n,j} : n < \omega\rangle$, then the sequences $\langle a_n : n < \omega\rangle$ and $\langle b_n : n < \omega\rangle$ satisfy the requirements of Lemma 3.1.

No systematic effective procedure was available in our sources ([8], [2]) to factorize ideals. So instead, we considered factorizations of integers. Even though the rings $\mathcal{O}_n$, $n < \omega$, are rarely unique factorization domains, this suffices. □

**Notation 3.10** Let $K$ be a number field, $\mathcal{O}$ its ring of integers, and let $v$ be a nonzero nonunit of $\mathcal{O}$. Let $v\mathcal{O} = (\mathfrak{M}_0)^{e_0} \cdots (\mathfrak{M}_{k_\mathcal{O}(v)-1})^{e_{k_\mathcal{O}(v)-1}}$, $k_\mathcal{O}(v) \geq 1$, $e_i \geq 1$, for $i < k_\mathcal{O}(v)$, be the factorization of $v\mathcal{O}$ into prime ideals ($\mathfrak{M}_i \neq \mathfrak{M}_j$ for $i \neq j$). For each $i < k_\mathcal{O}(v)$, let $h_i$ be the order of the equivalence class of $\mathfrak{M}_i$ in the ideal class group. Since the ideal class number $h$ is finite, each $h_i$ divides $h$ and there must exist $a_i \in \mathcal{O}$ such that $(\mathfrak{M}_i)^{h_i} = a_i\mathcal{O}$. For $i < k_\mathcal{O}(v)$, let us set $\lambda_i := (h/h_i)e_i$. Let $\boldsymbol{a_\mathcal{O}}(v) := \langle a_i : i < k_\mathcal{O}(v)\rangle$ and let $\boldsymbol{\lambda_\mathcal{O}}(v) := \langle \lambda_i : i < k_\mathcal{O}(v)\rangle$.

These definitions are noneffective, but up to order and multiplication by units, we can recover $\boldsymbol{a_\mathcal{O}}(v)$ in an effective manner.

**Lemma 3.11** *There is an effective uniform procedure which applied to a nonzero nonunit $v$ in $\mathcal{O}$ produces a sequence $\langle b_i : i < k\rangle$ of elements of $\mathcal{O}$ such that $k = k_\mathcal{O}(v)$ and for some permutation $\sigma$ of $k_\mathcal{O}(v)$, and all $i < k_\mathcal{O}(v)$, $b_i$ and $a_{\sigma(i)}$ are associates in $\mathcal{O}$ ("associate" meaning equal modulo multiplication by a unit).*

**Proof** Let us list without proof some easy properties of the sequences $\boldsymbol{a_\mathcal{O}}(v) := \langle a_i : i < k_\mathcal{O}(v)\rangle$ and $\boldsymbol{\lambda_\mathcal{O}}(v) := \langle \lambda_i : i < k_\mathcal{O}(v)\rangle$.

**Claim 3.12**

(i) *For $i < k_\mathcal{O}(v)$, $a_i$ is a nonunit,*
(ii) *for $i \neq j < k_\mathcal{O}(v)$, $a_i$ and $a_j$ are relatively prime,*
(iii) *$v^h$ and $\prod_{i<k_\mathcal{O}(v))} a_i^{\lambda_i}$ are associates.*

**Claim 3.13** *Let $\langle b_0, \ldots, b_{l-1}\rangle \in \mathcal{O}^l$, for $l < \omega$, be such that*

(i) *the $b_i$s are nonunits,*
(ii) *for $i \neq j$, $(b_i, b_j) = (1)$,*
(iii) *$v^h$ and $\prod_{i<l} b_i^{\mu_i}$ are associates for some sequence $\langle \mu_i : i < l\rangle \in (\omega \setminus \{0\})^l$.*

*Then necessarily $l \leq k_{\mathcal{O}}(v)$.*

**Claim 3.14** *Let $\langle b_0, \ldots, b_{k_{\mathcal{O}}(v)-1}\rangle$ satisfy (i), (ii), and (iii) of the previous claim (i.e., $l = k_{\mathcal{O}}(v)$). Then there exists a permutation $\sigma$ of $k_{\mathcal{O}}(v)$ such that, for any $i < k_{\mathcal{O}}(v)$, $a_i$ divides $b_{\sigma(i)}$.*

**Proof** By (ii) and (iii), one has $V_{\mathcal{O}}(v) = \mathring{\bigcup}_{i<k_{\mathcal{O}}(v)} V_{\mathcal{O}}(b_i)$. From $|V_{\mathcal{O}}(v)| = k_{\mathcal{O}}(v)$, we deduce $|V_{\mathcal{O}}(b_i)| = 1$, for each $i < k_{\mathcal{O}}(v)$. $V_{\mathcal{O}}(v) = \{\mathfrak{M}_i : i < k_{\mathcal{O}}(v)\}$. (ii) and (iii) imply the existence of a permutation $\sigma$ of $k_{\mathcal{O}}(v)$ such that $V_{\mathcal{O}}(b_i) = \{\mathfrak{M}_{\sigma(i)}\}$.

Hence by uniqueness of the factorization of $b_i\mathcal{O}$, there must exist $t_i \in \omega \setminus \{0\}$ such that $b_i\mathcal{O} = (\mathfrak{M}_{\sigma(i)})^{t_i}$. By definition, the order of the class of $\mathfrak{M}_{\sigma(i)}$ in the ideal class group is $h_{\sigma(i)}$. Therefore, $h_{\sigma(i)}$ divides $t_i$. We deduce that, for each $i < k_{\mathcal{O}}(v)$, there is $\nu_i \in \omega \setminus \{0\}$ such that $b_i\mathcal{O} = (\mathfrak{M}_{\sigma(i)})^{h_{\sigma(i)}\nu_i} = (a_{\sigma(i)}^{\nu_i})\mathcal{O}$. Hence $a_{\sigma(i)}$ divides $b_i$. Claim 3.14 follows. □

Let $N_{K/\mathbb{Q}}$ denote the norm relative to the field extension $K/\mathbb{Q}$. By $|N_{K/\mathbb{Q}}(\alpha)|$, we mean the absolute value of $N_{K/\mathbb{Q}}(\alpha)$ (elsewhere by $|\ |$ we mean the cardinality). To "compute" $\boldsymbol{a}_{\mathcal{O}}(v)$, we shall resort to Theorem 6.4.2 of [8].

**Theorem 3.15 ([8])** *Let $a \in \omega$. Then there are finitely many nonassociate elements $\alpha \in \mathcal{O}$ such that $|N_{K/\mathbb{Q}}(\alpha)| = a$. Those can be effectively computed.*

(If $K = \mathbb{Q}(\beta)$, for $\beta \in \widetilde{\mathbb{Z}}$, then one can check that the procedure is uniform in $\beta$).

Hence let $B$ be an effectively computed maximal set of nonassociate elements $\alpha$ such that $1 < |N_{K/\mathbb{Q}}(\alpha)| \leq |N_{K/\mathbb{Q}}(v^h)|$ (the minimum polynomial gives the absolute value of the norm, and by requiring $1 < |N_{K/\mathbb{Q}}(\alpha)|$, we exclude units). Setting $l_0 := \lfloor \log(|N_{K/\mathbb{Q}}(v^h)|)\rfloor$, we exhaustively test all sequences $\langle b_i : i < l\rangle$, $\langle \mu_i : i < l\rangle$, for $l \leq l_0$, $b_i \in B$, and $1 \leq \mu_i \leq l_0$, checking (in $\widetilde{\mathbb{Z}}$) whether

(ii) $\gcd(b_i, b_j) = 1$ for $i \neq j$,
(iii) $(\prod_{i<l} b_i^{\mu_i}) | v^h$ and $v^h | \prod_{i<l} b_i^{\mu_i}$ ( $|$ means "divides").

We know from Claim 3.12 that the sequences $\boldsymbol{a}_{\mathcal{O}}(v)$ (up to multiplication by units), $\boldsymbol{\lambda}_{\mathcal{O}}(v)$ will pass the test. Hence let us consider the set $S_{\max}$ of pairs $(\langle b_i : i < l\rangle, \langle \mu_i : i < l\rangle)$ which pass the test and such that $l$ is maximal. By Claim 3.13, the sequences in $S_{\max}$ have length $k_{\mathcal{O}}(v)$. On $S_{\max}$, we consider the following partial order

$$(\mathbf{b}, \boldsymbol{\mu}) \| (\mathbf{b}', \boldsymbol{\mu}') \quad \text{iff} \quad \begin{array}{l}\text{there is a permutation } \sigma \text{ of } |\mathbf{b}| \text{ such that} \\ \text{for any } i < |\mathbf{b}|,\ b_i \text{ divides } b'_{\sigma(i)}.\end{array}$$

By Claim 3.14, $(\boldsymbol{a}_{\mathcal{O}}(v), \boldsymbol{\lambda}_{\mathcal{O}}(v))$ is a minimum for $\|$ on $S_{\max}$ (up to multiplication by units for $\boldsymbol{a}_{\mathcal{O}}(v)$). We choose a minimum element $(\mathbf{b}, \boldsymbol{\mu})$ of $S_{\max}$ (according to a fixed recursive well ordering of $\widetilde{\mathbb{Z}}$). One can check that by definition of $S_{\max}$ and $\|$, $\mathbf{b} := \langle b_i : i < k_{\mathcal{O}}(v)\rangle$ satisfies the requirements of Lemma 3.11. □

**Definition 3.16** Let $\approx$ be the equivalence relation defined on finite sequences of algebraic integers as follows:

$$\mathbf{a} \approx \mathbf{b} \quad \text{iff} \quad |\mathbf{a}| = |\mathbf{b}| \text{ and there is a permutation } \sigma \text{ of } |\mathbf{a}| \text{ such that for all } i < |\mathbf{a}|,\ a_i \text{ and } b_{\sigma(i)} \text{ are associates.}$$

We can now develop the inductive argument which gives Lemma 3.1. Let $u$ be our initial algebraic integer in $\mathcal{O}_{n_0}$. By applying repeatedly Lemma 3.11, we construct recursively a sequence $\langle \boldsymbol{b_n} : n < \omega\rangle$ where $\boldsymbol{b_n}$ is a finite sequence of nonzero nonunits of $\mathcal{O}_{n_0+n}$ such that

(∗) $\boldsymbol{b_0} \approx \boldsymbol{a}_{\mathcal{O}_{n_0}}(u)$,

(∗∗) $\boldsymbol{b_{n+1}} \approx \boldsymbol{a}_{\mathcal{O}_{n_0+n+1}}(\boldsymbol{b_n}(0))$.

From Claim 3.12(ii), (iii), we deduce

(◇) $V_{\widetilde{\mathbb{Z}}}(u) = \mathring{\bigcup}_{i<|\boldsymbol{b_0}|} V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_0}(i))$,

(◇◇) for all $n < \omega$, $V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_n}(0)) = \mathring{\bigcup}_{i<|\boldsymbol{b_{n+1}}|} V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_{n+1}}(i))$.

**Claim 3.17** *The infinite intersection $\bigcap_{n<\omega} V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_n}(0))$ is reduced to a unique maximal ideal $\mathfrak{M}_u$.*

**Proof** In order to treat (∗) and (∗∗) simultaneously, let us set $\boldsymbol{b_{-1}} = \langle u\rangle$. For each $n < \omega$, (∗), (∗∗), and the definition of the (partial) function $\boldsymbol{a}_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}^{<\omega}$ imply the existence, for each $n < \omega$, of distinct prime ideals $\mathfrak{M}_{n,i} \in \mathrm{Max}(\mathcal{O}_{n_0+n})$, integers $e_{n,i} \geq 1$, $h_{n,i} < \omega$, for $i < |\boldsymbol{b_n}|$ such that

(a.1) $\boldsymbol{b_{n-1}}(0)\mathcal{O}_{n_0+n} = (\mathfrak{M}_{n,0})^{e_{n,0}} \cdots (\mathfrak{M}_{n,k})^{e_{n,k}} \qquad (k = |\boldsymbol{b_n}| - 1)$,

(a.2) for any $i < |\boldsymbol{b_n}|$, $h_{n,i}$ is the order of the equivalence class of $\mathfrak{M}_{n,i}$ and $(\mathfrak{M}_{n,i})^{h_{n,i}} = \boldsymbol{b_n}(i)\mathcal{O}_{n_0+n}$.

We claim that $\mathfrak{M}_u := \bigcup_{n<\omega} \mathfrak{M}_{n,0}$ is "the" maximal ideal lying in the intersection.

1. By (a.1) and (a.2), $(\mathfrak{M}_{n,0}\mathcal{O}_{n_0+n+1})^{h_{n,0}} \subseteq \mathfrak{M}_{n+1,0}$. Hence by primeness of $\mathfrak{M}_{n+1,0}$, $\mathfrak{M}_{n,0} \subseteq \mathfrak{M}_{n+1,0}$. Therefore, $\mathfrak{M}_u := \bigcup_{n<\omega} \mathfrak{M}_{n,0}$ is a prime ideal of $\widetilde{\mathbb{Z}}$ containing $u$.
2. Let $\mathfrak{M} \in \bigcap_{n<\omega} V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_n}(0))$. We check $\mathfrak{M} = \mathfrak{M}_u$. Since for each $n < \omega$, $(\mathfrak{M}_{n,0})^{h_{n,0}} = \mathbf{b}_n(0)\mathcal{O}_{n_0+n}$, we obtain $(\mathfrak{M}_{n,0})^{h_{n,0}} \subseteq \mathfrak{M} \cap \mathcal{O}_{n_0+n}$ and by primeness of $\mathfrak{M} \cap \mathcal{O}_{n_0+n}$, $\mathfrak{M}_{n,0} \subseteq \mathfrak{M}$. Therefore, $\mathfrak{M} = \mathfrak{M}_u$. □

**Claim 3.18** $V_{\widetilde{\mathbb{Z}}}(u) = \{\mathfrak{M}_u\} \mathring{\cup} \mathring{\bigcup}\{V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_n}(i)) : n < \omega,\ i \geq 1\}$.

**Proof** From (◇) and (◇◇) above, we derive that, for any $n < \omega$,

$$V_{\widetilde{\mathbb{Z}}}(u) = V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_n}(0)) \mathring{\cup} \mathring{\bigcup}\{V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_j}(i)) : j \leq n,\ i \geq 1\}.$$

Let $\mathfrak{M} \in V_{\widetilde{\mathbb{Z}}}(u) \setminus \bigcup\{V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_n}(i)) : n < \omega,\ i \geq 1\}$. Because of the above equality, $\mathfrak{M} \in \bigcap_{n<\omega} V_{\widetilde{\mathbb{Z}}}(\boldsymbol{b_n}(0))$, and hence by the previous claim, $\mathfrak{M}$ is $\mathfrak{M}_u$. □

For each $n < \omega$, one can compute the ideal class number $h(n)$ of $\mathcal{O}_{n_0+n}$ ([2], 6.5.9, [8], 6.5.1). For $b \in \widetilde{\mathbb{Z}}$ and $k \in \omega \setminus \{0\}$, we denote by $b^{1/k}$ the least root of $X^k - b$ (according to a fixed recursive well ordering of $\widetilde{\mathbb{Z}}$).

**Claim 3.19** $\mathfrak{M}_u = \bigcup_{n<\omega} (\boldsymbol{b_n}(0))^{1/h(n)}\widetilde{\mathbb{Z}}$.

**Proof** Let $n < \omega$ be fixed. Keeping the notation of (a.1), (a.2) in the proof of Claim 3.17, $h_{n,0}$ is the order of the class of $\mathfrak{M}_{n,0}$ and $(\mathfrak{M}_{n,0})^{h_{n,0}} = \boldsymbol{b_n}(0)\mathcal{O}_{n_0+n}$. Let $r \geq n_0 + n$ be such that $(\boldsymbol{b_n}(0))^{1/h_{n,0}} \in \mathcal{O}_r$. Then $\mathfrak{M}_{n,0}\mathcal{O}_r = (\boldsymbol{b_n}(0))^{1/h_{n,0}}\mathcal{O}_r$. This implies

$$\mathfrak{M}_u = \bigcup_{n<\omega}\mathfrak{M}_{n,0} = \bigcup_{n<\omega}\mathfrak{M}_{n,0} \cdot \widetilde{\mathbb{Z}} = \bigcup_{n<\omega}(\boldsymbol{b_n}(0))^{1/h_{n,0}} \cdot \widetilde{\mathbb{Z}}. \tag{2}$$

The sequence $\langle h_{n,0} : n < \omega\rangle$ has not been obtained in an effective manner, but the sequence $\langle h(n) : n < \omega\rangle$ of ideal class numbers works as well: for any $n < \omega$, $h_{n,0}$ divides $h(n)$. Hence in $\widetilde{\mathbb{Z}}$, $(\boldsymbol{b_n}(0))^{1/h_{n,0}}$ is associate to a power of $\mathbf{b}_n(0)^{1/h(n)}$. Combined with equality (2) and primeness of $\mathfrak{M}_u$, this gives $\mathfrak{M}_u \subseteq \bigcup_{n<\omega}(\boldsymbol{b_n}(0))^{1/h(n)}\widetilde{\mathbb{Z}} \subseteq \mathfrak{M}_u$. $\square$

Now to obtain Lemma 3.1, it suffices to set (with the notation of the lemma)

1. $\langle a_n : n < \omega \rangle := \langle (\boldsymbol{b_n}(0))^{1/h(n)} : n < \omega \rangle$,
2. $\langle b_n : n < \omega \rangle := \langle \prod_{1 \leq i < |\boldsymbol{b_n}|} \boldsymbol{b_n}(i) : n < \omega \rangle$.

For further use, let us note a consequence of Lemma 3.11: effective good factorization.

**Claim 3.20**

1. *Let $K = \mathbb{Q}(\alpha)$ be a number field and let $\mathcal{O}$ be its ring of integers. There is an effective procedure (uniform in $\alpha$) which applied to $b \in \mathcal{O} \setminus \{0\}$ and $c \in \mathcal{O}$ gives $d \in \mathcal{O}$ such that $V_{\mathcal{O}}(b) \setminus V_{\mathcal{O}}(c) = V_{\mathcal{O}}(d)$.*
2. *Therefore, there is an algorithm which, on inputs $(b, c) \in (\widetilde{\mathbb{Z}} \setminus \{0\}) \times \widetilde{\mathbb{Z}}$, produces $d \in \widetilde{\mathbb{Z}}$ such that $V_{\widetilde{\mathbb{Z}}}(b) \setminus V_{\widetilde{\mathbb{Z}}}(c) = V_{\widetilde{\mathbb{Z}}}(d)$.*

**Proof** Let $b \in \mathcal{O} \setminus \{0\}$ and $c \in \mathcal{O}$. We dismiss the easy cases:

1. $b$ is a unit or $c = 0$; we set $d = 1$.
2. $b$ is a nonunit and $c$ is a unit; we set $d = b$.

If both $b$ and $c$ are nonzero nonunits, then by Lemma 3.11, one can effectively obtain sequences $\mathbf{b} = \langle b_i : i < k_b\rangle$, $\mathbf{c} = \langle c_j : j < k_c\rangle$ such that $\mathbf{b} \approx \boldsymbol{a_{\mathcal{O}}}(b)$ and $\mathbf{c} \approx \boldsymbol{a_{\mathcal{O}}}(c)$. Setting $D := \{b_i : i < k_b \text{ and } \forall j < k_c\ (b_i, c_j) = (1)\}$, we deduce the equality $V_{\mathcal{O}}(b) \setminus V_{\mathcal{O}}(c) = V_{\mathcal{O}}(\prod D)$, and set $d := \prod D$ (by convention $\prod \varnothing := 1$). $\square$

**Positive characteristic** In positive characteristic, one can obtain analogs of the constructions in Examples 3.3 and 3.6 (we do not claim effectiveness, because we relied on results of [8] which require separability). Instead of considering $\mathbb{Z}$ and the prime numbers, one builds from the ring $F_p[t]$ and the monic irreducible polynomials of $F_p[t]$. The obtained models are localizations of $\widetilde{F_p[t]}$. Corresponding to Example 3.3, one has the following proposition.

**Proposition 3.21** *Let $p > 0$ be prime. One can define a localization $R$ of $\widetilde{F_p[t]}$ satisfying "$T^{\text{atomic}}_{\text{ring}} + \text{char} = p$" such that* $\text{Max}(R)$ *has the structure of* $\text{Max}(F_p[t])$.

**Remark 3.22** Let us note that there is also an equivalent of Example 3.6: a model where every monic irreducible polynomial of $F_p[t]$ belongs to infinitely many maximal ideals. We shall see later that the theory "$T^{\text{atomic}}_{\text{ring}} + \text{char} = p$" is complete. As opposed to the case of characteristic 0, the two examples are thus elementarily equivalent (but not isomorphic).

## 4 Model Completeness

We introduced in 2.10, the languages

1. $\mathcal{L}_{\text{ring}}^{\text{atomic}} := \mathcal{L}_{\text{ring}} \cup \{\,\underline{\text{rad}}\,, \text{size}_{=1}\}$ and
2. $\mathcal{L}' := \mathcal{L}_{\text{ring}} \cup \{S_{n,k,l} : n, k, l < \omega\}$.

The following proposition shows their relation to the model completeness of $T_{\text{ring}}^{\text{atomic}}$. We denote by Prime the set of rational prime numbers.

**Proposition 4.1** *Let* $p \in \text{Prime} \cup \{0\}$.

(a) *Relative to* $T_{\text{ring}}^{\text{atomic}} + \text{char} = p$, *each* $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ *formula is effectively equivalent to an existential* $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ *formula.*

(b) *With any* $\mathcal{L}'$ *formula* $\varphi(\mathbf{y})$, *one can associate effectively a disjunction of* $\mathcal{L}'$ *formulas* $\varphi_0(\mathbf{y}) \vee \cdots \vee \varphi_{r-1}(\mathbf{y})$ *such that*

 (i) $T_{\text{ring}}^{\text{atomic}} + \text{char} = p \vdash \varphi(\mathbf{y}) \longleftrightarrow \bigvee_{i<r} \varphi_i(\mathbf{y})$, *and*

 (ii) *each* $\mathcal{L}'$ *formula* $\varphi_i(\mathbf{y})$, *for* $i < r$, *is of the type*

$$\exists z \big( z^e + P_{e-1}(\mathbf{y}) z^{e-1} + \cdots + P_0(\mathbf{y}) = 0 \ \wedge \ \psi(\mathbf{y}, z) \big)$$

*where each* $P_j(\mathbf{y}) \in \mathbb{Z}[\mathbf{y}]$ *and* $\psi(\mathbf{y}, z)$ *is an* $\mathcal{L}'$ *quantifier-free formula.*

**Remark 4.2** We note that by (b), given any $\mathcal{L}_{\text{ring}}$ sentence $\sigma$, one can effectively obtain a finite set of algebraic integers $\{\alpha_i : i < k\}$ (set stable by automorphisms of $\widetilde{\mathbb{Z}}$) and an $\mathcal{L}'$ quantifier-free formula $\delta(\mathbf{x})$ such that in any model $R$ of $T_{\text{ring}}^{\text{atomic}}$ $+ \text{char} = 0$, $R \models \sigma \leftrightarrow \delta(\alpha_0, \ldots, \alpha_{k-1})$. Let $p$ prime $> 0$. One deduces a similar result for $T_{\text{ring}}^{\text{atomic}} + \text{char} = p$, with $\widetilde{F_p}$ instead of $\widetilde{\mathbb{Z}}$.

Replacing in the special existential formulas of [13] and [14], the "nonunit" predicate by the "size$_{=1}$" predicate, we consider "specific" existential $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formulas.

**Definition 4.3** A specific existential formula $\psi(\mathbf{y})$ is an $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formula of the following type: for $\mathbf{E}(\mathbf{x}, \mathbf{y}) \in {}^{s}\mathbb{Z}[\mathbf{x}, \mathbf{y}]$, $f(\mathbf{x}, \mathbf{y})$, $\alpha_i(\mathbf{x}, \mathbf{y})$, $\beta_i(\mathbf{x}, \mathbf{y})$, $\delta_j(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}[\mathbf{x}, \mathbf{y}]$, $i < I$, $j < J$, let

$$\begin{aligned} g(\mathbf{x}, \mathbf{y}) &:= f(\mathbf{x}, \mathbf{y}) \cdot \textstyle\prod_{i<I} \beta_i(\mathbf{x}, \mathbf{y}) \cdot \prod_{j<J} \delta_j(\mathbf{x}, \mathbf{y}), \\ \psi(\mathbf{y}) &:= \ \exists \mathbf{x} \left( \mathbf{E}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \ \wedge \ g(\mathbf{x}, \mathbf{y}) \neq 0 \ \wedge \textstyle\bigwedge_{i<I} \alpha_i(\mathbf{x}, \mathbf{y}) \, \underline{\text{rad}} \, \beta_i(\mathbf{x}, \mathbf{y}) \ \wedge \right. \\ & \qquad \left. \textstyle\bigwedge_{j<J} \text{size}_{=1}(\delta_j(\mathbf{x}, \mathbf{y})) \right). \end{aligned}$$

**Claim 4.4** *Relative to* $T_{\text{ring}}^{\text{atomic}}$, *every existential* $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ *formula is effectively equivalent to a disjunction of specific existential formulas.*

**Proof** We check that the negations of the predicates $\underline{\text{rad}}$ and $\text{size}_{=1}$ can be expressed by existential positive $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formulas. Let us first note

$$\begin{aligned} T_{\text{ring}}^{\text{atomic}} \vdash x \text{ nonunit } &\leftrightarrow \ \exists y \, (\text{size}_{=1}(y) \wedge x \, \underline{\text{rad}} \, y), \ \text{ and} \\ \neg(x \, \underline{\text{rad}} \, y) \ &\leftrightarrow \ \ \exists z \, \big( (z, x) = (1) \wedge (z, y) \neq (1) \big) \\ &\leftrightarrow \ \ \exists z, t \, \big( (z, x) = (1) \wedge t \text{ nonunit } \wedge (z, y) = (t) \big). \end{aligned}$$

Now in any model $R$ of $T_{\text{ring}}^{\text{atomic}}$, for $u \in R$, one has

$$V_R(u) \text{ is not an atom} \iff \begin{array}{l} V_R(u) = \varnothing \text{ or there is } v \in R \text{ such that} \\ (V_R(v) \text{ atom and } V_R(v) \subsetneq V_R(u)). \end{array}$$

Hence, $T_{\text{ring}}^{\text{atomic}} \vdash \neg(\text{size}_{=1}(x)) \leftrightarrow \exists y \left( xy = 1 \vee (\text{size}_{=1}(y) \wedge x \,\underline{\text{rad}}\, y \wedge \neg(y \,\underline{\text{rad}}\, x)) \right)$. Therefore, in $T_{\text{ring}}^{\text{atomic}}$, $\neg(\text{size}_{=1}(x))$ is (equivalent to) a formula of the right form.

Now in the definition of specific formulas, we require some terms to be $\neq 0$. Let us simply note that, by **Ru.6**,

$$\begin{aligned} x \,\underline{\text{rad}}\, y &\leftrightarrow (y = 0 \wedge x = 0) \vee (y \neq 0 \wedge x \,\underline{\text{rad}}\, y), \\ \text{size}_{=1}(x) &\leftrightarrow (x \neq 0 \wedge \text{size}_{=1}(x)). \end{aligned}$$

Combining all these elements, we deduce Claim 4.4. □

What makes possible the link between truth in the model of $T_{\text{ring}}^{\text{atomic}}$ and truth in its constructible Boolean algebra is the following.

**Lemma 4.5 ([14], 2.13)** *Let $R$ be a Bezout domain with algebraically closed fraction field. Let $\mathbf{b}$ be in $R$ and let $\psi(\mathbf{x})$ be an $\mathcal{L}_{\text{ring}}$ formula. Then the set $\{\mathfrak{M} \in \text{Max}(R) : R_{\mathfrak{M}} \models \psi(\mathbf{b})\}$ is constructible.*

**Notation 4.6** Let $R$, $\mathbf{b}$ and $\psi(\mathbf{x})$ be as in Lemma 4.5. Then one sets

$$\|\psi(\mathbf{b})\|_R := \{\mathfrak{M} \in \text{Max}(R) : R_{\mathfrak{M}} \models \psi(\mathbf{b})\}.$$

Because of Claim 4.4, our goal is to show that any specific existential $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formula is effectively equivalent to a universal $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formula.

Modulo an assumption about the irreducibility of the closed set defined by the equations $\mathbf{E} = \mathbf{0}$ in the specific formula $\varphi(\mathbf{y})$ (an assumption which will be lifted later by resorting to [13]'s splitting descriptions), the pattern of proof is as follows:

(1) to prove a "Feferman-Vaught transfer principle," obtaining $\mathcal{L}_{\text{ring}}$ formulas $\varphi_i(\mathbf{y})$, $i < k$, and an $\mathcal{L}_{\text{boole}}$ formula $\Phi(X_0, \ldots, X_{k-1})$ such that in any model $R$ of $T_{\text{ring}}^{\text{atomic}}$, for any $\mathbf{b}$ in $R$,

$$R \models \varphi(\mathbf{b}) \quad \text{iff} \quad B(R) \models \Phi(\|\varphi_0(\mathbf{b})\|_R, \ldots, \|\varphi_{k-1}(\mathbf{b})\|_R);$$

(2) by (effective) quantifier elimination in $T_{\text{boole}}^{\text{atomic}}$, to construct a quantifier-free $\mathcal{L}_{\text{boole}}^{\text{atomic}}$ formula $\Psi(X_0, \ldots, X_{k-1})$ equivalent to $\Phi(X_0, \ldots, X_{k-1})$ in $T_{\text{boole}}^{\text{atomic}}$;

(3) given $\Psi, \varphi_0, \ldots, \varphi_{k-1}$, to define a quantifier-free $\mathcal{L}'$ formula $\psi(\mathbf{y})$ such that in any model $R$ of $T_{\text{ring}}^{\text{atomic}}$, for any $\mathbf{b}$ in $R$,

$$B(R) \models \Psi(\|\varphi_0(\mathbf{b})\|_R, \ldots, \|\varphi_{k-1}(\mathbf{b})\|_R) \quad \text{iff} \quad R \models \psi(\mathbf{b});$$

(4) to check that any quantifier-free $\mathcal{L}'$ formula is equivalent to an existential (and hence also to a universal) $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formula.

All the steps will be effective.

For later use, let us set some notation.

**Notation 4.7** For an $\mathcal{L}_{\text{ring}}$ formula $\psi$, let $\psi^{(0)} := \neg\psi$, $\psi^{(1)} := \psi$.

Attributing values to the variables $\mathbf{y}$ in the specific formula, we are led to consider the following.

**Definition 4.8** Let $R$ satisfy $T_{\text{ring}}^{\text{atomic}}$. (a) We say that an $\mathcal{L}_{\text{ring}}^{\text{atomic}}(R)$ existential sentence $\exists \mathbf{x}\, \varphi(\mathbf{x})$ is suitable if $\varphi(\mathbf{x})$ is of the following form: for $W$ an absolutely irreducible closed set defined over $R$, for $f,, S_i, T_i, P_j \in R[\mathbf{X}]$, $i < m$, $j < n$, one has

$$\varphi^+(\mathbf{x}) := \mathbf{x} \in W \wedge \left(f(\mathbf{x}) \cdot \prod_{i<m} T_i(\mathbf{x}) \cdot \prod_{j<n} P_j(\mathbf{x})\right) \neq 0 \wedge \bigwedge_{i<m} S_i(\mathbf{x}) \,\underline{\mathrm{rad}}\, T_i(\mathbf{x}),$$
$$\varphi(\mathbf{x}) := \varphi^+(\mathbf{x}) \wedge \bigwedge_{j<n} \mathrm{size}_{=1}(P_j(\mathbf{x}));$$

(b) $\varphi$ being defined as above, for $\sigma \in^n 2$, we set (using Notation 4.7)

$$\varphi_\sigma(\mathbf{x}) := \varphi^+(\mathbf{x}) \wedge \bigwedge_{j<n} (P_j(\mathbf{x}) \text{ nonunit})^{(\sigma(j))}.$$

Considering atomic Boolean algebras, to improve legibility, we write "$u$ atom" for "$R_1(u) \wedge \neg R_2(u)$".

**Lemma 4.9** *Let $R$ be a model of $T_{\text{ring}}^{\text{atomic}}$, and let $\exists \mathbf{x}\, \varphi(\mathbf{x})$ be suitable. Then the following are equivalent:*

(a) $R \models \exists \mathbf{x} \varphi(\mathbf{x})$,
(b) $B(R) \models \;\exists \langle Y_\sigma : \sigma \in {}^n2\rangle$ *partition of* 1 *such that*
(1) $\bigwedge_{\sigma \in {}^n2} \left(Y_\sigma \subseteq \|\exists \mathbf{x} \varphi_\sigma(\mathbf{x})\|_R\right) \wedge$
(2) $\bigwedge_{j<n} \left((\sum_{\sigma(j)=1} Y_\sigma) \text{ atom}\right)$.

(In the definition of a partition, we do not require all elements to be $\neq 0$).

**Proof**

(a) $\Rightarrow$ (b) Let $R \models \varphi(\mathbf{a})$, for some $\mathbf{a}$ in $R$. By definition of $\varphi$,
(i) $R \models \varphi^+(\mathbf{a})$,
(ii) for each $j < n$, $V_R(P_j(\mathbf{a}))$ is an atom.

We set $Y_\sigma := \|\varphi_\sigma(\mathbf{a})\|_R$. One has $\vdash (\varphi^+(\mathbf{x}) \leftrightarrow \bigvee_{\sigma \in {}^n2} \varphi_\sigma(\mathbf{x}))$ and $\vdash \neg(\varphi_\sigma(\mathbf{x}) \wedge \varphi_\tau(\mathbf{x}))$, for $\sigma \neq \tau$. Also $R \models \varphi^+(\mathbf{a})$ implies $\|\varphi^+(\mathbf{a})\|_R = \mathrm{Max}(R)$. We deduce that $\langle Y_\sigma : \sigma \in {}^n2\rangle$ is a partition of $\mathrm{Max}(R)$. Obviously, for any $\sigma \in {}^n2$, $Y_\sigma \subseteq \|\exists \mathbf{x} \varphi_\sigma(\mathbf{x})\|_R$, and (1) holds. Also

$$\begin{aligned} V_R(P_j(\mathbf{a})) &= \|P_j(\mathbf{a}) \text{ nonunit}\|_R = \|\varphi^+(\mathbf{a}) \wedge P_j(\mathbf{a}) \text{ nonunit}\|_R \\ &= \|\bigvee_{\sigma(j)=1} \varphi_\sigma(\mathbf{a})\|_R = \sum_{\sigma(j)=1} Y_\sigma. \end{aligned}$$

and (2) holds.

(b) $\Rightarrow$ (a) Let $\langle Y_\sigma : \sigma \in {}^n2\rangle$ be a partition of $\mathrm{Max}(R)$ satisfying (1) and (2).

**Notation 4.10** By (2), for each $i < n$, there exists a unique $\sigma_i \in {}^n2$ such that $\sigma_i(i) = 1$ and $Y_{\sigma_i}$ is an atom. We set $\Sigma := \{\sigma_i : i < n\}$, and for each $\sigma \in {}^n2$, let $I_\sigma = \{i < n : \sigma(i) = 1\}$. For each $\sigma \in \Sigma$, since $Y_\sigma$ is an atom, let $u_\sigma \in R \setminus \{0\}$ be such that $V_R(u_\sigma) = Y_\sigma$.

If $\sigma \in \Sigma$, then $I_\sigma \neq \varnothing$. Also $\bigcup_{\sigma \in \Sigma} I_\sigma = n$. We also note (by (b)(2)) the following.

**Claim 4.11** *Let $\sigma \in \Sigma$. Then $|Y_\sigma| = 1$, and for any $\tau \neq \sigma$, either $I_\tau \cap I_\sigma = \varnothing$ or $Y_\tau = \varnothing$.*

We shall prove (a) by using the local-global argument of [14], Proposition 3.8. (The existence of a "solution" in each $R_{\mathfrak{M}}$, $\mathfrak{M} \in \mathrm{Max}(R)$ implies the existence of a "solution" in $R$.) A step toward realizing this program is the following.

**Claim 4.12** *For $\mathfrak{M} \in \mathrm{Max}(R)$, let $(*_{\mathfrak{M}})$ be the statement*

$$R_{\mathfrak{M}} \models \exists \mathbf{x}\left[\varphi^+(\mathbf{x}) \wedge \bigwedge_{\sigma \in \Sigma} \bigwedge_{j \in I_\sigma} \left(P_j(\mathbf{x}) \,\underline{\mathrm{rad}}\, u_\sigma \wedge u_\sigma \,\underline{\mathrm{rad}}\, P_j(\mathbf{x})\right)\right].$$

*Then $(*_{\mathfrak{M}})$ holds for every $\mathfrak{M} \in \mathrm{Max}(R)$.*

**Proof** Let $\mathfrak{M} \in \mathrm{Max}(R)$ be fixed. We check $(*_{\mathfrak{M}})$. Let us set $T := \{\sigma \in \Sigma : u_\sigma \notin \mathfrak{M}\}$, $I := \bigcup\{I_\sigma : \sigma \in T\}$, and $J := \bigcup\{I_\sigma : \sigma \in \Sigma \setminus T\}$ (by Claim 4.11, $J = n \setminus I$). If $\sigma \in T$, then $R_{\mathfrak{M}} \models (y \,\underline{\mathrm{rad}}\, u_\sigma) \wedge (u_\sigma \,\underline{\mathrm{rad}}\, y \longleftrightarrow y \text{ unit})$. Similarly, for $\sigma \in \Sigma \setminus T$, $R_{\mathfrak{M}} \models (y \,\underline{\mathrm{rad}}\, u_\sigma \longleftrightarrow y \text{ nonunit}) \wedge (u_\sigma \,\underline{\mathrm{rad}}\, y)$. Hence, in $R_{\mathfrak{M}}$ (adopting the convention that an empty conjunction always holds),

(i) $\bigwedge_{\sigma\in\Sigma}\bigwedge_{j\in I_\sigma} u_\sigma \,\underline{\mathrm{rad}}\, P_j(\mathbf{x}) \longleftrightarrow \bigwedge_{\sigma\in T}\bigwedge_{j\in I_\sigma} P_j(\mathbf{x}) \text{ unit}$
$\longleftrightarrow \bigwedge_{j\in I} P_j(\mathbf{x}) \text{ unit};$

(ii) $\bigwedge_{\sigma\in\Sigma}\bigwedge_{j\in I_\sigma} P_j(\mathbf{x}) \,\underline{\mathrm{rad}}\, u_\sigma \longleftrightarrow \bigwedge_{\sigma\in\Sigma\setminus T}\bigwedge_{j\in I_\sigma} P_j(\mathbf{x}) \text{ nonunit}$
$\longleftrightarrow \bigwedge_{j\in J} P_j(\mathbf{x}) \text{ nonunit}.$

Hence it suffices to prove $(**_{\mathfrak{M}})$:

$$R_{\mathfrak{M}} \models \exists\mathbf{x}\big[\,\varphi^+(\mathbf{x}) \wedge \textstyle\bigwedge_{j\in I} P_j(\mathbf{x}) \text{ unit} \wedge \bigwedge_{j\in J} P_j(\mathbf{x}) \text{ nonunit}\,\big].$$

Since $\langle Y_\sigma : \sigma \in {}^n2\rangle$ is a partition of $\mathrm{Max}(R)$, there is a unique $\bar\sigma \in {}^n2$ such that $\mathfrak{M} \in Y_{\bar\sigma}$.

**Subclaim 4.13** $\bar\sigma_{|I} \equiv 0$ and $\bar\sigma_{|J} \equiv 1$.

**Proof** We check $\bar\sigma_{|I} \equiv 0$. Let us suppose for a contradiction $\bar\sigma(j) = 1$ with $j \in I$. There must exist $\sigma \in T$ such that $j \in I_\sigma$. Then $j \in I_\sigma \cap I_{\bar\sigma}$. By Claim 4.11, necessarily $\sigma = \bar\sigma$. But $\mathfrak{M} \in Y_{\bar\sigma}$ and $\mathfrak{M} \notin V_R(u_\sigma) = Y_\sigma$. We reached a contradiction.

We prove $\bar\sigma_{|J} \equiv 1$. Let us assume $\bar\sigma(j) = 0$, for some $j \in I_\sigma$, with $\sigma \in \Sigma \setminus T$. Then $\mathfrak{M} \in V_R(u_\sigma) = Y_\sigma$ and $\mathfrak{M} \in Y_{\bar\sigma}$. Since the $Y_s$s define a partition, necessarily $\sigma = \bar\sigma$. But $\sigma(j) = 1$ and $\bar\sigma(j) = 0$. Again we obtained a contradiction. □

Now by definition, $\varphi_{\bar\sigma}(\mathbf{x}) := \varphi^+(\mathbf{x}) \wedge \bigwedge_{j<n}(P_j(\mathbf{x}) \text{ nonunit})^{(\bar\sigma(j))}$. Since $\bar\sigma_{|I} \equiv 0$, and $\bar\sigma_{|J} \equiv 1$, we deduce

$$\varphi_{\bar\sigma}(\mathbf{x}) \longleftrightarrow \big(\varphi^+(\mathbf{x}) \wedge \textstyle\bigwedge_{j\in I} P_j(\mathbf{x}) \text{ unit} \wedge \bigwedge_{j\in J}(P_j(\mathbf{x}) \text{ nonunit})\big).$$

Since we know $\mathfrak{M} \in Y_{\bar\sigma} \subseteq \|\exists\mathbf{x}\varphi_{\bar\sigma}(\mathbf{x})\|_R$, the equality

$$\|\exists\mathbf{x}\varphi_{\bar\sigma}(\mathbf{x})\|_R = \|\exists\mathbf{x}\big(\varphi^+(\mathbf{x}) \wedge \textstyle\bigwedge_{j\in I} P_j(\mathbf{x}) \text{ unit} \wedge \bigwedge_{j\in J}(P_j(\mathbf{x}) \text{ nonunit})\big)\|_R$$

gives

$$R_{\mathfrak{M}} \models \exists\mathbf{x}\big[\,\varphi^+(\mathbf{x}) \wedge \textstyle\bigwedge_{j\in I} P_j(\mathbf{x}) \text{ unit} \wedge \bigwedge_{j\in J} P_j(\mathbf{x}) \text{ nonunit}\,\big].$$

Therefore, $(**_{\mathfrak{M}})$ holds. Claim 4.12 follows. □

For each $\mathfrak{M} \in \mathrm{Max}(R)$, we thus have

$$R_{\mathfrak{M}} \models \exists\mathbf{x}\big[\mathbf{x} \in W \wedge (f(\mathbf{x})\cdot\textstyle\prod_{i<I} T_i(\mathbf{x})\cdot\prod_{j<n} P_j(\mathbf{x}) \neq 0) \wedge \bigwedge_{i<I} S_i(\mathbf{x}) \,\underline{\mathrm{rad}}\, T_i(\mathbf{x})$$
$$\wedge \textstyle\bigwedge_{\sigma\in\Sigma}\bigwedge_{j\in I_\sigma}(u_\sigma \,\underline{\mathrm{rad}}\, P_j(\mathbf{x}) \wedge P_j(\mathbf{x}) \,\underline{\mathrm{rad}}\, u_\sigma)\,\big].$$

Since the $u_\sigma$s are $\neq 0$, we can apply [14], 3.8, (**Ru.5** is not required in the hypotheses of [14], 3.8) and deduce that the right-hand side formula holds in $R$: let $\mathbf{a}$ in $R$ be such that

$$R \models \mathbf{a} \in W \wedge (f(\mathbf{a})\cdot\textstyle\prod_{i<I} T_i(\mathbf{a})\cdot\prod_{j<n} P_j(\mathbf{a}) \neq 0) \wedge \bigwedge_{i<I} S_i(\mathbf{a}) \,\underline{\mathrm{rad}}\, T_i(\mathbf{a})$$
$$\wedge \textstyle\bigwedge_{\sigma\in\Sigma}\bigwedge_{j\in I_\sigma}(u_\sigma \,\underline{\mathrm{rad}}\, P_j(\mathbf{a}) \wedge P_j(\mathbf{a}) \,\underline{\mathrm{rad}}\, u_\sigma).$$

We thus have $R \models \varphi^+(\mathbf{a})$ and for any $\sigma \in \Sigma,\ j \in I_\sigma,\ V_R(P_j(\mathbf{a})) = V_R(u_\sigma)$ which is an atom. Now since $n = \bigcup_{\sigma\in\Sigma} I_\sigma$, for any $j < n$, $R \models \mathrm{size}_{=1}(P_j(\mathbf{a}))$. This concludes the proof of Lemma 4.9. □

Let $\Phi(\langle X_\sigma : \sigma \in {}^n2\rangle)$ be the $\mathcal{L}_{\text{boole}}^{\text{atomic}}$ formula,

"$\exists\, \langle Y_\sigma : \sigma \in {}^n2\rangle$ partition of 1 s.t. $\big[\bigwedge_{\sigma\in{}^n2} Y_\sigma \subseteq X_\sigma \wedge \bigwedge_{j<n}\big(\sum_{\sigma(j)=1} Y_\sigma\big)$ atom $\big]$".

By (effective) quantifier elimination of $T_{\text{ring}}^{\text{atomic}}$ (Theorem 2.3) relative to the language $\mathcal{L}_{\text{boole}}^{\text{atomic}}$, one can construct a quantifier-free $\mathcal{L}_{\text{boole}}^{\text{atomic}}$ formula $\Psi(\langle X_\sigma : \sigma \in {}^n2\rangle)$ which is equivalent in $T_{\text{boole}}^{\text{atomic}}$ to the formula $\Phi(\langle X_\sigma : \sigma \in {}^n2\rangle)$. The next element is thus the following.

**Claim 4.14**

(a) *With any $\mathcal{L}_{\text{boole}}^{\text{atomic}}$ quantifier-free formula $\Psi(X_0, \dots, X_{s-1})$ and any $\mathcal{L}_{\text{ring}}$ formulas $\varphi_0(\mathbf{y}), \dots, \varphi_{s-1}(\mathbf{y})$, one can effectively associate an $\mathcal{L}'$ quantifier-free formula $\psi(\mathbf{y})$ such that in any model $R$ of $T_{\text{ring}}^{\text{atomic}}$, for any $\mathbf{b}$ in $R$,*

$(*)$ $\quad B(R) \models \Psi(\|\varphi_0(\mathbf{b})\|_R, \dots, \|\varphi_{s-1}(\mathbf{b})\|_R)$ *iff* $R \models \psi(\mathbf{b})$.

(b) *Every $\mathcal{L}'$ quantifier-free formula is (effectively) equivalent in $T_{\text{ring}}^{\text{atomic}}$ to an existential $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formula.*

**Proof** First, given any $\mathcal{L}_{\text{ring}}$ formulas $\varphi_i(\mathbf{y})$, for $i < k$, one can easily construct by induction on the length of an $\mathcal{L}_{\text{boole}}$ term $t(X_0, \dots, X_{k-1})$ an $\mathcal{L}_{\text{ring}}$ formula $\varphi^t(\mathbf{y})$ such that in any ring $R$, for any $\mathbf{b}$ in $R$,

$$t(\|\varphi_0(\mathbf{b})\|_R, \dots, \|\varphi_{k-1}(\mathbf{b})\|_R) = \|\varphi^t(\mathbf{b})\|_R.$$

We exhaust the different possibilities for the atomic formulas of $\mathcal{L}_{\text{boole}}^{\text{atomic}}$. Let $\Psi(\mathbf{X})$ be the formula $t(\mathbf{X}) = 0$, for some $\mathcal{L}_{\text{boole}}$ term $t$. We note that, in an atomic Boolean algebra, one has

$$t(\mathbf{X}) = 0 \leftrightarrow \neg R_1(t(\mathbf{X})).$$

Hence this case can be reduced to the following one: Let $\Psi(\mathbf{X})$ be the formula $R_n(t(\mathbf{X}))$, for $n \geq 1$.

To express $B(R) \models R_n(\|\varphi^t(\mathbf{b})\|_R)$, we return to van den Dries's argument in [13], 1.3 (replacing conjunctive by disjunctive normal form): by effective quantifier elimination in algebraically closed valuation rings, one can effectively obtain formulas $\langle \varphi_i : i < I\rangle$ such that $\|\varphi^t(\mathbf{b})\|_R = \|\bigvee_{i<I} \varphi_i(\mathbf{b})\|_R$, and each $\varphi_i(\mathbf{x})$ is of the form

$$\bigwedge_{j<k}(\alpha_j(\mathbf{x}) | \beta_j(\mathbf{x})) \wedge \bigwedge_{r<l}(\gamma_r(\mathbf{x}) \nmid \delta_r(\mathbf{x})),$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in {}^k\mathbb{Z}[\mathbf{x}], \boldsymbol{\gamma}, \boldsymbol{\delta} \in {}^l\mathbb{Z}[\mathbf{x}]$.

Moreover, by (effectively) increasing the number of disjuncts $\varphi_i$s and their lengths, we can assume $\vdash \neg(\varphi_i \wedge \varphi_j)$, for $i \neq j < I$. Hence $\|\varphi^t(\mathbf{b})\|_R$ is the disjoint union of the $\|\varphi_i(\mathbf{b})\|_R$s, for $i < I$. There are at least $n$ atoms in $\|\varphi^t(\mathbf{b})\|_R$ if and only if there is a sequence $\mathbf{m} = \langle m_i : i < I\rangle$ such that $\sum \mathbf{m} := \sum_{i<I} m_i = n$ and each $\|\varphi_i(\mathbf{b})\|_R$ contains at least $m_i$ atoms. Hence

$$B(R) \models R_n(\|\varphi^t(\mathbf{b})\|_R) \leftrightarrow \bigvee_{\sum \mathbf{m}=n} \bigwedge_{i<I} R_{m_i}(\|\varphi_i(\mathbf{b})\|_R).$$

Hence we have to prove that expressions of the following kind,

$$R_m(\|\bigwedge_{j<k}(\alpha_j(\mathbf{b}) | \beta_j(\mathbf{b})) \wedge \bigwedge_{r<l}(\gamma_r(\mathbf{b}) \nmid \delta_r(\mathbf{b}))\|_R),$$

can be formulated in an $\mathcal{L}'$ quantifier-free way.

But $\| \bigwedge_{j<k}(\alpha_j(\mathbf{b}) | \beta_j(\mathbf{b})) \wedge \bigwedge_{r<l}(\gamma_r(\mathbf{b}) \not| \delta_r(\mathbf{b}))\|_R$ is equal to the following:

$$\bigcap_{j<k}\|(\alpha_j(\mathbf{b})|\beta_j(\mathbf{b}))\|_R \cap \bigcap_{r<l}\|(\gamma_r(\mathbf{b}) \not| \delta_r(\mathbf{b}))\|_R,$$

$$\bigcap_{j<k} D_R(\alpha_j(\mathbf{b}):\beta_j(\mathbf{b})) \cap \bigcap_{r<l} V_R(\gamma_r(\mathbf{b}):\delta_r(\mathbf{b})),$$

$$D_R(\textstyle\prod_{j<k}(\alpha_j(\mathbf{b}):\beta_j(\mathbf{b}))) \cap V_R(\mathrm{gcd}_{r<l}(\gamma_r(\mathbf{b}):\delta_r(\mathbf{b}))).$$

So finally,

$$B(R) \models R_m(\|\textstyle\bigwedge_{j<k}(\alpha_j(\mathbf{b})|\beta_j(\mathbf{b})) \wedge \bigwedge_{r<l}(\gamma_j(\mathbf{b}) \not| \delta_r(\mathbf{b}))\|_R) \text{ iff } R \models S_{m,k,l}(\boldsymbol{\alpha}(\mathbf{b}), \boldsymbol{\beta}(\mathbf{b}), \boldsymbol{\gamma}(\mathbf{b}), \boldsymbol{\delta}(\mathbf{b})).$$

This concludes the proof of 4.14(a).

(b) To check that any quantifier-free $\mathcal{L}'$ formula is equivalent to an existential $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formula, it suffices to verify that both $S_{n,k,l}$ and its negation have existential $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ definitions with respect to $T_{\text{ring}}^{\text{atomic}}$. Let us first check that this holds for the predicates $S_n$, $n < \omega$ and for their negations. The definition of $S_n$ given in 2.7 is clearly $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ existential. Now in $T_{\text{ring}}^{\text{atomic}}$, for $n \geq 1$, one has the equivalence,

$$\neg S_n(x) \leftrightarrow \big[\, x \text{ unit } \vee \exists x_0, \ldots, x_{n-2}\big(\textstyle\bigwedge_{i<n-1}\mathrm{size}_{=1}(x_i) \wedge (\prod_{i<n-1} x_i)\, \underline{\mathrm{rad}}\, x\big)\,\big].$$

Hence $\neg S_n$ also admits an existential definition in $\mathcal{L}_{\text{ring}}^{\text{atomic}}$.

Let us deal now with the $S_{n,k,l}$s and their negations. By Definition 2.7, the above, and the fact that "$(x):(y)=(z)$" is $\mathcal{L}_{\text{ring}}$ existential in Bezout domains, one obtains an existential definition for $S_{n,k,l}$. To express $\neg S_{n,k,l}$, let us note that in any model $R$ of $T_{\text{ring}}^{\text{atomic}}$, for $u$ in $R \setminus \{0\}$, $D_R(u)$ contains infinitely many atoms. Hence, for $\mathbf{a}, \mathbf{b} \in {}^kR$, $\mathbf{c}, \mathbf{d} \in {}^lR$, $n \geq 1$, the following (i) and (ii) are equivalent:

(i) $B(R) \models \neg R_n\big(D_R(\prod_{i<k}(a_i:b_i)) \cap V_R(\mathrm{gcd}_{j<l}(c_j:d_j))\big)$,

(ii) $$\begin{cases} (\,\mathrm{gcd}_{j<l}(c_j:d_j) = \prod_{i<k}(a_i:b_i) = 0\,) \text{ or} \\ \begin{cases} \mathrm{gcd}_{j<l}(c_j:d_j) \neq 0 \text{ and } \exists e \in R\ V_R(e) = V_R(\mathrm{gcd}_{j<l}(c_j:d_j)) \setminus \\ V_R(\prod_{i<k}(a_i:b_i)) \text{ with } \neg R_n(V_R(e)) \text{ in } B(R). \end{cases} \end{cases}$$

Therefore, with respect to $T_{\text{ring}}^{\text{atomic}}$, $\neg S_{n,k,l}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})$, for $n \geq 1$, is equivalent to the formula,

$$\begin{aligned}\big[\mathrm{gcd}_{j<l}(z_j:t_j) = \textstyle\prod_{i<k}(x_i:y_i) = 0\big] \vee \big[\mathrm{gcd}_{j<l}(z_j:t_j) \neq 0 \wedge \\ \exists e\big(\,(e, \textstyle\prod_{i<k}(x_i:y_i)) = (1) \wedge \\ (e \cdot \textstyle\prod_{i<k}(x_i:y_i))\, \underline{\mathrm{rad}}\, (\prod_{i<k}(x_i:y_i) \cdot \mathrm{gcd}_{j<l}(z_j:t_j)) \wedge \\ (\textstyle\prod_{i<k}(x_i:y_i) \cdot \mathrm{gcd}_{j<l}(z_j:t_j))\, \underline{\mathrm{rad}}\, (e \cdot \prod_{i<k}(x_i:y_i)) \wedge \neg S_n(e)\big)\big].\end{aligned}$$

We thus deduce that, relative to $T_{\text{ring}}^{\text{atomic}}$, $\neg S_{n,k,l}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})$ also admits an existential $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ definition. This concludes the proof of Claim 4.14. □

Combining the previous results with "full splitting descriptions" of [13] and [14], we obtain the following claim.

**Claim 4.15** *Let $\varphi(\mathbf{y})$ be a specific existential $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formula with system of equations $\mathbf{E}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.*

(i) *Then one can effectively construct a quantifier-free* $\mathcal{L}'$ *formula* $\lambda(\mathbf{y})$ *satisfying the following equivalences: for any model* $R$ *of* $T_{\text{ring}}^{\text{atomic}}$*, any* $\mathbf{b}$ *such that the set* $\{\mathbf{x} \in \mathrm{Frac}(R)^N : \mathbf{E}(\mathbf{x}, \mathbf{b}) = \mathbf{0}\}$ *is absolutely irreducible,*

$$R \models \varphi(\mathbf{b}) \leftrightarrow \lambda(\mathbf{b}).$$

(ii) *Let* $p \in \mathrm{Prime} \cup \{0\}$. *Then one can obtain effectively a sequence of* $\mathcal{L}'$ *formulas* $\langle \lambda_p^i(\mathbf{y}) : i < I \rangle$ *of the form "*$\exists z\big(z^e + P_{e-1}(\mathbf{y})z^{e-1} + \cdots + P_0(\mathbf{y}) = 0 \wedge \psi(\mathbf{y}, z)\big)$*" with* $P_j(\mathbf{y}) \in \mathbb{Z}[\mathbf{y}]$ *and* $\psi(\mathbf{y}, z)$ *quantifier-free formula such that*

$$T_{\text{ring}}^{\text{atomic}} + \text{ char} = p \vdash \varphi(\mathbf{y}) \leftrightarrow \bigvee_{i<I} \lambda_p^i(\mathbf{y}).$$

(iii) *Let again* $p \in \mathrm{Prime} \cup \{0\}$. *Then one can effectively construct an* $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ *universal formula* $\mu_p(\mathbf{y})$ *such that*

$$T_{\text{ring}}^{\text{atomic}} + \text{ char} = p \vdash \varphi(\mathbf{y}) \leftrightarrow \mu_p(\mathbf{y}).$$

**Proof** (i) follows from 4.9, 2.3, and 4.14(a).

(ii) To split uniformly the algebraic set defined by the set of equations of $\varphi$ into irreducible components and apply (i), we appeal to van den Dries's arguments and refer to the proof of [13], 2.7(ii).

(iii) Similarly, we resort to the proof of [13], 2.7(i) and add the fact that any quantifier-free $\mathcal{L}'$ formula is equivalent to a universal $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formula. □

**Proof of Proposition 4.1** Let $p \in \mathrm{Prime} \cup \{0\}$. By Claim 4.4 and Claim 4.15(iii), every $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ existential formula is effectively equivalent in $T_{\text{ring}}^{\text{atomic}}$+ char $= p$ to a universal $\mathcal{L}_{\text{ring}}^{\text{atomic}}$ formula. This gives 4.1(a). From the model completeness of $T_{\text{ring}}^{\text{atomic}}$+ char $= p$ in $\mathcal{L}_{\text{ring}}^{\text{atomic}}$, Claim 4.4, and Claim 4.15(ii), we derive that any $\mathcal{L}'$ formula is equivalent in $T_{\text{ring}}^{\text{atomic}}$+ char $= p$ to a disjunction of formulas of the right form. 4.1(b) follows. □

## 5 Decidability of $T_{\text{ring}}^{\text{atomic}}$

Decidability of $T_{\text{ring}}^{\text{atomic}}$ can be deduced from the strong form of model completeness (Proposition 4.1(b)).

**Proposition 5.1** *Let* $p \in \mathrm{Prime}$. *The theory* $T_{\text{ring}}^{\text{atomic}}$+ char $= p$ *is complete.*

**Proof** All elements of $\widetilde{F_p}$ are 0 or units. Hence, if $A_0, A_1$ are two models of $T_{\text{ring}}^{\text{atomic}}$+ char $= p$, then (identifying $(\widetilde{F_p})^{A_0}$ and $(\widetilde{F_p})^{A_1}$), for $n, k, l \geq 1$, $\mathbf{a}, \mathbf{b}$ in ${}^k\widetilde{F_p}$, $\mathbf{c}, \mathbf{d}$ in ${}^l\widetilde{F_p}$, one easily checks

$$A_0 \models S_{n,k,l}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \ \Leftrightarrow\ A_1 \models S_{n,k,l}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}).$$

Hence, by Remark 4.2, $A_0$ and $A_1$ are elementarily equivalent. □

**Proposition 5.2** *The theory* $T_{\text{ring}}^{\text{atomic}}$+ char $= 0$ *is decidable.*

**Proof** We propose an algorithm which, applied to any $\mathcal{L}_{\text{ring}}$ sentence $\sigma$, decides whether there exists a model of $T_{\text{ring}}^{\text{atomic}}$+ (char $= 0$) + $\sigma$. This implies that the set $\{\tau : (T_{\text{ring}}^{\text{atomic}} + \text{char} = 0) \not\vdash \tau\}$ is recursive and hence that the theory $T_{\text{ring}}^{\text{atomic}}$ + char $= 0$ is decidable.

So let $\sigma$ be a fixed $\mathcal{L}_{\text{ring}}$ sentence. By Remark 4.2, one can obtain effectively a quantifier-free $\mathcal{L}'$ formula $\varphi(\mathbf{x})$ under disjunctive normal form and algebraic integers $\alpha_i$, $i < n$, such that in any model $R \supseteq \widetilde{\mathbb{Z}}$ of $T_{\text{ring}}^{\text{atomic}}$, one has $R \models \sigma \leftrightarrow \varphi(\boldsymbol{\alpha})$. It thus suffices to decide for each conjunction $C(\boldsymbol{\alpha})$ of $\varphi(\boldsymbol{\alpha})$ whether it holds in some model $R \supseteq \widetilde{\mathbb{Z}}$ of $T_{\text{ring}}^{\text{atomic}}$. We deal with a given conjunction $C(\boldsymbol{\alpha})$ as follows.

**Step (0)** Since $T_{\text{ring}}^{\text{atomic}} \vdash x = 0 \leftrightarrow \neg S_{1,1,1}(\langle x\rangle, \langle 1\rangle, \langle 0\rangle, \langle 1\rangle)$, we can assume all our conjuncts or their negation are of the form $S_{n,k,l}(\mathbf{P}(\boldsymbol{\alpha}), \mathbf{Q}(\boldsymbol{\alpha}), \mathbf{S}(\boldsymbol{\alpha}), \mathbf{T}(\boldsymbol{\alpha}))$, for $\mathbf{P}, \mathbf{Q} \in {}^{k}\mathbb{Z}[\mathbf{X}]$, $\mathbf{S}, \mathbf{T} \in {}^{l}\mathbb{Z}[\mathbf{X}]$, with $n, k, l < \omega$.

**Step (1)** One computes in $\widetilde{\mathbb{Z}}$ all polynomials in $\boldsymbol{\alpha}$ and the expressions $\prod_{i<k} \big(P_i(\boldsymbol{\alpha}) : Q_i(\boldsymbol{\alpha})\big)$ and $\gcd\big((S_0(\boldsymbol{\alpha}) : T_0(\boldsymbol{\alpha})), \ldots, (S_{l-1}(\boldsymbol{\alpha}) : T_{l-1}(\boldsymbol{\alpha}))\big)$ (which occur in the predicates $S_{n,k,l}$, for $n, k, l < \omega$). These computations are valid in any $R \supseteq \widetilde{\mathbb{Z}}$. Since $S_{n,k,l}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \longleftrightarrow S_{n,1,1}\big(\langle \prod_{i<k}(a_i : b_i)\rangle, \langle 1\rangle, \langle \gcd_{j<l}(c_j : d_j)\rangle, \langle 1\rangle\big)$, we have computed a new sequence $\boldsymbol{\beta}$ of algebraic integers and a conjunction $C'(\boldsymbol{\beta})$ equivalent in any $R \supseteq \widetilde{\mathbb{Z}}$ to $C(\boldsymbol{\alpha})$, whose conjuncts or negation are of the form $S_{n,1,1}(\langle \beta_i\rangle, \langle 1\rangle, \langle \beta_j\rangle, \langle 1\rangle)$.

**Step (2)** To decide whether a conjunct $S_{n,1,1}(\langle \beta_i\rangle, \langle 1\rangle, \langle \beta_j\rangle, \langle 1\rangle)$ of $C'(\boldsymbol{\beta})$ holds in a model $R$ of $T_{\text{ring}}^{\text{atomic}}$, we have to decide whether $R_n(V_R(\beta_j) \cap D_R(\beta_i))$ holds in the associated constructible algebra. The only problematic case is when $\beta_j \neq 0$ (for $n \geq 1$, $R_n(D_R(\beta_i)) \Leftrightarrow \beta_i \neq 0$). By 3.20, one can compute $\gamma \in \widetilde{\mathbb{Z}}$ such that $V_{\widetilde{\mathbb{Z}}}(\gamma) = V_{\widetilde{\mathbb{Z}}}(\beta_j) \setminus V_{\widetilde{\mathbb{Z}}}(\beta_i)$. By 3.9, this implies that for any $R \supseteq \widetilde{\mathbb{Z}}$, $V_R(\gamma) = V_R(\beta_j) \setminus V_R(\beta_i)$. Hence, if $R \supseteq \widetilde{\mathbb{Z}}$, $R \models S_{n,1,1}(\langle \beta_i\rangle, \langle 1\rangle, \langle \beta_j\rangle, \langle 1\rangle) \leftrightarrow S_n(\gamma)$. We have thus effectively obtained a sequence $\boldsymbol{\gamma}$ of algebraic integers and a conjunction $C''(\boldsymbol{\gamma})$ equivalent to $C'(\boldsymbol{\beta})$ in any model $R \supseteq \widetilde{\mathbb{Z}}$ of $T_{\text{ring}}^{\text{atomic}}$ such that all conjuncts of $C''(\boldsymbol{\gamma})$ or their negation are of the form $S_n(\gamma_i)$, for some $n < \omega$.

**Step (3)** In order to apply the consistency result 3.7, we need to turn the sequence $\boldsymbol{\gamma}$ of parameters into a sequence $\mathbf{t}$ of nonzero nonunits which are pairwise relatively prime in $\widetilde{\mathbb{Z}}$. Doing this, we shall get (effectively) a disjunction of conjunctions such that in any model $R \supseteq \widetilde{\mathbb{Z}}$ of $T_{\text{ring}}^{\text{atomic}}$, $R \models C''(\boldsymbol{\gamma}) \leftrightarrow \bigvee_{s<S} C_s(\mathbf{t})$. Lemma 3.7 will allow us to conclude for each $C_s(\mathbf{t})$. Hence let $g = |\boldsymbol{\gamma}|$. We can assume all $\gamma_i$s, for $i < g$, are $\neq 0$ ($S_n(0)$ always holds in a model $R$ of $T_{\text{ring}}^{\text{atomic}}$).

Exactly as in the proof of 3.8, for each nonempty $I \subseteq g$, one obtains effectively $\gamma_I \in \widetilde{\mathbb{Z}}$ such that $V_{\widetilde{\mathbb{Z}}}(\gamma_I) := \big(\bigcap_{i\in I} V_{\widetilde{\mathbb{Z}}}(\gamma_i)\big) \cap \big(\bigcap_{i\notin I} (V_{\widetilde{\mathbb{Z}}}(\gamma_i))^c\big)$. The $\gamma_I$s are nonzero and pairwise relatively prime in $\widetilde{\mathbb{Z}}$. By the same arguments as in 3.8, for $R \supseteq \widetilde{\mathbb{Z}}$, one has

$$\begin{cases} V_R(\gamma_I) = \big(\bigcap_{i\in I} V_R(\gamma_i)\big) \cap \big(\bigcap_{i\notin I} (V_R(\gamma_i))^c\big), \\ V_R(\gamma_i) = \mathring{\bigcup}_{i\in I} V_R(\gamma_I). \end{cases}$$

From the last equality, for $m < \omega$, $i < g$, one deduces the equivalence

$$V_R(\gamma_i) \text{ contains at least } m \text{ atoms} \quad \text{iff} \quad \begin{cases} \text{there is a sequence } \langle m_I : I \ni i\rangle \text{ such} \\ \text{that } \sum_{I\ni i} m_I = m \text{ and each } V_R(\gamma_I), \\ \text{with } i \in I, \text{ contains at least } m_I \text{ atoms.} \end{cases}$$

Hence, for $i < g$, $R \models S_m(\gamma_i) \longleftrightarrow \bigvee_{(\sum_{i\in I} m_I)=m} \bigwedge_{I\ni i} S_{m_I}(\gamma_I)$.

After some elementary handling, we obtain effectively

$$C''(\boldsymbol{\gamma}) \longleftrightarrow \bigvee_{s<S} C_s(\langle \gamma_I : \varnothing \neq I \subseteq g\rangle),$$

with each $C_s(\langle \gamma_I : \varnothing \neq I \subseteq g\rangle)$ of the form

$$\bigwedge\nolimits_{I\in P_0} S_{m_I}(\gamma_I) \wedge \bigwedge\nolimits_{I\in P_1} \neg S_{n_I}(\gamma_I).$$

To take care of each $C_s, s < S$, we first get rid of the $\gamma_I$s which are units in $\widetilde{\mathbb{Z}}$ ($T_{\text{ring}}^{\text{atomic}} \vdash x$ unit $\leftrightarrow \neg S_1(x)$).

Hence finally, if $\mathbf{t}$ is an enumeration of the set $\{\gamma_I$ nonunit in $\widetilde{\mathbb{Z}} : \varnothing \neq I \subseteq g\}$, then we are left with a conjunction $\bigwedge_{i\in J} S_{\mu_i}(t_i) \wedge \bigwedge_{i\in J'} \neg S_{\nu_i}(t_i)$ which by Lemma 3.7 (all $t_i$s are nonzero nonunits which are pairwise relatively prime in $\widetilde{\mathbb{Z}}$) is realized in a model $R$ of $T_{\text{ring}}^{\text{atomic}}$ if and only if for each $i \in J \cap J'$, $\mu_i < \nu_i$. Therefore, having decided for all the conjunctions $C_s(\langle \gamma_I : \varnothing \neq I \subseteq g\rangle)$, $s < S$, we have decided for their disjunction and hence for our initial conjunction $C(\boldsymbol{\alpha})$. □

From the two previous propositions, one infers the following.

**Proposition 5.3** *$T_{\text{ring}}^{\text{atomic}}$ is a decidable theory.*

In order to get axiomatizations of complete extensions of $T_{\text{ring}}^{\text{atomic}} + \text{char} = 0$, we shall apply the following.

**Claim 5.4** *Let $A_0, A_1 \supseteq \widetilde{\mathbb{Z}}$ be two models of $T_{\text{ring}}^{\text{atomic}}$ such that for any $n < \omega$, any $a \in \widetilde{\mathbb{Z}}$, $A_0 \models S_n(a) \Leftrightarrow A_1 \models S_n(a)$. Then $A_0$ and $A_1$ are elementarily equivalent.*

**Proof** Let $\sigma$ be an $\mathcal{L}_{\text{ring}}$ sentence. By arguments in Steps (0)–(2) of the proof of Proposition 5.2, one constructs a quantifier-free formula $\varphi(\mathbf{x})$ in $\mathcal{L}_{\text{ring}} \cup \{S_n : n < \omega\}$ and a sequence $\boldsymbol{\beta}$ of algebraic integers such that for any model $R \supseteq \widetilde{\mathbb{Z}}$ of $T_{\text{ring}}^{\text{atomic}}$, $R \models \sigma \leftrightarrow \varphi(\boldsymbol{\beta})$. Hence necessarily, if $A_0, A_1$ satisfy the hypotheses of the claim, $A_0 \models \sigma \Leftrightarrow A_1 \models \sigma$. □

We can now prove results announced in Section 3. Concerning Example 3.3, let us recall that, for each prime $p$, we had the (effective) decomposition,

$$V_{\widetilde{\mathbb{Z}}}(p) = \{\mathfrak{M}_p\} \mathbin{\mathring{\cup}} \mathring{\bigcup}_{n<\omega} V_{\widetilde{\mathbb{Z}}}(b_{p,n}).$$

The recursive model $R$ was obtained by turning all $b_{p,n}$s into units and hence getting all $V_R(p)$'s atoms.

**Proposition 3.5** *The theory of $R$ is recursively axiomatized as the theory $T$:*

$$\left(\begin{array}{l} T_{\text{ring}}^{\text{atomic}} + \textit{char} = 0 + \\ \textit{the quantifier-free diagram of } \widetilde{\mathbb{Z}} + \\ \textit{for each prime } p, \textit{ the axiom } \text{“size}_{=1}(p)\text{” } \textit{(formally its } \mathcal{L}_{\text{ring}} \textit{ equivalent)} + \\ \textit{for each prime } p, \textit{ each } n < \omega, \textit{ the axiom “} b_{p,n} \textit{ unit”}. \end{array}\right.$$

**Proof** Example 3.3 is clearly a model of $T$. To show completeness of $T$, we check the following: for any model $R$ of $T$, $\upsilon \in \widetilde{\mathbb{Z}}$, and $n < \omega$, one has the equivalence

$$R \models S_n(\upsilon) \quad \text{iff} \quad \begin{array}{l}\text{there exist distinct primes } p_0, \ldots, p_{n-1} \\ \text{such that for all } i < n,\ \upsilon \in \mathfrak{M}_{p_i}.\end{array}$$

Let $p \in$ Prime be fixed in this paragraph. Since $V_{\widetilde{\mathbb{Z}}}(p) = \{\mathfrak{M}_p\} \mathbin{\mathring{\cup}} \mathring{\bigcup}_{n<\omega} V_{\widetilde{\mathbb{Z}}}(b_{p,n})$ and since all $b_{p,n}$s are units in $R$, necessarily $V_R(p) \subseteq \{\mathfrak{M} \in \text{Max}(R) : \mathfrak{M}_p \subseteq \mathfrak{M}\}$. But

$V_R(p)$ is an atom since $R \models T$; hence there must exist a unique maximal ideal $\mathfrak{M}_p^R$ such that $V_R(p) = \{\mathfrak{M}_p^R\}$ and $\mathfrak{M}_p \subseteq \mathfrak{M}_p^R$.

Let now $\upsilon \in \widetilde{\mathbb{Z}}$. Since $V_R(\upsilon) = \bigcup_{p \in \text{Prime}} (V_R(\upsilon) \cap V_R(p))$ and since each intersection $V_R(\upsilon) \cap V_R(p)$ has size at most 1, for each $p \in$ Prime, we deduce the equivalences,

$$\begin{aligned} R \models S_n(\upsilon) \quad &\Leftrightarrow \quad |V_R(\upsilon)| \geq n \qquad \text{(Fact 2.9)} \\ &\Leftrightarrow \quad \text{there are distinct primes } p_0, \ldots, p_{n-1} \\ &\qquad \text{such that, for } i < n,\ |V_R(\upsilon) \cap V_R(p_i)| = 1 \\ &\Leftrightarrow \quad \text{there are distinct primes } p_0, \ldots, p_{n-1} \\ &\qquad \text{such that, for } i < n,\ V_R(\upsilon) \cap V_R(p_i) = \{\mathfrak{M}_{p_i}^R\} \\ &\Leftrightarrow \quad \text{there are distinct primes } p_0, \ldots, p_{n-1} \\ &\qquad \text{such that, for } i < n,\ \upsilon \in \mathfrak{M}_{p_i}. \end{aligned}$$

Let now $A_0, A_1$ be two models of $T$. For any $n < \omega$, $\upsilon \in \widetilde{\mathbb{Z}}$, we have

$$A_0 \models S_n(\upsilon) \ \Leftrightarrow \left( \begin{array}{c} \text{there are distinct primes } p_0, \ldots, p_{n-1} \\ \text{such that, for } i < n,\ \upsilon \in \mathfrak{M}_{p_i} \end{array} \right) \Leftrightarrow \ A_1 \models S_n(\upsilon).$$

Hence, by Claim 5.4, $A_0$ and $A_1$ are elementarily equivalent. $T$ is complete. □

In symmetry with the atomless case, one has the following.

**Proposition 5.5** *Up to elementary equivalence, there is a unique model $R$ of $T_{\text{ring}}^{\text{atomic}}$ + char = 0 whose algebraic part (i.e., $R \cap \widetilde{\mathbb{Q}}$) is $\widetilde{\mathbb{Z}}$ (respectively, $\widetilde{\mathbb{Q}}$).*

**Proof** We showed existence in Section 2; we check unicity. Let us assume first that $R$ is a model of $T_{\text{ring}}^{\text{atomic}}$ such that $R \cap \widetilde{\mathbb{Q}} = \widetilde{\mathbb{Z}}$. We verify that for any $\upsilon \in \widetilde{\mathbb{Z}}$ and any $n \geq 1$, $R \models S_n(\upsilon) \ \Leftrightarrow \ \upsilon$ nonunit in $\widetilde{\mathbb{Z}}$.

We show the implication from right to left. Let $\upsilon$ be a nonzero nonunit of $\widetilde{\mathbb{Z}}$. Then by **Ru.5** in $\widetilde{\mathbb{Z}}$, $V_{\widetilde{\mathbb{Z}}}(\upsilon)$ is infinite. Now every maximal ideal $\mathfrak{M}$ of $\widetilde{\mathbb{Z}}$ generates a proper ideal $\mathfrak{M}R$ of $R$ (using gcd and the fact that nonunits of $\widetilde{\mathbb{Z}}$ remain nonunits in $R$) which is included in a maximal ideal of $R$. Hence $V_R(\upsilon)$ is also infinite. By Fact 2.9, $R \models S_n(\upsilon)$, for any $n < \omega$.

We can now conclude from Claim 5.4 that, up to elementary equivalence, there is a unique model of $T_{\text{ring}}^{\text{atomic}}$ with algebraic part equal to $\widetilde{\mathbb{Z}}$. Concerning $\widetilde{\mathbb{Q}}$, either we argue as for Proposition 5.1, or we note that for $R$ model of $T_{\text{ring}}^{\text{atomic}}$ such that $R \cap \widetilde{\mathbb{Q}} = \widetilde{\mathbb{Q}}$, $\upsilon \in \widetilde{\mathbb{Z}}$ and $n \geq 1$ ($R \models S_n(\upsilon) \ \Leftrightarrow \ \upsilon = 0$). □

It is possible to develop this study in a more general setting including both the atomless (good Rumely domains) and the atomic cases: the situation where, in the associated constructible algebra, the set of atoms admits a sup. The gradual move from the atomless to the atomic case is expressed by the growing "size" of this sup: empty, finite, nowhere dense (closed, infinite), dense (open, nonempty), and finally the whole space.

Tarski (see [6], p. 74) proved that the theory of Boolean algebras such that the set of atoms has a sup, admits quantifier elimination in an appropriate language. It is then possible to convert this theorem into a result of model completeness for the corresponding theory of rings (in an adequate language), to construct models, and to show decidability. We chose to propose the atomic case because proofs in the general situation, though more complicated, follow the same pattern (yet construction of models is interesting).

## Note

1. The theorem is proposed as an exercise in [6], p. 73; a proof of the effective version is available at http://www.logique.jussieu.fr/~sureson.

## References


[1] Chang, C. C., and H. J. Keisler, *Model Theory. Studies in Logic and the Foundations of Mathematics*, vol. 73, North-Holland Publishing Co., Amsterdam, 1973. Zbl 0276.02032. MR 0409165. 402

[2] Cohen, H., *A Course in Computational Algebraic Number Theory*, vol. 138 of *Graduate Texts in Mathematics*, Springer-Verlag, Berlin, 1993. Zbl 0786.11071. MR 1228206. 408, 410

[3] Darnière, L., "Pseudo-algebraically closed rings," *Manuscripta Mathematica*, vol. 105 (2001), pp. 13–46. Zbl 1018.12008. MR 1885812. 401

[4] Ershov, Y. L., "Decidability of the elementary theory of relatively complemented distributive lattices and the theory of filters," *Algebra i Logika*, vol. 3 (1964), pp. 17–38. Zbl 0199.03103. 402

[5] Ershov, Y. L., "Near regularly-Prüfer rings," *Siberian Advances in Mathematics*, vol. 9 (1999), pp. 1–45. Zbl 0937.12005. MR 1695649. 401

[6] Hodges, W., *Model Theory*, vol. 42 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1993. Zbl 0789.03031. MR 1221741. 402, 421, 422

[7] Neukirch, J., *Algebraic Number Theory*, vol. 322 of *Fundamental Principles of Mathematical Sciences*, Springer-Verlag, Berlin, 1999. Translated from the 1992 German original by Norbert Schappacher. Zbl 0956.11021. MR 1697859. 408

[8] Pohst, M., and H. Zassenhaus, *Algorithmic Algebraic Number Theory*, vol. 30 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1989. Zbl 0685.12001. MR 1033013. 408, 409, 410, 411

[9] Prestel, A., and J. Schmid, "Existentially closed domains with radical relations: An axiomatization of the ring of algebraic integers," *Journal für die Reine und Angewandte Mathematik*, vol. 407 (1990), pp. 178–201. Zbl 0691.12013. MR 1048534. 399

[10] Rumely, R. S., "Arithmetic over the ring of all algebraic integers," *Journal für die Reine und Angewandte Mathematik*, vol. 368 (1986), pp. 127–33. Zbl 0581.14014. MR 850618. 399, 404, 408

[11] Tarski, A., "Grundzüge des Systementkalküls. I/II," *Fundamenta Mathematicae*, vol. 25/26 (1935/1936), pp. 503–26/283–301. Zbl 0012.38501. 401, 402

[12] Tarski, A., "Arithmetical classes and types of Boolean algebras," *Bulletin of the American Mathematical Society*, vol. 55 (1949), 1192, p. 64. 401, 402

[13] van den Dries, L., "Elimination theory for the ring of algebraic integers," *Journal für die Reine und Angewandte Mathematik*, vol. 388 (1988), pp. 189–205. Zbl 0659.12021. MR 944190. 401, 412, 413, 416, 417, 418

[14] van den Dries, L., and A. Macintyre, "The logic of Rumely's local-global principle," *Journal für die Reine und Angewandte Mathematik*, vol. 407 (1990), pp. 33–56. Zbl 0703.13021. MR 1048527. 399, 400, 401, 402, 405, 408, 412, 413, 414, 415, 417

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