Bounds on Weak Scattering

Gerald E. Sacks

In Memory of Jon Barwise

Abstract   The notion of a weakly scattered theory T is defined. T need not be scattered. For each $\mathcal{A}$ a model of T, let $\mathrm{sr}(\mathcal{A})$ be the Scott rank of $\mathcal{A}$. Assume $\mathrm{sr}(\mathcal{A}) \leq \omega_1^{\mathcal{A}}$ for all $\mathcal{A}$ a model of T. Let σ_2^T be the least Σ_2 admissible ordinal relative to T. If T admits effective k-splitting as defined in this paper, then $\exists \theta < \sigma_2^T$ such that $\mathrm{sr}(\mathcal{A}) < \theta$ for all $\mathcal{A}$ a model of T.

1 Introduction

This paper has two themes less disparate than they seem at first reading:

1. extending classical descriptive set theoretic results that impose bounds on suitably defined functions from ω^ω into ω_1;
2. extending and clarifying some early results on Scott ranks of countable structures sketched in [15].[1]

Let F be a function, possibly partial, from ω^ω into ω_1. A typical *classical bounding theorem* says the range of F is bounded by a countable ordinal if the graph of F has a suitable definition. For example, the graph of F is $\mathbf{\Sigma}_1^1$ with real parameter p; in this formulation the graph of F is viewed as a subset of $\omega^\omega \times \omega_1$ by requiring each value of F to be a well ordering of ω. Let $F(X)$ ambiguously denote the well ordering and also the ordinal represented by the well ordering. For each X, $F(X)$ is the unique solution of a Σ_1^1 formula with parameters p, X. Consequently $F(X)$ (the well ordering) is hyperarithmetic in p, X, and so

$$F(X) < \omega_1^{p,X}, \tag{1.1}$$

the least ordinal not recursive in p, X. The effective version of the theorem says that the bound on the range of F is an ordinal less than ω_1^p.

A recursion-theoretic approach to the effective bound originated by Kleene is as follows. (See Sacks [16] for details.) Suppose

$$(\forall \gamma < \omega_1^p)(\exists X)[F(X) > \gamma]. \tag{1.2}$$

Let R_e^p be the linear ordering of ω recursive in p via index e. Define W^p to be the set of all e such that R_e^p is a well ordering. Then

$$e \in W^p \longleftrightarrow \exists X \exists g[g \text{ is a 1-1 order-preserving map of } R_e^p \text{ into } F(X)]. \tag{1.3}$$

But then W^p is $\mathbf{\Sigma}_1^1$ with parameter p. This last is false according to a Kleene hierarchy result that says W^p is universal $\mathbf{\Pi}_1^1$ with parameter p, hence not $\mathbf{\Sigma}_1^1$ with parameter p.

A model theoretic approach to effective bounds is the path taken in this paper. A sketch may help to clarify later sections. Let $A(p)$ be the least Σ_1 admissible set with p as a member. Let Z be a $\Sigma_1^{A(p)}$ definable set of sentences of $\mathcal{L}_{\omega_1,\omega}$ coded by elements of $A(p)$ such that every model M of Z has the following properties:

1. The ordinals recursive in p form a proper initial segment of the ordinals in the sense of M.
2. There is an $X_0 \in M$ such that for all $\gamma < \omega_1^p$, $F(X_0) > \gamma$.
3. $p \in M$ and M is a Σ_1 admissible structure.

Assume the range of F is not bounded by an ordinal below ω_1^p. Then each $A(p)$-finite subset of Z (i.e., each subset of Z that is a member of $A(p)$) is consistent, and so Z has a model by Barwise Compactness. With the addition of "effective" type omitting, as in Grilliot [5] or Keisler [7], Z has a model M that omits ω_1^p but has nonstandard ordinals greater than all standard ordinals less than ω_1^p. Then

$$\omega_1^{p,X_0} \leq \omega_1^p; \tag{1.4}$$

otherwise, ω_1^p is recursive in $\langle p, X_0 \rangle$ and so $\omega_1^p \in M$. But then $\omega_1^{p,X_0} = \omega_1^p$ and $F(X_0) \geq \omega_1^{p,X_0}$ by property (2) of Z, which contradicts (1.1).

The search for a bounding theorem that extends the classical result seems hopeless at first. An extension has to talk about an F that allows $F(X) \geq \omega_1^{X,p}$, but $\omega_1^{X,p}$, as a function of X, is unbounded. Model theory comes to the rescue. Every countable structure $\mathcal{A}$ has a Scott rank [17], $\mathrm{sr}(\mathcal{A})$, an ordinal that can be as high as $\omega_1^{\mathcal{A}} + 1$ (see Section 2 for elaboration).

Let T be a countable theory. A reasonable starting assumption on T is

$$\forall \mathcal{A}[\mathcal{A} \models T \longrightarrow \mathrm{sr}(\mathcal{A}) \leq \omega_1^{\mathcal{A}}]. \tag{1.5}$$

An ingenious example (MA) devised by Makkai [11] shows that (1.5) is not enough. Examination of (MA) and its illuminative extensions in Knight and Young [8] leads to two further assumptions on T. The first, *effective k-splitting*, is technical and perhaps peripheral and is discussed further in Sections 9 and 10. The second, *weakly scattered*, is central. The theory T_M associated with (MA) satisfies (1.5) and has properties similar to effective k-splitting. In addition for every Σ_1 admissible countable α, T_M has a model $\mathcal{A}$ such that

$$\omega_1^{\mathcal{A}} = \alpha = \mathrm{sr}(\mathcal{A}). \tag{1.6}$$

Corollary 9.5 says if T is weakly scattered, satisfies (1.5), and has effective k-splitting, then there is a countable bound on the Scott ranks of the countable models of T; the effective version provides a bound less than the first Σ_2 admissible ordinal relative to T in contrast to the classical case above, where the effective bound on the range of F is less than ω_1^p, the first Σ_1 admissible ordinal relative to p.

The notion of weakly scattered is inspired by Morley's concept of scattered. Let $\mathcal{L}$ be a countable first-order language, $\mathcal{L}_0$ a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, and $T \subseteq \mathcal{L}_0$ a theory (i.e., a set of sentences) with a model. For (a) and (b) below, let $\mathcal{L}'$ be any countable fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_0$, and T' any finitarily consistent, ω-complete theory contained in $\mathcal{L}'$ and extending T. (The notions of finitary consistency and ω-completeness for fragments are reviewed at the beginning of Section 4.) T is said to be *scattered* if and only if (a) and (b) hold.

(a) For all $n > 0$ and all T', $S_n T'$, the set of all n-types over T', is countable.
(b) For all $\mathcal{L}'$, the set $\{T' \mid T' \subseteq \mathcal{L}'\}$ is countable.

The above definition of scattered is equivalent to the one in Morley's groundbreaking [13]. T is said to be *weakly scattered* if and only if (a) holds. By [13], a scattered theory can have at most ω_1 many countable models. In contrast, a weakly scattered theory can have 2^ω many countable models.

Knight [9] has announced a counterexample to Vaught's Conjecture (VC), a scattered first-order theory with ω_1 many countable models. VC has a precise formulation in Section 5.

In [15] the following bounding result was established: if T is scattered and satisfies (1.5), then T has only countably many countable models; furthermore, every countable model of T has a countable copy in $L(\beta, T)$ for some $\beta < \sigma_2^T$, the least α such that $L(\alpha, T)$ is Σ_2 admissible. Hence Vaught's Conjecture holds for T if T satisfies (1.5). The proofs given in [15] were somewhat sketchy, so missing details needed in later sections of this paper are given in Sections 3 through 5. If Vaught's Conjecture is false, then results for scattered theories yield information about models of counterexamples to VC. Theorem 4.9(vii) says if VC fails for T, then T has a model of cardinality ω_1 not elementarily equivalent in the sense of $\mathcal{L}_{\omega_1,\omega}$ to any countable model (Harnik and Makkai [6]). Theorem 5.3 describes an ω_1-sequence of atomic and saturated models that every counterexample must possess. Section 5 includes a related absoluteness result implicit in Morley [13]: VC(T), Vaught's Conjecture for T, is a $\Sigma_1^{L(\omega_1^{L(T)}, T)}$ predicate of T, hence, Σ_2^1.

Steel [18], as reported in [11], used an assumption stronger than (1.5) to prove VC(T). In Section 2 an arbitrary countable structure $\mathcal{A}$ is associated with a theory $T_{\omega_1^\mathcal{A}}^\mathcal{A}$ contained in a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ canonically generated from $\mathcal{A}$. By an argument of Nadel [14], $\mathcal{A}$ is a homogeneous model of $T_{\omega_1^\mathcal{A}}^\mathcal{A}$. Steel's assumption is equivalent to 'for every $\mathcal{A}$ a model of T, $T_{\omega_1^\mathcal{A}}^\mathcal{A}$ is ω-categorical'. Assumption (1.5) is equivalent to 'for every $\mathcal{A}$ a model of T, $\mathcal{A}$ is an atomic model of $T_{\omega_1^\mathcal{A}}^\mathcal{A}$'. Sacks and Young [12] produced a structure $\mathcal{A}$ such that $\mathcal{A}$ is an atomic model of $T_{\omega_1^\mathcal{A}}^\mathcal{A}$, but $T_{\omega_1^\mathcal{A}}^\mathcal{A}$ is not ω-categorical. (In addition, $\omega_1^\mathcal{A} = \omega_1^{CK}$ and $T_{\omega_1^\mathcal{A}}^\mathcal{A}$ is a Δ_1 subset of $L(\omega_1^{CK})$.)

Sections 7 through 9 are devoted to bounding for weakly scattered theories.

2 Scott Analysis and Rank

This section revisits [15] as promised in Section 1. Scott [17] showed that an arbitrary countable structure $\mathcal{A}$ with underlying first-order language $\mathcal{L}$ can be characterized up to isomorphism by a single sentence of $\mathcal{L}_{\omega_1,\omega}$. In essence there is a countable fragment $\mathcal{L}^{\mathcal{A}}$ of $\mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{A}$ is the atomic model of $T^{\mathcal{A}}$, the complete theory of $\mathcal{A}$ in $\mathcal{L}^{\mathcal{A}}$. Nadel [14] pointed the way to a canonical choice for $\mathcal{L}^{\mathcal{A}}$.

The admissible set $L(\omega_1^{\mathcal{A}}, \mathcal{A})$ is Gödel's L relativized to $\mathcal{A}$ as an element[2] and chopped off at $\omega_1^{\mathcal{A}}$, the least γ such that $L(\gamma, \mathcal{A})$ is Σ_1 admissible. Let

$$\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega} = \mathcal{L}_{\omega_1,\omega} \cap L(\omega_1^{\mathcal{A}}, \mathcal{A}). \tag{2.1}$$

Nadel [14] showed that

$\mathcal{A}$ is a homogeneous model of its complete theory $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$ in $\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$. $\qquad$ (2.2)

It follows that $\mathcal{A}$ is the atomic model of its complete theory in

$$\mathcal{L}_{\omega_1,\omega} \cap L(\omega_1^{\mathcal{A}} + 1, \mathcal{A}), \tag{2.3}$$

since the types over $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$ realized in $\mathcal{A}$ are first-order definable over $L(\omega_1^{\mathcal{A}}, \mathcal{A})$ and so become atoms of the complete theory of $\mathcal{A}$ contained in (2.3).

A Σ_1 recursion defines a canonical choice for $\mathcal{L}^{\mathcal{A}}$ and yields the definition of Scott rank for $\mathcal{A}$.

1. $\mathcal{L}^{\mathcal{A}}_0 = \mathcal{L}$.
2. $\mathcal{L}^{\mathcal{A}}_\lambda = \cup\{\mathcal{L}^{\mathcal{A}}_\delta \mid \delta < \lambda\}$ for limit λ.
3. $T^{\mathcal{A}}_\delta = $ complete theory of $\mathcal{A}$ in $\mathcal{L}^{\mathcal{A}}_\delta$.
4. $\mathcal{L}^{\mathcal{A}}_{\delta+1} = $ least fragment $\mathcal{L}^+$ of $\mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{L}^+ \supseteq \mathcal{L}^{\mathcal{A}}_\delta$, and for each $n > 0$, if $p(\overrightarrow{x})$ is a nonprincipal n-type of $T^{\mathcal{A}}_\delta$ realized in $\mathcal{A}$, then the conjunction

$$\wedge\{\mathcal{F}(\overrightarrow{x}) \mid \mathcal{F}(\overrightarrow{x}) \in p(\overrightarrow{x})\}$$

   is a member of $\mathcal{L}^+$.

Note that if $\mathcal{A}$ is isomorphic to $\mathcal{B}$, then $\mathcal{L}^{\mathcal{A}}_\delta = \mathcal{L}^{\mathcal{B}}_\delta$ and $T^{\mathcal{A}}_\delta = T^{\mathcal{B}}_\delta$ for all δ. For some $\delta < \omega_1$, all the n-types of $T^{\mathcal{A}}_\delta$ realized in $\mathcal{A}$ are principal. To see this, fix γ and suppose some nonprincipal type $p_{\gamma+1}$ of $T^{\mathcal{A}}_{\gamma+1}$ is realized in $\mathcal{A}$. Let p_γ be the restriction of $p_{\gamma+1}$ to T^A_γ. Since $p_{\gamma+1}$ is nonprincipal, there is a formula $\mathcal{G}(\overrightarrow{x})$ of $\mathcal{L}^{\mathcal{A}}_{\gamma+1}$ such that both

$$\exists \overrightarrow{x}[p_\gamma(\overrightarrow{x}) \wedge \mathcal{G}(\overrightarrow{x})] \text{ and } \exists \overrightarrow{x}[p_\gamma(\overrightarrow{x}) \wedge \neg\mathcal{G}(\overrightarrow{x})]$$

belong to $T^{\mathcal{A}}_{\gamma+1}$. Then there are n-tuples $\overrightarrow{b}$ and $\overrightarrow{c}$ of $\mathcal{A}$ such that

$$\mathcal{A} \models [p_\gamma(\overrightarrow{b}) \wedge \mathcal{G}(\overrightarrow{b})], \text{ and } \mathcal{A} \models [p_\gamma(\overrightarrow{c}) \wedge \neg\mathcal{G}(\overrightarrow{c})].$$

Thus a distinction between $\overrightarrow{b}$ and $\overrightarrow{c}$ is made by a formula of $\mathcal{L}^{\mathcal{A}}_{\gamma+1}$ but not by any formula of $\mathcal{L}^{\mathcal{A}}_\gamma$. Since $\mathcal{A}$ is countable, only countably many distinctions can be made.

Let $d_{\mathcal{A}}$ be the the least $\delta < \omega_1$ such that every distinction ever made is made by a formula of $\mathcal{L}^{\mathcal{A}}_\delta$. Then

$\mathcal{A}$ is the atomic model of $T^{\mathcal{A}}_{d_{\mathcal{A}}+1}$. $\qquad$ (2.4)

The *Scott Rank* of $\mathcal{A}$ is defined by

$$\mathrm{sr}(\mathcal{A}) = \text{ least } \alpha \; [\mathcal{A} \text{ is the atomic model of } T_\delta^A]. \tag{2.5}$$

If $\mathcal{A}$ is isomorphic to $\mathcal{B}$, then $\mathrm{sr}(\mathcal{A}) = \mathrm{sr}(\mathcal{B})$. Nadel's proof of (2.2), p. 273 of [14], sketched below, also shows

$$\mathcal{A} \text{ is a homogeneous model of } T_{\omega_1^\mathcal{A}}^\mathcal{A}. \tag{2.6}$$

Hence $d_\mathcal{A} \leq \omega_1^\mathcal{A}$, and so

$$\mathrm{sr}(\mathcal{A}) \leq \omega_1^\mathcal{A} + 1. \tag{2.7}$$

Note that $\mathcal{L}_\delta^\mathcal{A}$ and $T_\delta^\mathcal{A}$, as functions of $\delta < \omega_1^\mathcal{A}$, are $\Sigma_1^{L(\omega_1^\mathcal{A}, \mathcal{A})}$; that is, their graphs are Σ_1 definable subsets of $L(\omega_1^\mathcal{A}, \mathcal{A})$. Since the formulas of $\mathcal{L}_{\omega_1^\mathcal{A}}^\mathcal{A}$ and $T_{\omega_1^\mathcal{A}}^\mathcal{A}$ are "enumerated" in increasing order of complexity,

$$\mathcal{L}_{\omega_1^\mathcal{A}}^\mathcal{A} \text{ and } T_{\omega_1^\mathcal{A}}^\mathcal{A} \text{ are } \Delta_1^{L(\omega_1^\mathcal{A}, \mathcal{A})}. \tag{2.8}$$

To prove (2.6), let $p(\overrightarrow{x})$ be an n-type, and $q(\overrightarrow{x}, y)$ an $(n+1)$-type, of $T_{\omega_1^\mathcal{A}}^\mathcal{A}$, and $\overrightarrow{a}$, $\overrightarrow{b}$ n-tuples of $\mathcal{A}$. Suppose $p(\overrightarrow{x}) \subseteq q(\overrightarrow{x}, y)$ and

$$\mathcal{A} \models [p(\overrightarrow{a}) \wedge p(\overrightarrow{b}) \wedge \exists y q(\overrightarrow{a}, y)]. \tag{2.9}$$

For homogeneity, a $d \in \mathcal{A}$ is required so that $\mathcal{A} \models q(\overrightarrow{b}, d)$. Suppose no such d exists. Let $q_\delta(x, y)$ be the restriction of $q(x, y)$ to $\mathcal{L}_\delta^\mathcal{A}$. Then

$$\{q_\delta(x, y) \mid \delta < \omega_1^\mathcal{A}\} \text{ is } \Sigma_1^{L(\omega_1^\mathcal{A}, \mathcal{A})}. \tag{2.10}$$

For each $d \in \mathcal{A}$, there is a $\delta < \omega_1^\mathcal{A}$ such that $\mathcal{A} \models \neg q_\delta(\overrightarrow{b}, d)$. Since δ can be defined as a $\Sigma_1^{L(\omega_1^\mathcal{A}, \mathcal{A})}$ function of d, the Σ_1 admissibility of $L(\omega_1^\mathcal{A}, \mathcal{A})$ implies there is a $\delta_\infty < \omega_1^\mathcal{A}$ such that $\mathcal{A} \models \forall y \neg q_{\delta_\infty}(\overrightarrow{b}, y)$. But then

$$\mathcal{A} \models \forall y \neg q(\overrightarrow{a}, y). \tag{2.11}$$

A typical use of Scott rank in conjunction with Barwise Compactness and Grilliot type omitting is as follows.

Proposition 2.1 *Suppose $L(\alpha, T)$ is countable and Σ_1 admissible. If for each $\beta < \alpha$, T has a model of Scott rank $\geq \beta$, then T has a countable model such that*

$$\mathrm{sr}(\mathcal{A}) \geq \omega_1^{T, \mathcal{A}} = \alpha. \tag{2.12}$$

Note that the $\mathcal{A}$ of (2.12) must have Scott rank either α or $\alpha + 1$ by (2.7). Forcing the outcome to be $\alpha + 1$ is a problem addressed in this paper but far from resolved.

3 Small Δ_0^{ZF} Sets

The following is one of many variations (e.g., Makkai [10]) on a theme initiated by Barwise [2], an extension of a recursion theoretic fact needed for the enumeration of models of both scattered and weakly scattered theories. The variation below was mentioned and used in [15]. The recursion theoretic fact is as follows: if a set S of reals is Σ_1^1 and has cardinality less that 2^ω, then there exists a hyperarithmetic real H such that every member of S is Turing reducible to H; in addition, an index for H can be computed uniformly from an index for S. The latter uniformity is

key to establishing the Σ_1 character of the enumeration of models in Sections 4 and 8. Recall that a Δ_0^{ZF} formula is a formula in the language of set theory with only bounded quantifiers '$(\forall x \epsilon y)$' and '$(\exists u \epsilon v)$'. Let $D(x, y)$ be a Δ_0^{ZF} lightface formula and A a countable Σ_1 admissible set. Suppose $p, b \in A$. Define

$$S_{p,b} = \{x \mid x \epsilon V \wedge x \subseteq b \wedge D(x, p)\}. \tag{3.1}$$

Theorem 3.1   *If $S_{p,b} \notin A$, then the cardinality of $S_{p,b}$ is 2^ω.*

Proof   Let the language $\mathcal{L}$ consist of $\in$, bounded quantifiers $\forall x \in y$ and $\exists x \in y$, an individual constant $\underline{e}$ for each $e \in A$, and a special individual constant $\underline{c}$ different from all the $\underline{e}$s. Let Z be the following Δ_1^A set of sentences of $\mathcal{L}$.

1. The atomic diagram of A: $\underline{d} \in \underline{e} \leftrightarrow d \in e$; $\underline{d} \notin \underline{e} \leftrightarrow d \notin e$ for $d, e \in A$.
2. $\underline{c} \subseteq \underline{b}$, $D(\underline{c}, \underline{p})$, and $\underline{c} \neq \underline{e}$ for all $e \in A$.

Suppose Z is not consistent in the sense of $\mathcal{L}_{\omega_1, \omega}$. Then there is a $z_0 \in A$ such that $z_0 \subseteq Z$ and z_0 is not consistent. And z_0 consists of some $A_0 \in A$ such that A_0 is a subset of the atomic diagram of A, and

$$\underline{c} \subseteq \underline{b}, D(\underline{c}, \underline{p}), \text{ and } \{\underline{c} \neq \underline{e} \mid e \in f\} \tag{3.2}$$

for some $f \in A$. Since z_0 is inconsistent, there is a deduction $E \in A$ of

$$[\underline{c} \subseteq \underline{b} \wedge D(\underline{c}, \underline{p})] \longrightarrow \underline{c} \in \underline{f} \tag{3.3}$$

from A_0. But then $S_{p,b} \subseteq f$ and so $S_{p,b} \in A$.

Suppose Z is consistent. Then a Henkin-style construction in ω many stages yields a model of Z, hence, an actual $c \in (S_{p,b} - A)$. At stage j, a sentence σ of $\mathcal{L}$ is considered, and σ_j is either σ or $\neg\sigma$ so long as $Z \cup \{\sigma_i \mid i \leq j\}$ is consistent. If σ_j is an infinite disjunction (e.g., σ_j begins with '$\exists x \in \underline{e}$'), then some component of σ_j is added immediately.

The construction can be varied so 2^ω many cs are produced. Let t be a one-one map of ω onto $\{\underline{g} \mid g \in b\}$. After σ_j is chosen, and before σ_{j+1} is chosen, create a split as follows. Choose an n so that $(t(n) \in \underline{c})$ and $(t(n) \notin \underline{c})$ are each consistent with $Z \cup \{\sigma_i \mid i \leq j\}$. Then the construction takes 2^ω different paths, and different paths produce different cs. Such splits always exist. Otherwise there is a j such that $Z \cup \{\sigma_i \mid i \leq j\}$ is consistent and for each n there is a deduction $D_n \in \mathcal{A}$ from $Z \cup \{\sigma_i \mid i \leq j\}$ of either $(t(n) \in \underline{c})$ or $(t(n) \notin c)$. The Σ_1 admissibility of A puts all the D_ns in some $D \in A$. D decides which elements of $\underline{b}$ belong to $\underline{c}$. Hence there is an $e \in A$ such that $(\underline{c} = \underline{e})$ is deducible from $Z \cup \{\sigma_i \mid i \leq j\}$, a contradiction. $\square$

Corollary 3.2   *$S_{p,b}$ is countable $\longleftrightarrow$ $S_{p,b} \in A$.*

Theorem 3.3   *There exists a lightface Σ_1^{ZF} formula $\mathcal{F}(u, v, w)$ such that for any countable Σ_1 admissible set A and any $p, b, s \in A$,*

$$S_{p,b} \text{ is countable } \longrightarrow A \models \exists w \mathcal{F}(\underline{p}, \underline{b}, w); \tag{3.4}$$

$$(\forall s \in A)\{[A \models \mathcal{F}(\underline{p}, \underline{b}, \underline{s})] \longrightarrow s = S_{p,b}\}. \tag{3.5}$$

Proof The existence of $\mathcal{F}$ is implicit in the proof of Theorem 3.1. Thus Z is inconsistent if and only if $S_{p,b}$ is countable if and only if $S_{p,b} \in \mathcal{A}$. The statement

$$A \models \mathcal{F}(\underline{p}, \underline{b}, \underline{s}) \tag{3.6}$$

says

(i) there exist $A_0 \in A$ and E such that $A_0 \subseteq$ atomic diagram of A, and E is a deduction of (3.3) from A_0; and

(ii)
$$s = \{x \mid x \in f \wedge x \subseteq b \wedge D(x, p)\}. \tag{3.7}$$

$\square$

4 Enumeration of Models for Scattered Theories

Let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ for some countable first-order language $\mathcal{L}$ and $T \subseteq \mathcal{L}_0$ a theory with a model. Throughout this section T is scattered as defined in Section 1. For convenience assume T mentions all formulas of $\mathcal{L}_0$; thus $\mathcal{L}_0$ and $\mathcal{L}$ are recoverable from T.

4.1 Review of ω-completeness and finitary consistency for fragments   Let $\mathcal{L}'$ be
a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and $T' \subseteq \mathcal{L}'$ a set of sentences. T' is ω-complete in $\mathcal{L}'$ if and only if (1) and (2) hold.

1. For every sentence $\mathcal{F} \in \mathcal{L}'$, either $\mathcal{F} \in T'$ or $(\neg \mathcal{F}) \in T'$.
2. For any sentence $(\vee_i \mathcal{F}_i) \in T'$, there is an i such that $\mathcal{F}_i \in T'$.

Say T' is *finitarily consistent* if and only if no contradiction can be derived from T' using only the finitary rules of $\mathcal{L}_{\omega_1,\omega}$. The infinitary step being avoided is deriving an infinite conjunction by deriving each of its components. Say T' is *ω-consistent* if and only if for any sentence $(\vee_i \mathcal{F}_i) \in \mathcal{L}'$, if $T' \cup \{\vee_i \mathcal{F}_i\}$ is finitarily consistent, then there is an i such that $T' \cup \{\mathcal{F}_i\}$ is finitarily consistent.

Proposition 4.1   *If T' is finitarily consistent and ω-complete, then T' has a model.*

Proof Note that T' is ω-consistent. The model is constructed by extending T' to a finitarily consistent and ω-complete set of sentences that includes Henkin axioms. At each stage of the construction, the set of sentences up to that point is ω-consistent.

$\square$

Proposition 4.2   *Suppose for all $\beta \le \gamma < \lambda$, T_β is finitarily consistent and ω-complete in the fragment $\mathcal{L}_\beta$, $T_\beta \subseteq T_\gamma$, and $\mathcal{L}_\beta \subseteq \mathcal{L}_\gamma$. Then $\cup \{T_\beta \mid \beta < \lambda\}$ is finitarily consistent and ω-complete in the fragment $\cup \{\mathcal{L}_\beta \mid \beta < \lambda\}$.*

Morley [13] showed that the scatteredness of T implies the countable models of T can be arranged in a hierarchy of height at most ω_1 based on Scott rank with at most countably many models on each level. The current section revisits [15] and presents a Σ_1 enumeration of the countable models of T with a recursion-theoretic eye on some constructive details. The enumeration is a continuous tree $\mathcal{TR}(\mathcal{T})$ with at most ω_1 levels and at most countably many nodes on each level. Each node is a theory T' finitarily consistent and ω-complete in a fragment $\mathcal{L}_{T'}$ with $T \subseteq T'$ and $\mathcal{L}_0 \subseteq \mathcal{L}_{T'}$. Each T' has an atomic model, and the class of all such models is the class of all countable models of T.

The *enumeration* of $\mathcal{TR}(T)$ is as follows.

Level 0   Call T' a node if and only if T' is a finitarily consistent and ω-complete extension of T in the fragment $\mathcal{L}_0(= \mathcal{L}_{T'})$.

Level λ (limit)   Call T' a node if and only if there is a sequence $T_\beta(\beta < \lambda)$ such that T_β is on level β, $T_\beta \subseteq T_\gamma (\beta < \gamma < \lambda)$, and $T' = \cup\{T_\beta \mid \beta < \lambda\}$. $\mathcal{L}_{T'} = \cup\{\mathcal{L}_{T_\beta} \mid \beta < \lambda\}$.

Level $\delta + 1$ Suppose S is a node on level δ, that is, a finitarily consistent theory ω-complete in its fragment $\mathcal{L}_S$. If S is ω-categorical, then S has no successors on level $\delta + 1$. Otherwise, S has a nonprincipal n-type $p(\overrightarrow{x})$. Let $\mathcal{L}'_S$ be the least fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_S$ and containing the conjunction

$$\wedge\{\mathcal{F}(\overrightarrow{x}) \mid \mathcal{F}(\overrightarrow{x}) \in p(\overrightarrow{x})\} \tag{4.1}$$

for every nonprincipal n-type $p(\overrightarrow{x})$ of S for all $n > 0$. Say T' is a successor of S on level $\delta + 1$ if T' is a finitarily consistent and ω-complete extension of S in the fragment $\mathcal{L}'_S (= \mathcal{L}_{T'})$.

Proposition 4.3   *If $\beta < \omega_1$, then $\mathcal{TR}(T)$ has only countably many nodes on level β.*

Proof   By induction on β. Level 0 is countable by clause (b) of the definition of scattered. Suppose S is on level δ. Assume $\mathcal{L}_S$ is countable. The set of all nonprincipal n-types of S is countable by clause (a) of the definition of scattered; hence, $\mathcal{L}'_S$ is countable. The set of all successors of S on level $\delta + 1$ is countable by clause (b) of the definition of scattered.

Let T' be any node on the countable limit level λ. Let $\mathcal{L}_\lambda$ be the least fragment extending all the $\mathcal{L}_S$s for all theories S on all levels below λ. By induction, $\mathcal{L}_\lambda$ is countable. Let T'' be any finitarily consistent and ω-complete extension of T' in $\mathcal{L}_\lambda$. The set of all T''s is countable, so the set of all T's is countable. □

Let $\mathcal{TR}(T) \restriction \beta$ be the restriction of $\mathcal{TR}(T)$ to the levels *below* β.

Proposition 4.4

(i) *If $\beta < \alpha < \omega_1$ and $L(\alpha, T)$ is Σ_1 admissible, then*

$$(\mathcal{TR}(T) \restriction \beta) \in L(\alpha, T).$$

(ii) *There exists a lightface Σ_1^{ZF} formula $\mathcal{G}(u, v, w)$ such that for all scattered T, all countable Σ_1 admissible $L(\alpha, T)$, and all $b \in L(\alpha, T)$,*

$$(\mathcal{TR}(T) \restriction \beta) = b \Longleftrightarrow L(\alpha, T) \models \mathcal{G}(T, \beta, b).$$

Proof   By a $\Sigma_1^{L(\alpha,T)}$ recursion that relies on Theorem 3.3. Suppose

$$(\mathcal{TR}(T) \restriction (\delta + 1)) \in L(\alpha, T), \tag{4.2}$$

and theory S is on level δ. The set of nonprincipal types of S is the unique $s \in L(\alpha, T)$ that satisfies the $\Sigma_1 \mathcal{F}$ of Theorem 3.3 with p and b both equal to S. The statement "q is a nonprincipal type of S" is lightface Δ_0^{ZF} and corresponds to the formula $D(x, y)$ of (3.1). The fragment $\mathcal{L}'_S$ was defined just before Equation (4.1). The set of successors of S on level $\delta + 1$ is obtained from Theorem 3.3 with parameters $\langle p, b \rangle$ equal to $\langle S, \mathcal{L}'_S \rangle$. □

Let $\mathcal{A}$ be a countable model of T (a scattered theory as above). The Scott analysis of $\mathcal{A}$ differs little from its *tree analysis*:

$T(0, \mathcal{A}) = $ theory of $\mathcal{A}$ in $\mathcal{L}_0$, and $\mathcal{L}_{T(0,\mathcal{A})} = \mathcal{L}_0$.

$T(\lambda, \mathcal{A}) = \cup\{T(\beta, \mathcal{A}) \mid \beta < \lambda\}$.

$\mathcal{L}_{T(\lambda,\mathcal{A})} = \cup\{\mathcal{L}_{T(\beta,\mathcal{A})} \mid \beta < \lambda\}$.

$\mathcal{L}_{T(\delta+1,\mathcal{A})} = \mathcal{L}'_{T(\delta,A)}$ (defined similarly to $\mathcal{L}'_S$ on level $\delta + 1$ of $\mathcal{TR}(T)$ above).

$T(\delta + 1, \mathcal{A}) = $ theory of $\mathcal{A}$ in $\mathcal{L}_{T(\delta+1,\mathcal{A})}$.

Recall from Section 2 the definition of $d_\mathcal{A}$, the distinction rank of $\mathcal{A}$, and the argument that the Scott rank of $\mathcal{A}$ is either $d_\mathcal{A}$ or $d_\mathcal{A} + 1$. Clearly, there is a $\delta < \omega_1$ such that for all n, any distinction made between n-tuples of $\mathcal{A}$ by a formula of $\mathcal{L}_{T(\omega_1,\mathcal{A})}$ is made by a formula of $\mathcal{L}_{T(\delta,\mathcal{A})}$. The *tree rank of* $\mathcal{A}$ is defined by

$$\mathrm{tr}(\mathcal{A}) = \text{ least } \delta\,[\mathcal{A} \text{ is the atomic model of } T(\delta, \mathcal{A})]. \qquad (4.3)$$

Proposition 4.5 $\mathrm{tr}(\mathcal{A}) \leq \mathrm{sr}(\mathcal{A})$.

Proof $\mathcal{L}_\delta^A$ was defined just after Equation 2.3. By induction on δ, $\mathcal{L}_\delta^A \subseteq \mathcal{L}_{T(\delta,\mathcal{A})}$. Thus $T_{\mathrm{sr}(\mathcal{A})}^\mathcal{A} \subseteq T(\mathrm{sr}(\mathcal{A}), \mathcal{A})$. But $\mathcal{A}$ is an atomic, hence homogeneous, model of $T_{\mathrm{sr}(\mathcal{A})}^\mathcal{A}$, and so $\mathcal{A}$ is an atomic model of $T(\mathrm{sr}(\mathcal{A}), \mathcal{A})$. $\qquad\square$

Proposition 4.6   *Suppose $\mathcal{A} \models T$ and $\mathcal{L}(\alpha, < T, \mathcal{A} >)$ is Σ_1 admissible. Then*

$$\mathrm{tr}(\mathcal{A}) < \alpha \longrightarrow \mathrm{sr}(\mathcal{A}) < \alpha.$$

Proof   Suppose not. Then D, the set of all distinctions between n-tuples (all $n > 0$) of $\mathcal{A}$ made by formulas of $\mathcal{L}_{T(\mathrm{tr}(\mathcal{A}),\mathcal{A})}$, belongs to $\mathcal{L}(\alpha, \langle T, \mathcal{A}\rangle)$ by Proposition 4.4. And there is an unbounded $\Sigma_1^{L(\alpha,\langle T,\mathcal{A}\rangle)}$ map of D into α, a violation of the Σ_1 admissibility of $\mathcal{L}(\alpha, \langle T, \mathcal{A}\rangle)$. The map carries each distinction $d \in D$ to the least δ such that d is made by some formula of $\mathcal{L}_\delta^\mathcal{A}$. $\qquad\square$

A theory T can be scattered up to a point. The tree $\mathcal{TR}(T)$ is said to be *scattered below β* if the notion of scattered enumeration succeeds for T on all levels below β. To be more precise, $\mathcal{TR}(T)$ has only countably many nodes (perhaps none) on each level below β.

Proposition 4.7   *Suppose $\alpha < \omega_1$, $L(\alpha, T)$ is Σ_1 admissible, T is scattered below $(\alpha + 1)$, and T has a model of Scott rank $\geq \beta$ for each $\beta < \alpha$. Then there exists a theory T_α on level α of $\mathcal{TR}(T)$ such that T_α is $\Delta_1^{L(\alpha,T)}$.*

Proof   By Proposition 4.6, $\mathcal{TR}(T)$ has nodes on all levels below α if an $\mathcal{A}$ can be found that satisfies the hypotheses of Proposition 4.6 and also $\mathrm{sr}(\mathcal{A}) \geq \alpha$. To find $\mathcal{A}$ through Barwise Compactness, consider the following set Z of sentences.

(Z1)   Introduce a constant $\underline{e}$ to name each $e \in L(\alpha, T)$. Add the atomic diagram (in the sense of $\mathcal{L}_{\omega_1,\omega}$) of $L(\alpha, T)$ to Z. For each $\beta < \alpha$,

$$\forall x[x \in \underline{\beta} \longleftrightarrow \vee\{x = \underline{\gamma} \mid \gamma < \beta\}] \qquad (4.4)$$

is a typical member of $(Z1)$. Any model of $(Z1)$ is an end extension of $L(\alpha, T)$.

(Z2)   Introduce a new constant $\underline{d}$ and add sentences saying $\underline{d}$ is an ordinal greater than $\underline{\beta}$ for each $\beta < \alpha$.

(Z3)   Add $\mathcal{A} \models T$ and $\mathrm{sr}(\mathcal{A}) > \underline{\beta}$ for each $\beta < \alpha$.

(Z4)   Add the axioms for Σ_1 admissibility.

Let M be a model of Z that omits α but extends $L(\alpha, T)$ as in [5] or [7]. $L(\alpha, \langle T, \mathcal{A} \rangle)$ is Σ_1 admissible; otherwise $\alpha \in M$. (Z3) insures $\mathrm{sr}(\mathcal{A}) \geq \alpha$.

Let T' denote an arbitrary node below level α. Call T' unbounded if T' has extensions to theories on arbitrarily high levels below α. Then T can be regarded as an unbounded node.

Suppose T' is an unbounded node below level β for some $\beta < \alpha$; then T' has an unbounded extension on level β. Otherwise, the Σ_1 admissibility of $L(\alpha, T)$ implies T' is bounded.

There exists a $\beta_0 < \alpha$ and an unbounded node T_{β_0} on level β_0 such that for all $\beta \in (\beta_0, \alpha)$, T_{β_0} has a unique unbounded extension on level β. Otherwise, a tree $\mathcal{U}$ of unbounded nodes can be constructed such that $\mathcal{U}$ is isomorphic to the binary branching tree $2^{<\omega}$, and the branches of $\mathcal{U}$ define a continuum of nodes on some level $\alpha_0 \leq \alpha$ of $\mathcal{TR}(T) \restriction (\alpha + 1)$.

The set S_{ub} of unbounded nodes above T_{β_0} form an expanding sequence whose union is the desired T_α. To see S_{ub} is $\Delta_1^{L(\alpha,T)}$, let N_γ be the set of all nodes on level γ extending T_{β_0} for each $\gamma \in (\beta_0, \alpha)$. Then N_γ, as a function of γ, is $\Sigma_1^{L(\alpha,T)}$ by Proposition 4.4, and $(N_\gamma - S_{ub}) \in L(\alpha, T)$ since $N_\gamma \cap S_{ub}$ has just one element. There is a $\Sigma_1^{L(\alpha,T)}$ function that takes each node $e \in (N_\gamma - S_{ub})$ to a bound on the levels occupied by extensions of e. But then there is a strict upper bound $b < \alpha$ on the levels occupied by extensions of members of $(N_\gamma - S_{ub})$. Any such b singles out the unique member of $N_\gamma \cap S_{ub}$. $\square$

Proposition 4.8   *Suppose $\alpha \leq \omega_1$, $L(\alpha, T)$ is Σ_2 admissible, T is scattered below α, and T has models of arbitrarily high Scott rank less than α. Then there exists a theory T_α on level α of $\mathcal{TR}(T)$ such that T_α is $\Delta_1^{L(\alpha,T)}$.*

Proof   The proof is similar to that of Proposition 4.7. The only difference is in the handling of $\mathcal{U}$. Then and now $\mathcal{U}$ can be defined by a $\Sigma_2^{L(\alpha,T)}$ recursion of length ω, since the set of unbounded nodes is $\Pi_1^{L(\alpha,T)}$. But now the Σ_2 admissibility of $L(\alpha, T)$ implies $\mathcal{U} \in L(\alpha, T)$, and so the branches of $\mathcal{U}$ define a continuum of nodes on some level $\alpha_0 < \alpha$ of $\mathcal{TR}(T)$. $\square$

Two $\mathcal{L}$-structures are said to be $\mathcal{L}_{\omega_1,\omega}$-equivalent if they satisfy the same sentences of $\mathcal{L}_{\omega_1,\omega}$. (Recall that if $\mathcal{A}$ is countable and $\mathcal{L}_{\omega_1,\omega}$-equivalent to $\mathcal{B}$, then $\mathcal{A}$ is $\mathcal{L}_{\infty,\omega}$-equivalent to $\mathcal{B}$.)

Theorem 4.9   *Suppose Vaught's Conjecture fails for T. Then there exist T_β, $\mathcal{A}_\beta$, and $\mathcal{L}_\beta (\beta \leq \omega_1)$ such that*

   (i) *if $\beta < \omega_1$, then T_β is an ω-complete theory in the countable fragment $\mathcal{L}_\beta$;*

   (ii) *if $\beta \leq \gamma \leq \omega_1$, then $T_\beta \subseteq T_\gamma$, $\mathcal{A}_\beta \subseteq \mathcal{A}_\gamma$, and $\mathcal{L}_\beta \subseteq \mathcal{L}_\gamma$;*

   (iii) *if $\lambda(limit) \leq \omega_1$, then $T_\lambda = \cup\{T_\beta \mid \beta < \lambda\}$ and $\mathcal{A}_\lambda = \cup\{\mathcal{A}_\beta \mid \beta < \lambda\}$;*

   (iv) *T_{ω_1} is $\Delta_1^{L(\omega_1,T)}$ definable;*

   (v) *if $\beta \leq \omega_1$, then $\mathcal{A}_\beta$ is an atomic model of T_β;*

   (vi) *if $\beta < \omega_1$, then $\mathcal{A}_{\beta+1}$ realizes a nonprincipal type of T_β;*

   (vii) *(Harnik and Makkai [6]) The cardinality of $\mathcal{A}_{\omega_1}$ is ω_1, and $\mathcal{A}_{\omega_1}$ is not $\mathcal{L}_{\omega_1,\omega}$-equivalent to any countable model.[3]*

Proof A uncountable model $\mathcal{A}_{\omega_1}$ of T is constructed so that it is not $\mathcal{L}_{\omega_1,\omega}$-equivalent to any countable model. By Proposition 4.8, there is a theory T_{ω_1} on level ω_1 of $\mathcal{TR}(\omega_1)$ such that T_{ω_1} is $\Delta_1^{L(\omega_1,T)}$. Thus $T_{\omega_1} = \cup\{T_\gamma \mid \gamma < \omega_1\}$, and $(\gamma \leq \delta) \to (T_\gamma \subseteq T_\delta)$. p, the parameter used in the $\Delta_1^{L(\alpha,T)}$ definition of T_{ω_1}, belongs to $L(\alpha_0, T)$ for some $\alpha_0 < \omega_1$. Define

$$K = \{\beta \mid \alpha_0 < \beta < \omega_1 \wedge L(\beta, T) \preccurlyeq_1 L(\omega_1, T)\}.$$

(Recall that $X \preccurlyeq_1 Y$ means X is a Σ_1^{ZF} substructure of Y.) Let $\{\gamma_\delta \mid \delta < \omega_1\}$ be an increasing enumeration of K. Then $L(\gamma_\delta, T)$ is Σ_1 admissible, and so

$$T_{\gamma_\delta} = T_{\omega_1} \cap L(\gamma_\delta, T)$$

by Proposition 4.4(i). Also T_{γ_δ} is $\Delta_1^{L(\gamma_\delta,T)}$ definable via the same Δ_1 definition that works for T_{ω_1}, since $p \in L(\gamma_\delta, T) \preccurlyeq_1 L(\omega_1, T)$.

Structures $\mathcal{A}_\delta(\delta \leq \omega_1)$ and inclusion maps $i_{\beta,\delta} : \mathcal{A}_\beta \longrightarrow \mathcal{A}_\delta(\beta < \delta)$ are defined by recursion on δ. The map $i_{\beta,\delta}$ will be elementary with respect to the language $\mathcal{L}_{\gamma_\beta}$; that is, any sentence of $\mathcal{L}_{\gamma_\beta}$ with parameters in $\mathcal{A}_\beta$ and true in $\mathcal{A}_\beta$ will also be true in $\mathcal{A}_\delta$.

Stage 0   Structure $\mathcal{A}_0$ is the countable atomic model of T_{γ_0}.

Stage $\delta + 1$ Assume $\mathcal{A}_\delta$ is the countable atomic model of T_{γ_δ}. Extend $\mathcal{A}_\delta$ to $\mathcal{A}_{\delta+1}$, the countable atomic model of $T_{\gamma_{\delta+1}}$, so that the inclusion map $i_{\delta,\delta+1}$ is $\mathcal{L}_{\gamma_\delta}$-elementary.

Stage λ (limit $\leq \omega_1$)   Let

$$\mathcal{A}_\lambda = \cup\{\mathcal{A}_\delta \mid \delta < \lambda\}.$$

For all $\delta < \delta' < \lambda$, assume the inclusion map $i_{\delta,\delta'}$ is $\mathcal{L}_{\gamma_\delta}$-elementary. Then for each $\delta < \lambda$, $\mathcal{A}_\lambda$ is an $\mathcal{L}_{\gamma_\delta}$-elementary extension of $\mathcal{A}_\delta$ and so is a model of T_{γ_δ}. Thus $\mathcal{A}_\lambda$ is a model of $T\gamma_\lambda$.

To see that $\mathcal{A}_\lambda$ is an atomic model of $T\gamma_\lambda$, let $\overrightarrow{a}$ be an n-tuple of $\mathcal{A}_\lambda$. For some $\delta < \lambda$, $\overrightarrow{a}$ is an n-tuple of $\mathcal{A}_\delta$ and realizes some atom $\mathcal{F}(\overrightarrow{x})$ of T_{γ_δ}. Then $\mathcal{F}(\overrightarrow{x})$ is an atom of T_λ, because $L(\gamma_\delta, T) \preccurlyeq_1 L(\lambda, T)$. Hence, $\overrightarrow{a}$ realizes $\mathcal{F}(\overrightarrow{x})$ in $\mathcal{A}_\lambda$, since $i_{\delta,\lambda}$ is $\mathcal{L}_\delta$-elementary.

If $\mathcal{A}_{\omega_1}$ were $\mathcal{L}_{\omega_1,\omega}$-equivalent to some countable model, then it would be an atomic model of T_{γ_δ} for some $\delta < \omega_1$. But $\mathcal{A}_{\delta+1}$, hence $\mathcal{A}_{\omega_1}$, realizes a nonprincipal type of T_{γ_δ}. □

5 Absoluteness of Vaught's Conjecture

Let $VC(T)$ be the predicate 'Vaught's conjecture holds for T'. Morley's work [13] implies that $VC(T)$ is absolute. The enumeration tree, $\mathcal{TR}(T)$, of Section 4 is applied below to make the statement of $VC(T)$ more precise and to see in some detail how T can satisfy Vaught's Conjecture. Suppose an attempt is made to develop $\mathcal{TR}(T)$ and the attempt fails to produce a tree with only countably many nodes on each level and ω_1 many nonempty levels. Then there must be a countable β such that one of the following holds:

   (1) $\beta = 0$ and T has uncountably many finitarily consistent, ω-complete extensions in $\mathcal{L}_0$;

(2) $\beta = \delta + 1$, some theory S is on level δ, and for some n, the set of n-types of S is uncountable;
(3) $\beta = \delta + 1$, some theory S is on level δ, for all n the set of n-types of S is countable, and the set of all finitarily consistent, ω-complete extensions of S in $\mathcal{L}'_S$ is uncountable. $\mathcal{L}'_S$ is defined just before (4.1).
(4) $\beta = \lambda$ and the set of nodes on level λ is uncountable.
(5) level β is empty.

Define the *Vaught Rank* of T, $\mathrm{vr}(T)$ to be the least countable β that satisfies one of (1)–(5) above. (If there is no such β, let $\mathrm{vr}(T)$ be ω_1.) Define the predicate $VC(T)$ by $\mathrm{vr}(T) < \omega_1$. Suppose $\mathrm{vr}(T) = \beta < \omega_1$. If $\beta = 0$, then T has 2^ω finitarily consistent, ω-complete extensions in $\mathcal{L}_0$ by Theorem 3.1, hence, 2^ω many countable models. The same holds in cases (3) and (4). If (5) holds, then T has only countably many countable models, and each one is the atomic model of a theory on some level of $\mathcal{TR}(T)$ below level β. Suppose case (2) holds. Then for some n, there are $2^\omega n$-types of S by Theorem 3.1, hence, 2^ω many countable models of T.

Recall that

$$\omega_1^{L(T)} = \text{ least } \gamma \ [L(T) \models (\gamma \text{ is uncountable })]. \tag{5.1}$$

Proposition 5.1   *The predicate, Vaught's Conjecture holds for T, is*

$$\Sigma_1^{L(\omega_1^{L(T)}, T)}, \text{ hence } \Sigma_2^1.$$

Proof   By Proposition 4.4, $\mathcal{TR}(T) \subseteq L(\omega_1, T)$ and is $\Sigma_1^{L(\omega_1, T)}$. The statement $VC(T)$ says "at some level $\gamma < \omega_1$, either (a) $\mathcal{TR}(T)$ ends or (b) *blows up*; that is, a perfect kernel of theories or types is manifest." Let α_0 be the least $\alpha > \gamma$ such that $L(\alpha, T)$ is Σ_1 admissible.

Suppose (a) holds. Then Levy-Shoenfield Absoluteness implies $\alpha_0 < \omega_1^{L(T)}$, and there is an $\mathcal{L}_{\omega_1, \omega}$ sentence $\mathcal{K} \in L(\alpha_0, T)$ that expresses the fact that every model of T is an atomic model of some theory on some level at or below γ of $\mathcal{TR}(T)$.

Suppose (b) holds. Theorem 3.1 implies the existence of a perfect kernel of theories or types. A coding of some such perfect kernel by a real is constructible from any counting of α_0. The proof of Theorem 3.1 relies on the consistency of a certain set Z of axioms. Z is $\Sigma_1^{L(\alpha_0, T)}$, and the consistency of Z is $\Pi_1^{L(\alpha_0, T)}$. Hence Levy-Shoenfield Absoluteness implies $\alpha_0 < \omega_1^{L(T)}$, and so a code for the perfect kernel belongs to $L(\omega_1^{L(T)}, T)$. $\square$

Proposition 5.2   *Suppose T is a counterexample to Vaught's Conjecture. Then there is a theory T_{ω_1} on level ω_1 of $\mathcal{TR}(T)$ such that T_{ω_1} is $\Delta_1^{L(\omega_1, T)}$. For all countable β, T_β, the restriction of T_{ω_1} to level β, has an atomic model whose Scott rank is β.*

Proof   By Proposition 4.8. $\square$

Suppose $L(\alpha, T)$ is Σ_1 admissible, $\mathcal{A}$ is a countable model of T, and $\omega_1^{\mathcal{A}} = \alpha$. According to (2.6), $\mathcal{A}$ is a homogenous model of $T_\alpha^{\mathcal{A}}$; $\mathcal{A}$ is said to be *α-saturated* if every n-type ($n \geq 1$) of $T_\alpha^{\mathcal{A}}$ is realized in $\mathcal{A}$.

Theorem 5.3   *Suppose T is a counterexample to Vaught's Conjecture. Then there is a $\Delta_1^{L(\omega_1, T)}$ theory T_{ω_1} on level ω_1 of $\mathcal{TR}(T)$ and a closed unbounded set $C \subseteq \omega_1$ such that $\forall \alpha \in C: T_\alpha$, the restriction of T_{ω_1} to level α, has an atomic model $\mathcal{A}_\alpha$ of*

Scott rank α and an α-saturated model $\mathcal{B}_\alpha$ of Scott rank $\alpha + 1$. The atomic models form an expanding chain and each inclusion $\mathcal{A}_\beta \subset \mathcal{A}_\gamma$ ($\beta < \gamma$) is elementary with respect to the language of T_β.

Proof Proposition 4.8 provides T_{ω_1}. Let $p \in L(\omega_1, T)$ be the parameter needed for the $\Delta_1^{L(\omega_1, T)}$ definition of T_{ω_1}. For any α, let α^+ be the least $\beta > \alpha$ such that $L(\beta, T)$ is Σ_1 admissible.

For $x \in L(T)$, let $H_1(x)$ be the Σ_1 hull of x in $L(T)$. Recall that

$$x \subseteq H_1(x) \preccurlyeq_1 L(T)$$

and that x and $H_1(x)$ have the same cardinality in $L(T)$. An expanding sequence of countable Σ_1 hulls, $H^\delta (\delta < \omega_1)$, is defined by recursion on δ.

H^0 is $H_1(\{\mathrm{tc}(p), \omega_1, \mathrm{tc}(T)\})$. (tc is transitive closure.) Note that $\omega_1^+, \omega \in H^0$; if $d < e < \omega_1$ and $e \in H^0$, then $d \in H^0$. Let c_0 be the lub of the countable ordinals in H^0. Let $L(\beta_0, T)$ be the transitive collapse of H^0. Then

$$c_0 = \omega_1^{L(\beta_0, T)} \text{ and } L(c_0^+, T) \subseteq L(\beta_0, T). \tag{5.2}$$

Stage $\delta + 1$ Assume H^δ is countable in V. Then $H^\delta \cap \omega_1$ is a proper initial segment of ω_1. Let c_δ be the least countable ordinal not in H^δ. $H^{\delta+1}$ is $H_1(H^\delta \cup \{c_\delta\})$.

Stage λ (limit)   Let H^λ be $\cup\{H_\delta \mid \delta < \lambda\}$. Then $C = \{c_\delta \mid \delta < \omega_1\}$ is a closed unbounded set. Let $L(\beta_\delta, T)$ be the transitive collapse of H^δ. Then

$$c_\delta = \omega_1^{L(\beta_\delta, T)} \text{ and } L(c_\delta^+, T) \subseteq L(\beta_\delta, T). \tag{5.3}$$

Let T_{c_δ} be the restriction of T_{ω_1} to level c_δ of $\mathcal{TR}(T)$. Then T_{c_δ} is $\Delta_1^{L(c_\delta, T)}$ via parameter p, and N, the set of nonprincipal types of T_{c_δ}, is nonempty and countable in V. Then $T_{c_\delta} \in L(c_\delta^+, T)$, and so $N \in L(c_\delta^+, T)$ by Theorem 3.1. Hence the structure $L[c_\delta, T; T_{c_\delta}, N]$ (i.e., $L(c_\delta, T)$ with $x \in T_{c_\delta}$ and $x \in N$ as additional atomic predicates) is Σ_1 admissible because no subset of c_δ in $L(\beta_\delta, T)$ can define a counting of $\omega_1^{L(\beta_\delta, T)}$. Now the construction of M in the proof of Theorem 6.1 can be imitated to produce a model $\mathcal{B}$ of T_{c_δ} such that $\mathcal{B}$ realizes all the types in N and $\omega_1^{\mathcal{B}} = c_\delta$. The atomic $\mathcal{A}_\beta$s are supplied by Theorem 4.9.   □

6 Bounds on Scattered Theories

Once again $\mathcal{L}$ is a countable first-order language, $\mathcal{L}_0$ is a countable fragment of $\mathcal{L}_{\omega_1, \omega}$, and $T \subseteq \mathcal{L}_0$ has a model. $\mathcal{L}$ and $\mathcal{L}_0$ are effectively recoverable from T_0. T is scattered below β as was defined just before Proposition 4.7.

Theorem 6.1   *Suppose $\alpha < \omega_1$, $L(\alpha, T)$ is Σ_2 admissible, T is scattered below α, and for each $\beta < \alpha$, T has a model of Scott rank $\geqslant \beta$. Then T has a model $\mathcal{A}$ such that $\omega_1^{\mathcal{A}} = \alpha$ and $\mathrm{sr}(\mathcal{A}) = \alpha + 1$.*

Proof By Proposition 4.8, $\mathcal{TR}(A)$ has a theory T_α on level α such that T_α is Δ_1^α and T_α is $\cup\{T_\beta \mid \beta < \alpha\}$, where T_β is a node on level β. Let Z be the following set of sentences.

(Z1)   The atomic diagram of $L(\alpha, T)$ in the sense of $\mathcal{L}_{\omega_1, \omega}$.

(Z2)   Add $(\underline{d} > \underline{\beta})$ for all $\beta < \alpha$. $\underline{d}$ is a constant not occurring in (Z1).

(Z3)   Let $T_{\underline{d}}$ be a theory on level $\underline{d}$ of $\mathcal{TR}(T)$. Add $\mathcal{A}$ is the countable atomic model of $T_{\underline{d}}$ and $\mathcal{F} \in T_{\underline{d}}$ for each sentence $\mathcal{F} \in T_{\alpha}$.

(Z4)   Add $(b(\overrightarrow{x})$ is an atom of $T_{\underline{d}})$ for each $b(\overrightarrow{x})$ that is an atom of T_{α}; that is, $b(\overrightarrow{x})$ generates a principal type of T_{α}.

(Z5)   Add the axioms of Σ_1 admissibility. The set Z is $\Sigma_2^{L(\alpha,T)}$, since the set of atoms of T_{α} is $\Pi_1^{L(\alpha,T)}$.

Suppose $\beta < \alpha$, $L(\beta, T)$ is Σ_1 admissible, and Z_{β} is $Z \cap L(\beta, T)$. A fundamental fact of forcing in the setting of set theory is the Levy collapse of a cardinal to V preserves replacement; furthermore, the preservation of Σ_n replacement needs only Σ_n replacement in V. To check the consistency of Z_{β}, augment $L(\alpha, T)$ by adding a generic counting of $L(\beta, T)$ to $L(\alpha, T)$ that preserves the Σ_2 admissibility of $L(\alpha, T)$. The set Z_{β} can be modeled by the augmented $L(\alpha, T)$. By Proposition 4.4, $T_{\beta} \subseteq L(\beta, T)$. Interpret $\underline{d}$ as β. Interpret $\mathcal{A}$ as the atomic model of T_{β}. Such an $\mathcal{A}$ belongs to the augmented $L(\alpha, T)$ because there T_{β} is countable. If $b(\overrightarrow{x})$ is an atom of T_{α} and belongs to $L(\beta, T)$, then $b(\overrightarrow{x})$ is an atom of T_{β}.

The set Z has a model M that is a proper end extension of $L(\alpha, T)$ but omits α. $\omega_1^{\mathcal{A}} \leq \alpha$; otherwise, α is recursive in $\mathcal{A}$, and then $\alpha \in M$. By design $\mathcal{A} \models T_{\beta}$ for all $\beta < \alpha$; hence, $\text{sr}(\mathcal{A}) \geq \alpha$ by Proposition 4.5, and so $\omega_1^{\mathcal{A}} = \alpha$ by (2.6).

Suppose $\text{sr}(\mathcal{A}) = \alpha$. Then $\alpha \in M$ as follows. By supposition $\mathcal{A}$ is the atomic model of T_{α}. The rank of an atom $b(\overrightarrow{x})$ of T_{α} is the least $\beta < \alpha$ such that $b(\overrightarrow{x})$ is an atom of T_{β}. Let f be the function that carries each $\overrightarrow{a} \in \mathcal{A}$ to the rank of an atom of T_{α} that generates the principal type realized by $\overrightarrow{a}$ in $\mathcal{A}$. Thanks to (Z4) f is definable from $T_{\underline{d}}$, and so $f \in M$. Then $\text{lub}(\text{range} f) = \alpha \in M$.   □

Corollary 6.2 ([15])   *Suppose for every countable model $\mathcal{A}$ of T, the Scott rank of $\mathcal{A}$ is less than or equal to $\omega_1^{\mathcal{A}}$. Then Vaught's Conjecture holds for T.*

Proof   Suppose $VC(T)$ fails. Then T is scattered below ω_1, and $\mathcal{TR}(T)$ has nodes on every countable level. Choose an $\alpha < \omega_1$ such that $L(\alpha, T)$ is Σ_2 admissible. Then T has a countable model $\mathcal{A}$ such that $\omega_1^{\mathcal{A}} = \alpha$ and $\text{sr}(\mathcal{A}) = \alpha + 1$.   □

A more effective version of Corollary 6.2 is as follows. Define

$$\sigma_2^T = \text{ least } \alpha \, [L(\alpha, T) \text{ is } \Sigma_2 \text{ admissible}]. \tag{6.1}$$

$\text{vr}(T)$, the Vaught rank of T, was defined at the beginning of Section 6.

Corollary 6.3   *Suppose T does not have a countable model $\mathcal{A}$ such that*

$$\omega_1^{\mathcal{A}} = \sigma_2^T \text{ and } \text{sr}(\mathcal{A}) = \sigma_2^T + 1. \tag{6.2}$$

Then $\text{vr}(T) < \sigma_2^T$.

Proof   If $\text{vr}(T) \geqslant \sigma_2^T$, then T is scattered below σ_2^T and $\mathcal{TR}(T)$ has nodes on every level below σ_2^T.   □

As a warmup to the main bounding results of the paper (Section 8), the above is recast as an effective bounding theorem.

Corollary 6.4   *Suppose T is scattered and*

$$\text{sr}(\mathcal{A}) \leq \omega_1^{\mathcal{A}} \text{ for every countable } \mathcal{A} \models T. \tag{6.3}$$

Then $\exists \beta < \sigma_2^T$ such that

$$\mathrm{sr}(\mathcal{A}) < \beta \text{ for every } \mathcal{A} \models T. \tag{6.4}$$

Let $SA(T)$ say for every countable model $\mathcal{A}$ of T, the theory $T_{\omega_1^A}^{\mathcal{A}}$ is ω-categorical. Steel [18], as reported in Makkai [11], showed that $VC(T)$ follows from $SA(T)$. Theorem 6.5 is an effective version of Steel's result.

The set $L(\alpha, T)$ is said to be *recursively Mahlo* if $L(\alpha, T)$ is Σ_1 admissible and every $\Delta_1^{L(\alpha,T)}$ closed unbounded subset of α has a member β such that $L(\beta, T)$ is Σ_1 admissible. Define

$$\mathrm{rm}(T) = \text{ least } \gamma \ [L(\gamma, T) \text{ is recursively Mahlo}]. \tag{6.5}$$

Note that $\mathrm{rm}(T) < \sigma_2^T$.

Theorem 6.5 *Suppose T is scattered and*

$$T_{\omega_1^A}^{\mathcal{A}} \text{ is } \omega\text{-categorical for every countable } \mathcal{A} \models T. \tag{6.6}$$

Then $\exists \beta < \mathrm{rm}(T)$ such that

$$\mathrm{sr}(\mathcal{A}) < \beta \text{ for every countable } \mathcal{A} \models T. \tag{6.7}$$

Proof Suppose there is no such β. Let α be $\mathrm{rm}(T)$. Then Proposition 4.7 supplies a $\Delta_1^{L(\alpha,T)}$ theory T_α on level α of $\mathcal{TR}(T)$. Then $T_\alpha = \cup\{T_\beta \mid \beta < \alpha\}$, and T_β, as a function of β, is $\Sigma_1^{L(\alpha,T)}$.

There is a $\Sigma_1^{L(\alpha,T)}$ function f_0 such that $T_\beta \subseteq L(f_0(\beta), T)$ for all $\beta < \alpha$. Iteration of f_0 leads to a $\Delta_1^{L(\alpha,T)}$ closed unbounded set

$$C_0 = \{\gamma \mid T_\gamma \subseteq L(\gamma, T)\}. \tag{6.8}$$

A similar argument produces a $\Delta_1^{L(\alpha,T)}$ closed unbounded set C_1 such that

$$\forall \gamma \in C_1\big[(T_\alpha \cap L(\gamma, T)) \text{ is } \Delta_1^{L(\gamma,T)}\big]. \tag{6.9}$$

Then there is a $\Delta_1^{L(\alpha,T)}$ closed unbounded set K such that

$$\forall \gamma \in K\big[T_\gamma \subseteq L(\gamma, T) \text{ and } T_\gamma \text{ is } \Delta_1^{L(\gamma,T)}\big]. \tag{6.10}$$

Hence, for some $\gamma_0 \in K$, $L(\gamma_0, T)$ is Σ_1 admissible. Consequently T_{γ_0} has a model $\mathcal{B}$ such that $\omega_1^{\mathcal{B}} = \gamma_0$. But then $T_{\omega_1^{\mathcal{B}}}^{\mathcal{B}}$, hence T_{γ_0}, is ω-categorical and so has no extension to a node on level α. $\square$

7 Iterated Classical Bounding

In this section classical bounding (reviewed in Section 1) is translated into the language of Σ_1 admissible sets and revised to allow for iterated use in Σ_1 recursive definitions in Section 8.

Let $B(x)$ be a Δ_0^{ZF} formula with parameter p_0. The formula $B(x)$ is *β-bounded* if and only if

$$\forall c[B(c) \iff L[\beta, p_0; c] \models B(\underline{c})]. \tag{7.1}$$

The set $L[\beta, p_0; c]$ is the result of iterating first-order definability with $y \in c$ as an additional atomic predicate through the ordinals less than β starting with the transitive closure (tc) of $\{p_0\}$. Assume $B(x)$ is β-bounded. Define

$$c_\beta = c \cap L[\beta, p_0; c]. \tag{7.2}$$

Then $B(c) \iff B(c_\beta)$. For all z let A_z be the least Σ_1 admissible set with z as a member; thus

$$A_z = L(\omega_1^z, \mathrm{tc}(\{z\})). \tag{7.3}$$

Let $\mathcal{F}(u, v)$ be a Σ_1^{ZF} formula with parameter p_1, and let p be $\{p_0, p_1\}$. Suppose for all c, if $B(c)$, then there exists a unique $\delta \in A_{\{p,\beta,c_\beta\}}$ such that

$$A_{\{p,\beta,c_\beta\}} \models \mathcal{F}(\underline{c_\beta}, \underline{\delta}); \tag{7.4}$$

designate δ by $\delta_{p,\beta,c}$.

Theorem 7.1

(i) *There exists a $\delta_{p,\beta} \in A_{\{p,\beta\}}$ such that for all c,*

$$B(c) \longrightarrow \delta_{p,\beta,c} \leq \delta_{p,\beta}. \tag{7.5}$$

(ii) *$\delta_{p,\beta}$ can be construed as a partial function of p and β whose restriction to any Σ_1 admissible A has a Σ_1^A definition uniformly in A; that is, one Σ_1 formula works for all A.*

Proof   Let Z be the following $\Sigma_1^{A_{\{p,\beta\}}}$ set of sentences. Let $\alpha = \omega_1^{\{p,\beta\}}$.

(Z1)   Introduce constants $\underline{c}$ and $\underline{c_\beta}$, and put $\underline{c_\beta} = \underline{c} \cap L[\beta, p_0; \underline{c}]$ and $B(\underline{c_\beta})$ in Z.

(Z2)   Add constants that name the elements of

$$L(\alpha, \mathrm{tc}(\{p, \beta, c_\beta\})) \tag{7.6}$$

and sentences of $\mathcal{L}_{\omega_1,\omega}$ that define each element in terms of elements of lower definability rank.

(Z3)   Let $\mathcal{F}(u, v)$ be $\exists w \mathcal{G}(u, v, w)$ for some Δ_0^{ZF} formula $\mathcal{G}(u, v, w)$.   Add $\neg \mathcal{G}(\underline{c_\beta}, \underline{\delta}, \underline{r})$ for all $\delta < \alpha$ and every $\underline{r}$ that names an element of (7.6).

(Z4)   Add axioms for Σ_1 admissibility.

Suppose Z is consistent. Assume for a moment that

$$Z \text{ is countable.} \tag{7.7}$$

As in the proof of Proposition 4.7, Z has a model M that is a proper end extension of (7.6) but omits α. Then (7.6) is Σ_1 admissible, and so

$$A_{\{p,\beta,c_\beta\}} = L(\alpha, \mathrm{tc}(\{p, \beta, c_\beta\})). \tag{7.8}$$

But then $A_{\{p,\beta,c_\beta\}} \models \neg \mathcal{F}(\underline{c_\beta}, \underline{\delta})$ for all $\delta < \alpha$, a contradiction since

$$\delta_{p,\beta,c_\beta} \in A_{\{p,\beta,c_\beta\}}.$$

Thus Z is inconsistent.

To remove assumption (7.7), generically extend the universe V to V' so that Z is countable in V'. Then Z is inconsistent in V', hence in V, by the absoluteness of provability in the sense of $\mathcal{L}_{\infty,\omega}$.

Since Z is $\Sigma_1^{A_{\{p,\beta\}}}$, there must be an inconsistent $W \subseteq Z$ such that

$$W \in A_{\{p,\beta\}}.$$

The set W consists of $(W1)$, $(W2)$, and $(W3)$:

(W1)   (Z1) and (Z4) above.

(W2)   Some $A_0 \in A_{\{p,\beta\}}$ such that $A_0 \subseteq$ set of sentences of (Z2).

(W3)   For some $\delta_1 < \alpha$, the sentence $\neg \mathcal{G}(c_\beta, \underline{\delta}, \underline{r})$ for all $\delta < \delta_1$ and every $\underline{r}$ of (Z2) that names an element of $L(\delta_1, \mathrm{tc}(\{p, \beta, c_\beta\}))$.

Then there is a deduction $D \in A_{\{p,\beta\}}$ from (W1) and (W2) of

$$\vee \{\mathcal{F}(\underline{c_\beta}, \delta) \mid \delta < \delta_1\}. \tag{7.9}$$

Let ρ_0 be the least ρ such that there is such a $D \in L(\rho, \mathrm{tc}(\{p, \beta\}))$; let $\delta_{\{p,\beta\}}$ be the least δ_1 associated with any such $D \in L(\rho_0, \mathrm{tc}(\{p, \beta\}))$. Then

$$\delta_{p,\beta,c} \leq \delta_{p,\beta} \tag{7.10}$$

for any c such that $B(c)$ holds. The Σ_1^{ZF} formula $\mathcal{H}$ that defines $\delta_{p,\beta}$ as a partial function of p, β uniformly owes its existence to the effective nature of deducibility in $\mathcal{L}_{\omega_1,\omega}$. The formula $\mathcal{H}$ singles out a deduction in $A_{\{p,\beta\}}$ that establishes the value of $\delta_{p,\beta}$ and can be formulated to succeed in every Σ_1 admissible A, because $p, \beta \in A$ implies $A_{\{p,\beta\}}$ is a Σ_1^A definable (uniformly) subclass of A. $\qquad\square$

8   Enumeration of Models under Weak Scattering

Let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ for some countable first-order language $\mathcal{L}$ and $T \subseteq \mathcal{L}_0$ a theory with a model. Assume T is weakly scattered as defined in Section 1. For convenience assume T mentions all formulas of $\mathcal{L}_0$; thus $\mathcal{L}_0$ and $\mathcal{L}$ are recoverable from T. Since T need not be scattered, there is no hope of enumerating theories in $L(\omega_1, T)$ whose atomic models are exactly the countable models of T. But some useful vestiges of the constructive features of scattering carry over to weak scattering, and $L(\omega_1, T)$ manages to say a great deal about the countable models of T.

First consider $\mathcal{RH}(T)$, the *raw hierarchy* for the countable models of T. On level 0 of $\mathcal{RH}(T)$, put every T_0 such that $T \subseteq T_0$ and T_0 is a finitarily consistent, ω-complete theory of $\mathcal{L}_0$. (If needed, see the beginning of Section 4 for a review.)

Suppose T_δ is on level δ of $\mathcal{RH}(T)$. Define

$$\delta- = \begin{cases} \delta - 1 & \text{if } \delta \text{ is a successor,} \\[2mm] \delta & \text{if } \delta \text{ is not a successor.} \end{cases} \tag{8.1}$$

Let $\mathcal{L}_0(T_{0-})$ be $\mathcal{L}_0$. Assume T_δ extends a unique $T_{\delta-}$ on level $\delta-$ and $\mathcal{L}_\delta(T_{\delta-})$ is countable. If all n-types ($n \geq 1$) of T_δ are principal, then $\mathcal{L}_{\delta+1}(T_\delta)$ is undefined and T_δ has no extensions on level $\delta + 1$. Otherwise, let $\mathcal{L}_{\delta+1}(T_\delta)$ be the least fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_\delta(T_{\delta-})$ and having as a member the conjunction

$$\wedge \{\mathcal{F}(\overrightarrow{x}) \mid \mathcal{F}(\overrightarrow{x}) \in p(\overrightarrow{x})\} \tag{8.2}$$

for every nonprincipal n-type $p(\overrightarrow{x})$ of T_δ ($n \geq 1$). Since T is weakly scattered, the set $\mathcal{L}_{\delta+1}(T_\delta)$ is countable.

On level $\delta + 1$ of $\mathcal{RH}(T)$ put every $T_{\delta+1}$ that extends T_δ and is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\delta+1}(T_\delta)$. Put T_λ on level λ if there is a sequence $T_\delta(\delta < \lambda)$ such that

  (a) T_δ is on level δ;
  (b) $T_\beta \subseteq T_\gamma$ if $\beta \leq \gamma$; and
  (c) $T_\lambda = \cup\{T_\delta \mid \delta < \lambda\}$.

Define $\mathcal{L}_\lambda(T_\lambda)$ to be $\cup\{\mathcal{L}_\delta(T_{\delta-}) \mid \delta < \lambda\}$.

It is straightforward to verify that $\mathcal{A}$ is a countable model of T if and only if $\mathcal{A}$ is the atomic model of T_δ for some countable δ. Define the *raw tree rank* of $\mathcal{A}$ by

$$\mathrm{rtr}(\mathcal{A}) = (\text{least } \delta)[\mathcal{A} \text{ is the atomic model of some } T_\delta]. \tag{8.3}$$

Propositions 4.5 and 4.6 hold when tr is rtr. Thus

$$\mathrm{rtr}(\mathcal{A}) \leq \mathrm{sr}(\mathcal{A}), \tag{8.4}$$

and if $L(\alpha, \langle T, \mathcal{A} \rangle)$ is Σ_1 admissible, then

$$\mathrm{rtr}(\mathcal{A}) < \alpha \longrightarrow \mathrm{sr}(\mathcal{A}) < \alpha. \tag{8.5}$$

What matters more is what can be expressed inside $L(\alpha, T)$ when $\alpha \leq \omega_1$ and $L(\alpha, T)$ is Σ_1 admissible. Let A_δ be the set of all T_δs on level δ of $\mathcal{RH}(T)$. The set A_δ will be defined by a β-bounded Δ_0^{ZF} formula (7.1), and its definition as such, denoted by $\ulcorner A_\delta \urcorner$, will belong to $L(\alpha, T)$ when $\delta < \alpha$. The fragment $\mathcal{L}_\delta(T_{\delta-})$ will be constructible from $T_{\delta-}$ via an ordinal $\rho_\delta < \alpha$ for all $T_{\delta-} \in A_{\delta-}$. The pair $\ulcorner A_\delta \urcorner$ and ρ_δ will be defined by a simultaneous $\Sigma_1^{L(\alpha,T)}$ recursion uniformly in α, that is, the same Σ_1 formula will work for all $\alpha \leq \omega_1$ such that $L(\alpha, T)$ is Σ_1 admissible.

Consider an arbitrary T_δ on level δ of $\mathcal{RH}(T)$. There exists a natural *recovery process* that can be applied to T_δ to recover the unique sequence T_γ ($\gamma < \delta$) such that

$$\begin{aligned} &T_\gamma \text{ is on level } \gamma, \\ &\gamma_1 \leq \gamma_2 \longrightarrow T_{\gamma_1} \subseteq T_{\gamma_2}, \text{ and} \\ &T_\lambda = \cup\{T_\gamma \mid \gamma < \lambda\} \text{ for all limit } \lambda \leq \delta. \end{aligned} \tag{8.6}$$

The recovery proceeds as follows. It begins with T_0 is $T_\delta \cap \mathcal{L}_0$. If γ is a successor, then

$$T_\gamma = T_\delta \cap \mathcal{L}_\gamma(T_{\gamma-}). \tag{8.7}$$

If γ is a limit, then $T_\gamma = \cup\{T_\beta \mid \beta < \lambda\}$.

The recovery process can be used to decide whether or not an arbitrary set c is a theory on level δ of $\mathcal{RH}(T)$. The answer is yes if and only if c passes the following tests at all levels $\gamma \leq \delta$.

Level 0   Let c_0 be $c \cap \mathcal{L}_0$; c_0 is an extension of T and a finitarily consistent, ω-complete theory of $\mathcal{L}_0$.

Level $\gamma + 1 \leq \delta$ Let $\mathcal{L}_{\gamma+1}(c_\gamma)$ be the least fragment extending $\mathcal{L}_\gamma(c_{\gamma-})$ and having as a member the conjunction

$$\wedge\{\mathcal{F}(\vec{x}) \mid \mathcal{F}(\vec{x}) \in p(\vec{x})\} \tag{8.8}$$

for every nonprincipal n-type $p(\vec{x})$ of $c_{\gamma-}$. Let $c_{\gamma+1}$ be $c \cap \mathcal{L}_{\gamma+1}(c_\gamma)$. $c_{\gamma+1}$ extends c_γ and is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\gamma+1}(c_\gamma)$.

Level λ (limit) $\leq \delta$ Let c_λ be $\cup\{c_\gamma \mid \gamma < \lambda\}$; let $\mathcal{L}_\lambda(c_\lambda)$ be $\cup\{\mathcal{L}_\gamma(c_{\gamma-}) \mid \gamma < \lambda\}$.

In short, c is a theory on level δ of $\mathcal{RH}(T)$ if and only if c satisfies the recovery process on all levels $\gamma \leq \delta$ and $c = c_\delta$. It will follow below that A_δ is β-bounded Δ_0^{ZF} definable (7.1), where β is large enough to define the recovery process.

An effective version of the recovery process is woven into the $\Sigma_1^{L(\alpha,T)}$ recursive definitions of ρ_δ and $\ulcorner A_\delta \urcorner$ for $0 < \delta < \alpha$. The set $\mathcal{L}_\delta(T_{\delta-})$ is constructible from

$T_{\delta-}$ via the ordinal ρ_δ for all $T_{\delta-} \in A_{\delta-}$, and $\ulcorner A_\delta \urcorner$ is a β-bounded Δ_0^{ZF} definition of A_δ. The definition $\ulcorner A_\delta \urcorner$ specifies the value of β and the Δ_0^{ZF} formula.

Stage 0 $\mathcal{L}_0(T_{0-})$ is $\mathcal{L}_0$; A_0 is the set of all finitarily consistent, ω-complete theories of $\mathcal{L}_0$ extending T. Since $\mathcal{L}_0$ is recoverable from T, the set A_0 is β-bounded Δ_0^{ZF} definable with $\beta = 0$ and parameter T.

Stage $\delta + 1$ Assume the recursion has produced sequences

$$\{\rho_\gamma \mid \gamma \leq \delta\}, \ \{\ulcorner A_\gamma \urcorner \mid \gamma \leq \delta\} \in L(\alpha, T) \tag{8.9}$$

such that $\ulcorner A_\gamma \urcorner$ is a β-bounded Δ_0^{ZF} definition of A_γ, and $\mathcal{L}_\gamma(T_{\gamma-})(\gamma \leq \delta)$ is first-order definable over

$$L[\rho_\gamma, \mathcal{L}_0; T_{\gamma-}]. \tag{8.10}$$

(The definition of (8.10) follows (7.1).) Consider an arbitrary $T_\delta \in A_\delta (\delta > 0)$. Use the recovery process to construct the unique $T_{\delta-} \in A_{\delta-}$ such that

$$T_{\delta-} \subseteq T_\delta \subseteq \mathcal{L}_\delta(T_{\delta-}). \tag{8.11}$$

The recovery is effective thanks to the sequence $\rho_\gamma (\gamma \leq \delta)$. Now $\mathcal{L}_{\delta+1}(T_\delta)$ can be defined as above (8.2) but with an effective twist. Let ST_δ be the set of all n-types $(n \geqslant 1)$ of T_δ. Since T is weakly scattered, Corollary 3.2 implies

$$ST_\delta \in L(\omega_1^{T_\delta}, T_\delta), \tag{8.12}$$

the least Σ_1 admissible set with T_δ as a member. Let

$$\gamma_{T_\delta} = (\text{least } \gamma)[ST_\delta \in L(\gamma, T_\delta)]. \tag{8.13}$$

By Theorem 3.3, the ordinal γ_{T_δ}, as a function of T_δ, is uniformly Σ_1; the same Σ_1^{ZF} formula singles out γ_{T_δ} in $L(\omega_1^{T_\delta}, T_\delta)$ for every $T_\delta \in A_\delta$ and for all δ. By Theorem 7.1(i), there is a γ_δ such that

$$(\forall T_\delta \in A_\delta)[\gamma_{T_\delta} \leq \gamma_\delta < \alpha]. \tag{8.14}$$

Hence $ST_\delta \in L(\gamma_\delta, T_\delta)$ for all $T_\delta \in A_\delta$. Theorem 7.1(ii) implies that γ_δ, as a function of δ, has a uniform Σ_1 definition utilizing the parameters occurring in $\ulcorner A_\delta \urcorner$ and the uniform Σ_1 definition of γ_{T_δ}. Any n-type $p(\overrightarrow{x}) \in ST_\delta$ for any $T_\delta \in A_\delta$ is constructible from T_δ via some ordinal less than γ_δ.

A set $\mathcal{P}_\delta$ of first-order definitions can be assembled at level γ_δ of $L(\alpha, T)$ as follows. Let

$$\{p_j^{\mathcal{T}_\delta} \mid j \in \mathcal{J}_\delta\} \tag{8.15}$$

be the set of all first-order definitions over $L(\gamma, T)$ for all $\gamma < \gamma_\delta$ with parameter $\mathcal{T}_\delta$. For each $T_\delta \in A_\delta$, the object $p_j(T_\delta)$ is the set defined by $p_j(\mathcal{T}_\delta)$ when the parameter $\mathcal{T}_\delta$ is assigned the value T_δ. The set (8.15) has a natural well-ordering W_δ definable at level γ_δ, since each $p_j^{\mathcal{T}_\delta}$ is specified by its level $\gamma < \gamma_\delta$ and its Gödel number $e < \omega$ as a formula of ZF. The type $d_\delta(\mathcal{T}_\delta)$, the *default type for $\mathcal{T}_\delta$*, is defined by its action on $T_\delta \in A_\delta$:

$$j(T_\delta) \ = \ (\text{least } j \text{ in sense of } W_\delta)[p_j(T_\delta) \text{ is an } n\text{-type of } T_\delta]; \tag{8.16}$$

$$d_\delta(\mathcal{T}_\delta) \ = \ p_{j(T_\delta)}(T_\delta). \tag{8.17}$$

The formula $p_j^{\mathcal{T}_\delta}$ is a slight variant of $p_j(\mathcal{T}_\delta)$ and is defined by its action on $T_\delta \in A_\delta$.

$$
p_j^{T_\delta} = \begin{cases} p_j(T_\delta) & \text{if } p_j(T_\delta) \text{ is an } n\text{-type of } T_\delta, \\[2ex] d_\delta(T_\delta) & \text{the default type, otherwise.} \end{cases} \tag{8.18}
$$

Let $\mathcal{P}_\delta = \{p_j^{\mathcal{T}_\delta} \mid j \in \mathcal{J}_\delta\}$. Then

1. for all $T_\delta \in A_\delta$ and $p(\overrightarrow{x}) \in ST_\delta$, there is a $j \in \mathcal{J}_\delta$ such that $p_j^{T_\delta}$ defines $p(\overrightarrow{x})$ at level γ_δ of $L(\alpha, T)$, and
2. $p_j^{T_\delta} \in ST_\delta$ for all $T_\delta \in A_\delta$ and all $j \in \mathcal{J}_\delta$.

It can happen for some $T_\delta \in A_\delta$ and $j, k \in \mathcal{J}_\delta$ that $j \neq k$ but $p_j^{T_\delta} = p_k^{T_\delta}$. Such repetitions are the price paid to have $\mathcal{P}_\delta \in L(\gamma_\delta + 1, T)$.

The ordinal $\rho_{\delta+1} < \alpha$ is chosen just large enough to develop the sequence $\rho_\gamma \, (\gamma \leq \delta)$ needed for the recovery of $T_{\delta-}$ from $T_\delta (\delta > 0)$ and the ordinal γ_δ needed to assemble $\mathcal{P}_\delta$. The set $\mathcal{L}_{\delta+1}(T_\delta)$ is first-order definable over $L[\rho_{\delta+1}, \mathcal{L}_0; T_\delta]$; its definition begins with $\mathcal{L}_\delta(T_{\delta-})$, adds the conjunction of all formulas in $p_j^{\mathcal{T}_\delta}$ for each $p_j^{\mathcal{T}_\delta} \in \mathcal{P}_\delta$, and closes under the finitary operations that generate a fragment of $\mathcal{L}_{\omega_1, \omega}$.

To complete stage $\delta+1$, construe $A_{\delta+1}$ to be the set of all x such that the effective version of the recovery process applied to x reports that x is a theory on level $\delta + 1$ of $\mathcal{RH}(T)$. The effective version uses the sequence $\rho_\gamma \, (0 < \gamma \leq \delta + 1)$ to define $\mathcal{L}_\gamma(T_{\gamma-})$ from $T_{\gamma-}$ for all $T_{\gamma-} \in A_{\gamma-}$. Thus $A_{\delta+1}$ is β-bounded Δ_0^{ZF} definable with β equal to $\rho_{\delta+1}$, and $\ulcorner A_{\delta+1} \urcorner \in L(\alpha, T)$. The parameter specified by $\ulcorner A_{\delta+1} \urcorner$ is T.

Stage λ (limit)   Assume for $0 < \gamma < \lambda$ that $\mathcal{L}_\gamma(T_{\gamma-})$ is constructible from $T_{\gamma-}$ via ρ_γ for all $T_{\gamma-} \in A_{\gamma-}$. Use the effective version of the recovery process to define A_λ as a β-bounded Δ_0^{ZF} class. For $T_\gamma \in A_\lambda$, effectively recover the unique sequence $T_\gamma \, (\gamma < \lambda)$ such that T_λ is $\cup\{T_\gamma \mid \gamma < \lambda\}$, and then define $\mathcal{L}_\lambda(T_\lambda)$ to be $\cup\{\mathcal{L}_\gamma(T_{\gamma-}) \mid 0 < \gamma < \lambda\}$.

Makkai [10] showed that if T is a counterexample to Vaught's Conjecture, then T has a model of cardinality ω_1 that is $\mathcal{L}_{\infty, \omega}$ equivalent to a countable model. The following are variants of his results.

Suppose A is a countable Σ_1 admissible set and $T \in A$. Assume $T \subseteq \mathcal{L}_0$, $\mathcal{L}_0$ is a countable fragment of $\mathcal{L}_{\omega_1, \omega}$, and $\mathcal{L}$ is a countable first-order language. Also assume every symbol of $\mathcal{L}$ is mentioned in T so that $\mathcal{L}$ is recoverable from T. Let $\mathcal{L}'$ denote an arbitrary fragment of $\mathcal{L}_{\omega_1, \omega}$ that extends $\mathcal{L}$, and T' an arbitrary finitarily consistent, ω-complete theory contained in $\mathcal{L}'$ and extending T. Call T *weakly scattered in A* if and only if $ST' \in A$ for all $T' \in A$. According to Theorem 3.3, we have the following.

Theorem 8.1   *Suppose $\mathcal{A}$ is a countable model of T. Assume T is weakly scattered in $L(\omega_1^{T, \mathcal{A}}, \langle T, \mathcal{A} \rangle)$, and*

$$
\mathrm{sr}(\mathcal{A}) \geq \omega_1^{T, \mathcal{A}}.
$$

Then $\mathcal{A}$ is $\mathcal{L}_{\infty, \omega}$ equivalent to a model of T of cardinality ω_1.

Proof   Let $\alpha = \omega_1^{T, \mathcal{A}}$. Thus $\omega_1^{\mathcal{A}} = \alpha$, since $\omega_1^{\mathcal{A}} + 1 \geq \mathrm{sr}(\mathcal{A})$. Let $T_\beta^{\mathcal{A}} (\beta \leq \mathrm{sr}(\mathcal{A}))$ be the Scott analysis of $\mathcal{A}$ as defined in Section 2. By Theorem 3.3, $ST_\beta^{\mathcal{A}} \in L(\alpha, \langle T, \mathcal{A} \rangle)$

(and so $T_\beta^{\mathcal{A}}$ has a countable atomic model) for all β such that $\beta + 1 < \mathrm{sr}(\mathcal{A})$. The set Z is $\Sigma_1^{L(\alpha, \langle T, \mathcal{A} \rangle)}$ and consists of the following sentences:

(Z1)   the atomic diagram (in the sense of $\mathcal{L}_{\omega_1, \omega}$) of $L(\alpha, \langle T, \mathcal{A} \rangle)$;

(Z2) $\underline{d}$ is a countable ordinal and $\underline{d} \geq \delta$ (all $\delta < \omega_1^{T, \mathcal{A}}$);

(Z3) $\forall y[y < \underline{d} \rightarrow T_y^{\mathcal{A}}$ has a countable atomic model];

(Z4)   axioms of Σ_1 admissibility.

The set Z is consistent since it can be modeled by V (the real world). Every model of Z is an end extension of $L(\alpha, \langle T, \mathcal{A} \rangle)$. Let M be a model of Z that omits α. Thus M has nonstandard ordinals greater than every ordinal less than α. Hence $\mathrm{sr}(\mathcal{A}) \geq \alpha$ in V and $\alpha \notin M$, so $\mathrm{sr}(\mathcal{A}) \geq \gamma$ for some nonstandard $\gamma \in M$.

Now work inside M. Let $T_\delta^{\mathcal{A}}$ ($\delta \leq \gamma$) be the Scott analysis of $\mathcal{A}$ up to level γ. Choose a nonstandard $\beta < \gamma$. Then $T_\beta^{\mathcal{A}}$ has a countable atomic model $\mathcal{A}_\beta$. There is a map

$$i_{\beta\gamma} : \mathcal{A}_\beta \rightarrow \mathcal{A} \qquad (8.19)$$

that is elementary with respect to all formulas of $\mathcal{L}_\beta^{\mathcal{A}}$ (defined in Section 2). Note that $i_{b\gamma}$ is not onto, since $\mathcal{A}_\beta$ is not isomorphic to $\mathcal{A}$ in M.

But $\mathcal{A}_\beta$ is isomorphic to $\mathcal{A}$ in V. Now $\omega_1^{\mathcal{A}_\beta} \leq \alpha$ since $\alpha \notin M$; also $\mathrm{sr}(\mathcal{A}_\beta) \geq \delta$ for all $\delta < \alpha$. Hence $\mathrm{sr}(\mathcal{A}_\beta) \geq \alpha$, and so $\omega_1^{\mathcal{A}_\beta} \geq \alpha$. Thus both $\mathcal{A}_\beta$ and $\mathcal{A}$ are homogeneous models of $T_\alpha^{\mathcal{A}}$ by (2.6). To see they realize the same types of $T_\alpha^{\mathcal{A}}$, choose $p_\alpha \in ST_\alpha^{\mathcal{A}}$ and first suppose $\mathcal{A}_\beta \models p_\alpha(\bar{b})$. In M, note that $\mathcal{A}_\beta \models p_\beta(\bar{b})$ for some type p_β of $T_\beta^{\mathcal{A}}$ and that $\mathcal{A} \models p_\gamma(i_{\beta\gamma}(\bar{b}))$ for some type p_γ of $T_\gamma^{\mathcal{A}}$. Then

$$p_\alpha \subseteq p_\beta \subseteq p_\gamma, \qquad (8.20)$$

since $i_{\beta\gamma}$ is $\mathcal{L}_\beta^{\mathcal{A}}$ elementary. Hence $\mathcal{A} \models p_\alpha(i_{\beta\gamma}(\bar{b}))$. It follows that

$$i_{\beta\gamma} \text{ is } \mathcal{L}_{\omega_1, \omega} \text{ elementary}, \qquad (8.21)$$

since the types of $T_\alpha^{\mathcal{A}}$ realized in $\mathcal{A}_\beta$ are atoms of $\mathcal{L}_{\omega_1, \omega}$.

Now suppose $\mathcal{A} \models p_\alpha(\bar{a})$. In M, the tuple $\bar{a}$ realizes p_γ in $\mathcal{A}$, a type of $T_\gamma^{\mathcal{A}}$. Choose a nonstandard $\delta < \beta$. Let p_β be the restriction of p_γ to $\mathcal{L}_\beta^A$, and let p_δ be the restriction to $\mathcal{L}_\delta^{\mathcal{A}}$. Then $p_\alpha \subseteq p_\delta \subseteq p_\beta \subseteq p_\gamma$. So

$$\mathcal{A} \models \exists \bar{x} \, p_\delta(\bar{x}). \qquad (8.22)$$

But then $\exists \bar{x} \, p_\delta(\bar{x}) \in T_{\delta+1} \subseteq T_\beta$, so p_δ, hence p_α, is realized in $\mathcal{A}_\beta$.

Thanks to the above there exist structures $\mathcal{B}_0$ and $\mathcal{B}_1$, both isomorphic to $\mathcal{A}$, such that $\mathcal{B}_0 \subsetneqq \mathcal{B}_1$ and the inclusion map i is $\mathcal{L}_{\omega_1, \omega}$ elementary. A strictly expanding $\mathcal{L}_{\omega_1, \omega}$ elementary chain $\mathcal{B}_\delta (\delta \leq \omega_1)$ is defined by iterating i. For $\delta < \omega_1$, assume $\mathcal{B}_\delta$ is isomorphic to $\mathcal{A}$. Then enlarge $\mathcal{B}_\delta$ to $\mathcal{B}_{\delta+1}$, another copy of $\mathcal{A}$. For limit $\lambda \leq \omega_1$, let $\mathcal{B}_\lambda$ be the union of the $\mathcal{B}_\delta$s ($\delta < \lambda$). $\mathcal{B}_{\omega_1}$ is an $\mathcal{L}_{\omega_1, \omega}$ elementary extension of $\mathcal{B}_0$, hence $\mathcal{L}_{\omega_1, \omega}$-equivalent to $\mathcal{A}$, consequently $\mathcal{L}_{\infty, \omega}$-equivalent to $\mathcal{A}$. $\square$

Corollary 8.2   *Suppose T is weakly scattered. For each $\beta < \omega_1^T$, assume T has a model of Scott rank $\geq \beta$. Then T has a countable model $\mathcal{A}$ such that*

$$\mathrm{sr}(\mathcal{A}) \geq \omega_1^{T, \mathcal{A}} = \omega_1^T,$$

and every such $\mathcal{A}$ is $\mathcal{L}_{\infty,\omega}$ equivalent to a model of T of cardinality ω_1.

9 Bounds on Weakly Scattered Theories

Once again let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ for some countable first-order language $\mathcal{L}$ and $T \subseteq \mathcal{L}_0$ a weakly scattered theory with a model. Assume $L(\alpha, T)$ is Σ_1 admissible. Consider B_α, a $\Delta_1^{L(\alpha,T)}$ set of sentences designed so that every model of B_α constitutes a node on level α of $\mathcal{RH}(T)$, the raw hierarchy for T. The axioms of B_α are

1. $T \subseteq T_0$ and T_0 is a finitarily consistent, ω-complete theory of $\mathcal{L}_0$,
2. T_δ has a nonprincipal n-type for some n (all $\delta < \alpha$),
3. $T_\delta \subseteq T_{\delta+1}$ and $T_{\delta+1}$ is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\delta+1}(T_\delta)$ (all $\delta < \alpha$),
4. $T_\lambda = \cup\{T_\delta \mid \delta < \lambda\}$ and $\mathcal{L}_\lambda(T_\lambda) = \cup\{\mathcal{L}_\delta(T_{\delta-}) \mid \delta < \lambda\}$ (all limit $\lambda < \alpha$).

Then B_α is $\Delta_1^{L(\alpha,T)}$ because Section 8 shows how to construct $\mathcal{L}_\delta(T_{\delta-})$ from $T_{\delta-}$ via the ordinal ρ_δ defined by a $\Sigma_1^{L(\alpha,T)}$ recursion on $\delta < \alpha$.

Sets $\mathcal{P}_\delta$ and $\mathcal{J}_\delta$ were defined just after (8.14). Define 'p is on level δ' by

$$p = p_j^{\mathcal{T}_\delta} \text{ for some } j \in \mathcal{J}_\delta. \tag{9.1}$$

A *split at level δ* is a sentence of the form p is on level δ, and there exist r and r' on level $\delta + 1$ such that $r \neq r'$ and both r and r' extend p. The sentence in abbreviated form is $\langle p, r, r' \rangle$. A split is a sentence of $\mathcal{L}_{\omega_1,\omega} \cap L(\alpha, T)$, because $\mathcal{P}_\delta$, $\mathcal{P}_{\delta+1} \in L(\alpha, T)$. The triple $\langle p, r, r' \rangle$ is a *k-split* if p has arity k. Let K be a set of k-splits. The set K is *unbounded* if and only if

$$\forall \beta < \alpha (\exists \delta > \beta)[K \text{ has a } k\text{-split on level } \delta]. \tag{9.2}$$

K has the *predecessor property* if and only if there is a partial function $f(p, \gamma)$ such that if $\gamma < \delta$ and $\langle p, r, r' \rangle \in K$ and asserts p splits at level δ, then $f(p, \gamma)$ is defined and belongs to $\mathcal{J}_\gamma$, and

$$B_\alpha \vdash \left[\langle p, r, r' \rangle \longrightarrow \left(p_{f(p,\gamma)}^{\mathcal{T}_\gamma} \text{ is extended by } p \right) \right]. \tag{9.3}$$

If such an f exists, then there is one that is $\Sigma_1^{L(\alpha,T)}$ definable, since the $\Delta_1^{L(\alpha,T)}$ definability of B_α implies the deduction claimed by (9.3) can be found in $L(\alpha, T)$.

The *effective k-splitting hypothesis* holds for T at α if and only if there exists an unbounded $\Delta_1^{L(\alpha,T)}$ set K of k-splits such that K has the predecessor property and $B_\alpha \cup K$ is consistent (in the sense of $\mathcal{L}_{\omega_1,\omega}$ restricted to $L(\alpha, T)$)) if B_α is. Consider Makkai's example [11] (also [8]) mentioned in Section 1. It can be formulated as a fragment $\mathcal{L}_0$ and a theory $T_M \subseteq \mathcal{L}_0$, both arithmetically definable, with the following properties:

(1) T_M is not weakly scattered;
(2) every countable model $\mathcal{A}$ of T_M has Scott rank at most $\omega_1^{\mathcal{A}}$;
(3) for every countable Σ_1 admissible $L(\alpha)$, there is a countable model $\mathcal{A}$ of T_M such that $\omega_1^{\mathcal{A}} = \alpha = \text{sr}(\mathcal{A})$.

Despite (1) it is possible to develop a crude hierarchy for T_M with a superficial resemblance to the raw hierarchy $\mathcal{RH}(T)$ of Section 8. For $\delta < \omega_1$ put theory $T' \supseteq T_M$ on level δ if there exists a countable model $\mathcal{A}$ of T_M such that $\text{sr}(\mathcal{A}) = \delta$ and $T' = T_{\text{sr}(\mathcal{A})}^{\mathcal{A}}$ (as defined in Section 2). Since T_M is not weakly scattered, it is not possible to give

a bounded description of all types associated with all theories on level δ, as was done with $\mathcal{P}_\delta$ in Section 8. Nonetheless, some of the types on level δ have properties that lend credence to the effective k-splitting hypothesis. The model $\mathcal{A}$ of (3) above is a tree with ω many levels and infinite paths. Some nodes of $\mathcal{A}$ have foundation rank (fr) $< \infty$. Foundation rank $\omega\delta + m$ corresponds to atoms of $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ of rank δ. Associated with level δ of $\mathcal{CH}(T_M)$, the crude hierarchy for T_M, are types of the form

$$x \text{ is on level } \delta \text{ of } \mathcal{A} \text{ and } \mathrm{fr}(x) \geq \omega\delta + m \tag{9.4}$$

that split on level $\delta + 1$ of $\mathcal{CH}(T)$. On level $\gamma < \delta$, (9.4) has a predecessor similar to (9.4) with δ replaced by γ.

Theorem 9.1   *Suppose T is weakly scattered and $L(\alpha, T)$ is countable and Σ_2 admissible. For each $\beta < \alpha$, suppose T has a model of Scott rank at least β. If for some k, the effective k-splitting hypothesis holds for T at α, then T has a countable model $\mathcal{A}$ such that*

$$\omega_1^{\mathcal{A}} = \alpha \text{ and } \mathrm{sr}(\mathcal{A}) = \alpha + 1.$$

Proof   By Barwise Compactness, T has a model $\mathcal{A}$ such that $L(\alpha, \langle T, \mathcal{A}\rangle)$ is Σ_1 admissible and $\mathrm{sr}(\mathcal{A}) \geq \alpha$. Then $\mathrm{rtr}(\mathcal{A}) \geq \alpha$ by (8.5) and so B_α is consistent. Let K be an unbounded $\Delta_1^{L(\alpha, T)}$ set of k-splits with a $\Sigma_1^{L(\alpha, T)}$ predecessor function $f(\gamma, p)$. A model of $B_\alpha \cup K$ is constructed so that T_α has a nonprincipal type q_α and the structure

$$L[\alpha, T; T_\alpha, q_\alpha] \tag{9.5}$$

is Σ_1 admissible with respect to Σ_1 formulas that include T_α and q_α as atomic predicates. Then, as in the type omitting proof of Theorem 6.1, T has a model $\mathcal{A}_1$ realizing q_α and such that $\omega_1^{\mathcal{A}_1} = \alpha$. The universe of (9.5) is the result of iterating first-order definability through the ordinals less than α starting with T and with T_α, q_α as additional atomic predicates. The construction of (9.5) is Henkinesque and gradually decides all sentences of rank less than α in a standard language $\mathcal{L}_{\alpha, T} \in \Delta_1^{L(\alpha, T)}$ that names all elements of (9.5) and is able to express how each one is defined from those of lower definability rank. The language $\mathcal{L}_{\alpha, T}$ does not have symbols T_α or q_α but does have symbols T_β and q_β for all $\beta < \alpha$. There is one twist. The Σ_1 admissibility of (9.5) is not obtained by an effective type omitting argument that omits α as in the proof of Theorem 6.1 but by direct manipulation of ranked sentences of $\mathcal{L}_{\alpha, T}$. The twist avoids Henkin constants.

Let S_n be the set of sentences chosen by the end of stage n. S_n will be $\Sigma_2^{L(\alpha, T)}$ definable. S_0 requires some preparation. Consider $p_j^{\mathcal{T}_\gamma}$ for some $j \in \mathcal{J}_\gamma$. $p_j^{\mathcal{T}_\gamma}$ is said to be *K-unbounded* if the set of all δ such that

$$\exists \langle p, r, r'\rangle \big[\langle p, r, r'\rangle \in K, p \text{ is on level } \delta, f(p, \gamma) = p_j^{\mathcal{T}_\gamma}\big] \tag{9.6}$$

is unbounded in α. Thus $B_\alpha \cup K$ implies $p_j^{\mathcal{T}_\gamma}$ has unboundedly many extensions that split in K. K-unboundedness is a $\Pi_2^{L(\alpha, T)}$ property. K-bounded means 'not K-unbounded'.

Claim 9.2   *For all γ there is a K-unbounded type on level γ.*

Proof of claim   Suppose not. Then for each $j \in \mathcal{J}_\gamma$, there is a least β_j such that for all $\delta \geq \beta_j$, (9.6) is false. The ordinal β_j, as a function of j, is $\Sigma_2^{L(\alpha, T)}$, hence

bounded by some $\beta_\infty < \alpha$. But then K is bounded by β_∞. A set $U \subseteq K$ is said to be *bounded* if

$$\exists \beta < \alpha (\forall \delta > \beta)[U \text{ does not have a } k\text{-split on level } \delta].$$

$\square$

Definition 9.3 (Definition of S_0)   Start with $B_\alpha \cup K$. Add

1. sentences of $\mathcal{L}_{\alpha,T}$ that express how each element of (9.5) is defined from elements of lower rank;
2. q_β is a type on level $\beta (\beta < \alpha)$;
3. q_β is extended by $q_\gamma (\beta < \gamma < \alpha)$;
4. $q_\beta \neq p (\beta < \alpha$ and p is K-bounded).

Note that 'q_β is a type on level β' is a ranked sentence, in particular, a disjunction, by the remarks following (8.14).

S_0 is $\Sigma_2^{L(\alpha,T)}$ definable since K-boundedness is $\Sigma_2^{L(\alpha,T)}$. To check the consistency of S_0, let M be a model of $B_\alpha \cup K$ that specifies the structure of $L(\alpha, T; T_\alpha)$ but says nothing about q_γ for any $\gamma < \alpha$. Fix $\tau < \alpha$. Suppose $\gamma < \tau$; then M can be interpreted as a model of those sentences in S_0 that mention q_γ only for $\gamma < \tau$. Choose a K-unbounded p_τ on level τ with the aid of Claim 9.2. Define

$$U_\tau = \{s \mid \exists t, t'[\langle s, t, t'\rangle \in K] \text{ and } f(s, \tau) = p_\tau\}, \tag{9.7}$$
$$U_\gamma^r = \{s \mid s \in U_\tau \wedge f(s, \gamma) = r\}(\gamma < \tau). \tag{9.8}$$

Fix $\gamma < \tau$. There must be a K-unbounded r on level γ. Suppose not. Then U_γ^r is bounded for every r on level γ. But

$$U_\tau = \cup\{U_\gamma^r \mid r \text{ is on level } \gamma\}. \tag{9.9}$$

Hence U_τ is bounded by the Σ_2 admissibility argument used to prove (9.2), and so p_τ is K-bounded.

For each $\gamma < \tau$, choose a K-unbounded r_γ on level γ. To see that for each $\gamma < \tau$,

$$B_\alpha \cup K \vdash r_\gamma \text{ is extended by } p_\tau, \tag{9.10}$$

let $s \in U_\gamma^{r_\gamma}$. Then $s \in U_\tau$. Assume $B_\alpha \cup K$. Then s extends $f(s, \tau) = p_\tau$ and s extends $f(s, \gamma) = r_\gamma$. Hence p_τ extends r_γ.

It follows from (9.10) that

$$B_\alpha \cup K \vdash r_{\gamma_1} \text{ is extended by } r_{\gamma_2} \tag{9.11}$$

when $\gamma_1 < \gamma_2 < \tau$. Now M, as promised above, can be interpreted as a model of that part of S_0 that mentions q_γ only for $\gamma < \tau$ by setting the interpretation of q_γ in M equal to that of r_γ.

Definition 9.4 (Definition of S_{n+1})   Assume S_n is consistent and $\Sigma_2^{L(\alpha,T)}$. There are two cases.

Case (a)   Suppose $\mathcal{F} = \vee\{\mathcal{F}_i \mid i \in I\}$ is a ranked sentence such that $S_n \cup \{\mathcal{F}\}$ is consistent. S_{n+1} is $S_n \cup \{\mathcal{F}_{i'}\}$ for some $i' \in I$ such that $S_n \cup \{\mathcal{F}_{i'}\}$ is consistent.

Case (b)   The purpose of this case is to establish Δ_0 bounding, hence Σ_1 replacement, for (9.5). Let $\mathcal{D}(x, y)$ be a Δ_0^{ZF} formula with constants naming elements of (9.5). Fix $\rho < \alpha$, and regard $\mathcal{D}(x, y)$ as possibly defining a many-valued function $d(x)$ from ρ into α that is Δ_0 in the sense of (9.5). For each $\delta < \rho$, define

$$H_\delta = \{\neg D(\delta, \gamma) \mid \gamma < \alpha\}. \tag{9.12}$$

Subcase (b1)   Suppose there is a $\delta < \rho$ such that $S_n \cup H_\delta$ is consistent. Let δ' be such a δ, and put S_{n+1} equal to $S_n \cup H_{\delta'}$. Then $d(\delta')$ will be undefined.

Subcase (b2)   Suppose (b1) fails. Then for each $\delta < \rho$,

$$S_n \vdash \vee\{D(\delta, \gamma) \mid \gamma < \alpha\}; \tag{9.13}$$

so by Barwise Compactness there is a $c(\delta) < \alpha$ such that

$$S_n \vdash \vee\{D(\delta, \gamma) \mid \gamma < c(\delta)\}. \tag{9.14}$$

$c(\delta)$ can be defined via deductions from S_n as a $\Sigma_2^{L(\alpha,T)}$ function of δ. Let c be $\sup\{c(\delta) \mid \delta < \rho\}$. Then $c < \alpha$ and $d(\delta)(\delta < \rho)$ will be bounded by c.

Define $S = \cup\{S_n \mid n < \omega\}$. By Case (a), S specifies (9.5). q_α is a nonprincipal type of T_α, because for every $\beta < \alpha$, S_0 and Claim 9.2 compel q_β to be K-unbounded and consequently to split. (An instance of Case (a) results in the choice of a K-unbounded p such that $(q_\beta = p)$ belongs to S.) By Case (b), (9.5) is Σ_1 admissible. It follows, as in the proof of Theorem 6.1, that T has a model $\mathcal{A}_1$ that realizes q_α and such that $\omega_1^{\mathcal{A}_1} = \alpha$. Hence $\mathrm{sr}(\mathcal{A}) = \alpha + 1$. □

Corollary 9.5 (Bounding)   *Suppose T is weakly scattered and for some k satisfies the effective k-splitting hypothesis at α. If $L(\alpha, T)$ is Σ_2 admissible and*

$$(\forall \text{ countable } \mathcal{A})[\mathcal{A} \models T \longrightarrow \mathrm{sr}(\mathcal{A}) \leq \omega_1^{\mathcal{A}}], \tag{9.15}$$

then

$$(\exists \beta < \alpha)(\forall \mathcal{A})[\mathcal{A} \models T \longrightarrow \mathrm{sr}(\mathcal{A}) < \beta]. \tag{9.16}$$

10 Further Results and Open Questions

Weakening the assumption of effective k-splitting in Section 9 is under study. At this writing it appears likely that the predecessor (9.3) property can be dropped from the assumption: all that is needed is an unbounded $\Delta_1^{L(\alpha,T)}$ set of k-splits consistent with B_α; then the existence of a predecessor function can be proved. (See Goddard [4].) There is a price to pay: the type structure $p_j^{T_\delta}$ ($\delta < \alpha$) of a weakly scattered theory T has to be treated with greater delicacy. A further weakening, less likely but more than plausible, is to rule out the existence of RN-models of T. Call $\mathcal{A}$ an *RN-model* of T if and only if (i) $\mathrm{sr}(\mathcal{A}) = \omega_1^{\mathcal{A}}$, (ii) $T_{\omega_1^{\mathcal{A}}}^{\mathcal{A}}$ is ω-categorical, and (iii) for each n there is a $\beta < \omega_1^{\mathcal{A}}$ such that each principal n-type of $T_{\omega_1^{\mathcal{A}}}^{\mathcal{A}}$ of arity n is generated by a formula of rank less than β. ($T_{\omega_1^{\mathcal{A}}}^{\mathcal{A}}$ is defined in Section 2.) Makkai [11] produces an $\mathcal{A}$ that satisfies (i) and (ii) but not (iii).

It appears that iterated forcing has a role to play above and also in the construction of an α-saturated model of T when T is weakly scattered and has countable models of unbounded Scott rank. But that is another story. (See Chan [3].)

Notes

1. [15] was a hasty writeup of a talk given at the 1971 meeting of the International Congress of Logic, Methodology and Philosophy of Science. Some details absent from [15] but needed here are presented below.

2. Strictly speaking, the relativization is to the transitive closure of $\mathcal{A}$.

3. As final corrections were being made to the galleys, it was discovered that Theorem 4.9 here is also reproved as Theorem 3.8, p. 84 of Baldwin [1] in this special issue.

References

[1] Baldwin, J., "The Vaught Conjecture: Do Uncountable Models Count?," *Notre Dame Journal of Formal Logic*, vol. 48 (2007), pp. 79–92 (electronic). 30

[2] Barwise, J., *Admissible Sets and Structures. An Approach to Definability Theory*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1975. Zbl 0316.02047. MR 0424560. 9

[3] Chan, A., *Models of High Rank for Weakly Scattered Theories*, Ph.D. thesis, Massachusetts Institute of Technology, 2006. 29

[4] Goddard, C., *Improving a Bounding Result for Weakly-Scattered Theories*, Ph.D. thesis, Massachusetts Institute of Technology, 2006. 29

[5] Grilliot, T. J., "Omitting types: Application to recursion theory," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 81–89. Zbl 0244.02016. MR 0344099. 6, 14

[6] Harnik, V., and M. Makkai, "A tree argument in infinitary model theory," *Proceedings of the American Mathematical Society*, vol. 67 (1977), pp. 309–14. Zbl 0384.03019. MR 0472506. 7, 14

[7] Keisler, H. J., *Model Theory for Infinitary Logic. Logic with Countable Conjunctions and Finite Quantifiers*, vol. 62 of *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Co., Amsterdam, 1971. Zbl 0222.02064. MR 0344115. 6, 14

[8] Knight, J. F., and J. M. Young, "Computable structures of rank ω_1^{CK}," preprint, 2004. 6, 26

[9] Knight, R. W., "The Vaught Conjecture: A Counterexample," manuscript, 2002. http://www.maths.ox.ac.uk/ knight/stuff/example2.ps 7

[10] Makkai, M., "An 'admissible' generalization of a theorem on countable Σ_1^1 sets of reals with applications," *Annals of Pure and Applied Logic*, vol. 11 (1977), pp. 1–30. Zbl 0376.02031. MR 0491142. 9, 24

[11] Makkai, M., "An example concerning Scott heights," *The Journal of Symbolic Logic*, vol. 46 (1981), pp. 301–18. Zbl 0501.03018. MR 613284. 6, 7, 19, 26, 29

[12] Millar, J., and G. E. Sacks, "Models with high Scott rank," in preparation. 7

[13] Morley, M., "The number of countable models," *The Journal of Symbolic Logic*, vol. 35 (1970), pp. 14–18. Zbl 0196.01002. MR 0288015. 7, 11, 15

[14] Nadel, M., "Scott sentences and admissible sets," *Annals of Pure and Applied Logic*, vol. 7 (1974), pp. 267–94. Zbl 0301.02050. MR 0384471. 7, 8, 9

[15] Sacks, G. E., "On the number of countable models," pp. 185–95 in *Southeast Asian Conference on Logic (Singapore, 1981)*, vol. 111 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1983. Zbl 0563.03015. MR 723338. 5, 7, 8, 9, 11, 18, 30

[16] Sacks, G. E., *Higher Recursion Theory*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990. Zbl 0716.03043. MR 1080970. 6

[17] Scott, D., "Logic with denumerably long formulas and finite strings of quantifiers," pp. 329–41 in *Theory of Models (Proceedings of the 1963 International Symposium, Berkeley)*, North-Holland, Amsterdam, 1965. Zbl 0166.26003. MR 0200133. 6, 8

[18] Steel, J. R., "On Vaught's conjecture," pp. 193–208 in *Cabal Seminar 76–77 (Proceedings of the Caltech-UCLA Logic Seminar, 1976–77)*, vol. 689 of *Lecture Notes in Mathematics*, Springer, Berlin, 1978. Zbl 0403.03027. MR 526920. 7, 19

Acknowledgments

Deprtment of Mathematics
Harvard University
Cambridge MA 02138
and
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Ave
Cambridge MA 02139-4307

sacks@math.harvard.edu
sacks@math.mit.edu