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Book Review

Mark van Atten. *On Brouwer*. Wadsworth Philosophers Series. Wadsworth/Thomson Learning, Belmont, 2004. viii + 95 pages.

1 Introduction

Mark van Atten's *On Brouwer* in the Wadsworth Philosophers Series is the most important introduction to Brouwerian intuitionism since the appearance of Arend Heyting's *Intuitionism. An Introduction* in 1956. Within the very small compass of 84 text pages, van Atten manages to convey a strong sense of the spirit of Brouwer's enterprise. Although, by his own admission, he does not attempt to convey the entire extent of Brouwer's philosophical views, leaving aside in particular "his mysticism, his philosophy of natural language, and the applicability of intuitionistic mathematics to the natural sciences" (p. vii), nonetheless, van Atten's presentation of Brouwer's views in the philosophy of mathematics proper does not suffer unduly from the limited number of pages. In considerable part this is a function of van Atten's strong focus on the "creating subject" as a central notion which ties together the development of Brouwer's career in intuitionistic mathematics. "At the beginning, the notion of the creating subject was already present but mostly implicitly so; the development of Brouwer's intuitionism consisted in the unfolding of this notion" (ibid.).

Van Atten's text is divided into six chapters, all but the fourth of which is suitable for a guided introduction to intuitionism at the upper undergraduate or postgraduate level; indeed, the author of this essay has used van Atten's volume in this way to (apparently) good effect. The fourth chapter, "Brouwer's Proof of the Bar Theorem," is considerably more challenging, despite van Atten's insistence that although Brouwer's proof of the bar theorem "is a bit technical, it is not particularly difficult" (p. 41). Although it is a pity to omit the chapter, since it is in a considerable sense the centerpiece of the volume, the omission does not seriously impair the flow of reading and there are only a few places in the last two chapters where points rely on references back to the proof of the bar theorem and its corollary, the fan theorem.

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In some sense, the book naturally breaks into three sections of two chapters each: in the first two chapters, van Atten considers the two acts of intuitionism (Chapter 1) and the role of proof and logic within intuitionism in general terms (Chapter 2). These two chapters lay out the basic picture of Brouwer's intuitionistic philosophy of mathematics insofar as it is typically considered a "foundation" for mathematics. The next two chapters consider choice sequences (Chapter 3) and the proof of the bar theorem (Chapter 4); here we are given a brisk presentation of the foundations (more in the mathematical than the philosophical sense) of intuitionistic analysis. The last two chapters open out onto larger issues raised by the intuitionist program, particularly when viewed in light of the centrality of the creating subject. Chapter 5 reviews so-called creating subject arguments, which exemplify Brouwer's commitment to the notion of the creating subject and evince the mathematical power of this commitment. Chapter 6 focuses on the issue of intersubjectivity and so addresses potential objections that intuitionism constitutes a form of philosophical, and hence mathematical, solipsism. This is surely the most controversial of van Atten's six chapters, and in it he proposes, in particular, that Brouwer's creating subject be interpreted in terms of the Husserlian notion of transcendental subject. Given the stress van Atten places on Brouwer's commitment to the notion of creating subject, I think it is not unfair to say that in this chapter he proposes a Husserlian interpretation of the intuitionistic program, suitably qualified; I will return to this interesting set of issues at the end of this essay.

2 Intuitionistic Foundations

Van Atten's treatment of the two acts of intuitionism is somewhat marred by an indefiniteness in the way he specifies their relation to each other, though admittedly the issue is a delicate one given that this is an issue which spans over a considerable period in the course of Brouwer's career. After citing a characterization of the two acts of intuitionism given by Brouwer late in his development (1952), van Atten first remarks that "part of the second act, the choice sequences, was added a decade after the first act had in effect been performed by Brouwer," using this to illustrate the open-ended and ongoing nature of intuitionism. He then goes on to remark that it took Brouwer "some time to discern that the intuition of two-ity," given in the first act of intuitionism, "also allows for choice sequences," which are given according to the second act of intuitionism in Brouwer's presentation. No dates are given here. The possibility of generating more general "mathematical species" is also recognized in the second act of intuitionism, and at this point, at least, van Atten draws no connection between the first act of intuitionism and the possibility of generating these more general mathematical species, among which spreads would be included. But in Chapter 5, van Atten remarks that "as we saw in chapter 1, he [Brouwer] came to say that the second act of intuitionism (in which choice sequences are recognized as mathematical objects) is an immediate consequence of the first..." (p. 69). This makes it sound like the possibility of generating the more general mathematical species is also a consequence of the first act of intuitionism, though van Atten does not say so explicitly.

Be this all as it may, van Atten naturally orients the first chapter around the intuitionistic notion of a mental mathematical construction, beginning informally with the constructions of $2 + 3$ and the construction of 5, noting that they are "the same" (p. 1). This leads almost immediately to the recognition of an intuitionistic criterion for the identity of constructions, and van Atten points out that "the primary notion of identity in intuitionism is that of intensional identity: if two objects are given by the same construction they are intensionally identical . . ." (p. 2). Two points are worth noting here. First, the identity of *objects* is defined in terms of sameness of construction: it is by a fundamental appeal to intuition that we see, for example, that the constructions of $2 + 3$ and of 5 are "the same." The second point, which is a consequence of the first, is that construction precedes the ontology of objects: in this sense intuitionism is a *praxiological* rather than an *ontological* theory (see [4], p. 119). This point is much easier to make early on in the discussion of intuitionism, since later on when we deal, for example, with choice sequences, there is potentially a tendency to conflate the construction of the sequence with the sequence as a mathematical object; linguistically it is almost inevitable that we disrespect this distinction at some point.[1] Brouwer was himself sensitive to this point, in particular when he introduces the cardinality 'denumerably unfinished' as a *façon de parler*. In a passage from 1907 which van Atten quotes, Brouwer insists that "from a strictly mathematical point of view this set [i.e., a 'denumerably unfinished' one] does not exist as a whole, nor does its power exist; however we can introduce these words here as an expression for a known intention" (cited, p. 7). Choice sequences, equally unfinished, should be treated analogously.

This becomes even more delicate when we move from the domain of construction and object to the domain of intuitionistic proof. Here again, Brouwer will recognize "infinite" proofs, but it takes a bit of unpacking to see what this means. First, the distinction between construction and proof is introduced in terms of a construction being direct and a proof indirect:

> To experience a truth is to experience that a certain construction has succeeded. This experience may be either direct, in case the subject has actually carried out the construction, or indirect, in case the subject sees that a certain construction method will, if followed through, give the direct experience sought after. Such indirect experiences, which are not merely 'expected' if that is meant to exclude 'guaranteed', are the domain of proofs and logic, which will be the subject of chapter 2. (p. 10)

Van Atten's treatment of proof in Chapter 2 amalgamates work done by Brouwer and his student Heyting, and, although van Atten carefully distinguishes between the contributions of the two, I will argue below that an important consequence of this division of labor for van Atten's interpretation of intuitionism does in fact go unrecognized. As I take the above-cited passage to indicate, the guiding thread for van Atten's interpretation of Brouwer's approach to the nature of proof is the distinction between proofs, as indirect, and as therefore depending in a foundational sense upon constructions, which are the direct and hence foundational mathematical acts. Yet in a passage which van Atten cites at the beginning of Chapter Two, Heyting declares,

> If mathematics consists of mental constructions, then every mathematical theorem is the expression of a result of a successful construction. The proof of the theorem consists in this construction itself, and the steps of the proof are the same as the steps of the mathematical construction. (Heyting, p. 107, cited, p. 16)

In Heyting's characterization a strong connection is expressed between the proof and the underlying construction which seems to verge on identity itself. Several pages later van Atten provides a characterization which may begin to brook the apparent tension: "for an intuitionist, a proof consists of steps that preserve constructability..." (p. 19). This again insists on the indirect nature of proof, which relies on an antecedent construction which it transforms in such a way that constructability is preserved, yet the "outcome" is then itself a construction, and we might take this as the "construction itself" whose steps are "the same" as the steps of the proof. But for the steps to be literally *the same* we would require that the proof be considered as a mathematical construction operating on an antecedent construction *as its object*. This, however, requires a transformation of the status of the antecedent construction from mathematical *action* to mathematical *object*, according to the distinction upon which I have insisted above.

In fact, van Atten seems to accord this transition a place within what he calls "ordinary" proof, that is, proof in the primitive sense: "in a proof, one shows that certain relations hold between certain mathematical objects" (p. 17). In general, this is to be distinguished from what Brouwer calls the proof's "canonical form": "A complete decomposition of an ordinary proof into elementary inferences results in a mental proof that Brouwer calls its canonical form" (p. 17). The distinction between ordinary proof and its canonical form is important, in particular, to deal with the fact that ordinary proofs, understood as mental objects along intuitionistic lines, generally involve an infinite number of "terms." If, in particular, a proof contains a universally quantified proposition $\forall x\, P(x)$ it must specify a method for constructing $P(0)$, $P(1)$, $P(2)$, ... "A canonical proof contains all these proofs instead of their summary statement $\forall x\, P(x)$, and whatever in the ordinary proof is inferred from that summary is in the canonical proof inferred from these infinitely many premises" (p. 17). Thus the canonical proof seems to have the status of a conditional schematic construction with a potentially infinite number of premises.

But it is not in terms of such a conditional construction that van Atten goes on to characterize the canonical proof, but rather in terms of Husserlian intentionality:

> a canonical proof reflects the inner structure of the complex thought that is the ordinary proof. The sense in which a canonical proof contains infinitely many elements is that of implicit reference, or, in Husserl's term, intentional implication. The decomposition of an ordinary proof into a canonical one consists in an analysis of the intentional structure of the former. (p. 17)

This interpretation of Brouwer's notion of canonical proof is, in particular, one which squares nicely with Heyting's approach to intuitionistic proof, known as the "proof interpretation," which relies fundamentally on a notion of intentional fulfillment. After citing Heyting's 1931 paper, "Die Intuitionistische Grundlegung der Mathematik" (which, as van Atten notes, Brouwer helped to have placed in "one of the top journals" (p. 23)), van Atten glosses the conception of intentional fulfillment and the concept of truth formulated in terms of it as follows: "the intention expressed by [the proposition] p is fulfilled exactly if it is known how to prove p by a construction. 'p is true' means 'the intention expressed by p has been fulfilled' " (p. 21).

For convenience, let us refer to van Atten's proposal as the *intentional fulfillment* interpretation of intuitionistic proof and truth. It has quite a number of desirable features. Perhaps most fundamentally, it is strongly anchored in the writings of Brouwer, who first explored the notion of canonical proof, and Heyting, with whose proof interpretation of intuitionism it dovetails. Equally importantly, it satisfies the dual desiderata that it supports a tensed conception of mathematical truth and a commitment to the potential infinite which does not extend to a commitment to actual infinity. Indeed, one might even suggest that both of these famous rallying points of intuitionism recede to the level of secondary consequences of the fundamental commitment to the intentional fulfillment conception of truth, and so I will not tarry over van Atten's defense of them. What we see emerging is perhaps even better described as mathematical intentionalism rather than intuitionism: pride of place are attributed to the notion of the creating subject and canonical proof.

As Michael Dummett noted in his 1973 essay, "The philosophical basis of intuitionistic logic," which significantly contributed in bringing intuitionism to the attention of the community of Anglo-American philosophy, "the notion of canonical proof thus lies in some obscurity; and this state of affairs is not indefinitely tolerable, because, unless it is possible to find a coherent and relatively sharp explanation of the notion, the viability of the intuitionist explanations of the logical constants must remain in doubt" ([5], p. 242). As Dummett also notes, when Heyting's proof interpretation defines, for example, a proof of an implication $p \rightarrow q$ in terms of transforming a proof of p into a proof of q, it does so in terms of *canonical* proof: it is the *fulfillment*, in the intentional sense, of $p \rightarrow q$ which is characterized. In developing a "relatively sharp explanation" of canonical proof van Atten thus does intuitionism a considerable service, and the fact that he puts the tools of Husserlian phenomenology at the service of this task gives his efforts a potential philosophical pedigree. But the close alignment of the fulfillment interpretation with Heyting's own proof interpretation should lead us to say of van Atten's interpretation the same thing that Heyting said of the formalization deriving from his proof interpretation. Van Atten, citing Heyting's 1930 paper, remarks, "Heyting never claimed that his formalization was the definitive intuitionistic logic, and moreover admitted that there could not be one, as 'the possibilities of thought cannot be reduced to a finite number of rules set up in advance' " (p. 23). Heyting's formalization of intuitionistic logic *cannot* be definitive simply because there *is* no definitive intuitionistic logic; van Atten's interpretation of intuitionistic mathematics fails to be definitive for a related reason. This is that on the intentional fulfillment interpretation of truth we are dealing with a view on which mathematical construction does not antecede logical implication: the intentional fulfillment interpretation of truth accommodates both on an equal footing, as van Atten's characterization of the intentional fulfillment interpretation makes clear. Just what it is that allows van Atten to "clear up" the notion of canonical proof is precisely what makes it impossible to take van Atten's interpretation as an *expression* of Brouwer's intuitionism.[2] Like Heyting's proof interpretation, it does however provide us with a powerful vantage for interpreting it. I will return to these points for further consideration at the end of this essay.

3 Intuitionistic Analysis

In the third and fourth chapters of his book, van Atten rightly takes the field of intuitionistic analysis as a proving ground for the notion of canonical proof and the status of the creating subject. He begins with a chapter on choice sequences, which, as he points out, "are not acceptable to classical mathematics," but are, on the other hand, "at the basis of Brouwer's analysis of the continuum" (p. 30). The radical novelty of Brouwer's approach to the continuum lies in his providing an account of it which does see the continuum as composed of points but nonetheless supports a nontrivial—though of course nonclassical—mathematical analysis of the continuum. As van Atten points out, Brouwer's approach has the third virtue that it is "constructive to boot" (p. 34). This project is grounded in the notion of choice sequence, though requiring the notion of a mathematical spread—also guaranteed by the second act of intuitionism—as well.

Van Atten lays stress upon the fact that, as constructions of the creating subject, choice sequences are not "incomplete," but rather always "in process." A real number is not at all to be thought of as the ideal limits of a choice sequence, but rather "it is the proceeding sequence itself" (p. 31). Roughly, these particular choice sequences consist of nested intervals on the continuum whose length shrinks to zero, and so van Atten stresses that we should not think of real numbers as corresponding to points. Rather, "as Husserl has described in his analyses of intuitive time, continua are not built up from atoms. For example, a moment in time is not a dimensionless point, but a little 'halo' " (ibid.), and the same holds of choice sequences as well: at any given time they define a value only up to such a "halo" and so are not atomic constituents of the continuum at all. Dedekind's "cuts" and Cantor's transfinite continuum fail as accounts of the continuum precisely because they fail to account for the nature of the continuum as continuous. "As Hermann Weyl put it, in the classical continuum of real numbers, the points are *exactly* as isolated from one another as the natural numbers" (citing Weyl 1918, p. 33).

In addition, choice sequences can embody more than the information carried by the entries in the sequence, because in many cases the ongoing sequence may be seen as generated according to certain "laws" associated with its production. With the consideration of such laws we return to the intensional identity criterion mentioned previously: two choice sequences, as constructions, may only be identified if they agree as intensional objects, and this requires in addition to their present entries also the laws according to which the sequences have been and will continue to be produced. Focusing on this future production helps to motivate why intensional information must be considered. If you and I have produced the same entries in our respective choice sequences so far, but if my law of production tells me all future entries will be the number 2, and your law tells you that all future entries will be the number 3, then these sequences are *already* divergent. For example, we already know that only a finite number of entries in your series differ from 3, and we already know in fact that this is *false* of my choice sequence. Consequently, van Atten characterizes a choice sequence as follows:

> In full generality, we can think of a choice sequence as a sequence of tuples
>
> $$(n_i, R^i_1, R^i_2, \ldots, R^i_{k_i})$$
>
> where at stage i the subject chooses an object n_i and a finite number of restrictions of orders $1 \ldots k_i$. An individual choice sequence has all of its properties solely in virtue of the subject's decision to construct it in a particular way. For this reason, choice sequences are highly intensional objects. (p. 34)

Strictly speaking, of course, choice sequences are constructions, not mathematical objects at all, but when we come to think of them as mathematical objects (focusing, so to speak, on the essential content of the construction),[3] the identity criterion for these objects is "highly intensional."

Still, all this would amount to but an interesting curiosity, as van Atten himself recognizes, if we were not able to erect a practice of mathematical analysis upon the account of the mathematical continuum in terms of choice sequences. For this purpose, Brouwer relies on a number of principles which collectively go by way of the name 'continuity principles'. The problem is this: if there were no guarantee that a functional value could ever be assigned to a choice sequence on the basis of the information embodied in the choice sequence at some specific point in time, then there would be no possibility of defining functions over choice sequences—including functions defined over the real numbers. In the case where there is no first-order intensional information (the so-called lawless sequences), we can see that this would amount to requiring that functional values be given on the basis of *some* finite string of choices for any particular choice sequence (with the finite value in general varying depending on the choice sequence in question). This, in turn, amounts to requiring that any choice sequence with the same string of finite entries be assigned the same functional value. Such is the content of the weak continuity principle (WC-N):

$$\forall\alpha\exists A(\alpha, x) \Rightarrow \forall\alpha\exists m\forall\beta\big[\overline{\beta}m = \overline{\alpha}m \rightarrow A(\beta, x)\big]$$

> where α and β range over choice sequences of natural numbers, m and x over natural numbers, and $\overline{\alpha}m$ stands for $\langle\alpha(0), \alpha(1), \ldots, \alpha(m-1)\rangle$, the initial segment of α of length m. (p. 35)

But what about the case of choice sequences which (to return to object oriented language) contain nontrivial intensional information? Should we expect these, too, to satisfy WC-N?

Van Atten provides an argument that, in fact, we should take WC-N to hold for choice sequences generally (just how generally will become clear below). He begins, however, by showing that simple-minded arguments will not do. Roughly, this is for the reason mentioned above: in the presence of intensional information we may know features of the future development of the choice sequence that depend directly on the intensional information provided. Consequently, van Atten provides a rather sophisticated argument, which is oriented in terms of Husserl's notion of noetic-noematic correlation. Applied to choice sequences, this orientation leads to the question: "in what ways can the freedom that the subject enjoys in the process of generating a choice sequence be reflected in the properties of the sequence itself?" (p. 36).

Here begins van Atten's rather delicate argument. He begins by enlisting the notion of a "provisional" restriction, which he associates with Troelstra and van Dalen's previous notion of a "hesitant sequence." A provisional restriction is one that holds until declared otherwise. Now, in the formulation of WC-N let us require that the relation A "refers to a choice sequence only through the graph of that sequence, i.e., only by direct reference to what value appears at what place" (p. 37); as van Atten points out, this will be sufficient for analysis (for a much more detailed discussion of the graph restriction, see [3], pp. 335–39). As we have seen above, a justification in this case is readily available. Thus we must show that further restriction is inessential. But van Atten argues that such restrictions cannot be essential "as they are not stable" (ibid.). Suppose we have a provisional restriction on a choice sequence, and consider a proof that a particular functional value obtains for this sequence. Such a proof is a proof that this functional value obtains *and will always obtain* for this sequence. As van Atten puts it, "if we have a proof of $A(\alpha, x)$, then this relation should hold forever" (ibid.). This proof appeals to a construction method with respect to the choice sequence x, but this cannot depend on any appeal to the provisional restriction, for then we could possibly yield different results at different stages in the construction of the sequence. But if this is true for sequences with provisional restrictions, then it must be true for sequences with nonprovisional restrictions, for in general such sequences cannot be extensionally distinguished from one another at any given time. We may now indicate the generality of the result: it holds for any "universe" in which for any choice sequence its provisional counterparts are also allowed. In particular, this holds for the "universe" of all choice sequences.

The arguments for and against the continuity principle form one of the most conceptually interesting parts of the intuitionistic enterprise and are all discussed in much greater detail in [3]. In closing this summary presentation I would only like to reiterate that without some form of the continuity principle intuitionistic analysis is a nonstarter. In psychological motivation at least, this makes all arguments for some version of the continuity principle look like ex post facto justifications, and within the arguments themselves we find this ex post facto character reflected in the assumption that *there are* some nontrivial proofs in the field of intuitionistic analysis.

Let us progress then to the canonical example of such a proof: this will take us more deeply into consideration of the notion of the canonical form of a proof as well! This cannot help us with the previous dilemma, since it will make appeal to the continuity principle, a strengthening of the weak continuity principle which van Atten argues for in Chapter 3, and to the (as I will insist) closely associated idea of a bar. The proof referred to is of course the proof of the Bar Theorem; van Atten will follow Brouwer's 1927 presentation, given in translation in van Heijenoort's anthology ([6], pp. 457–63).

Van Atten begins by identifying the concepts requisite for the statement of the theorem, which include that of a spread; after citing Brouwer's 1925 characterization, van Atten re-presents the notion in terms of the subsidiary notions of spread law and correlation law. This leads to the notion of admitted sequences, and the admitted sequences comprise a tree (the root node of which van Atten situates at the top). The spread may then be characterized in terms of the admitted sequences, and "the tree of admitted sequences may be called the underlying tree of the spread" (p. 43). As van Atten notes, this is not Brouwer's description; Brouwer speaks instead of "the species of choice sequences upon which the spread is based" (cited, p. 43), and given the extent to which van Atten's later presentation of the notion of canonical proof leans on explicit appeal to tree structures the distinction is important. Van Atten then tacitly extends the notion of correlation law, defined previously on sequences, to map "the underlying tree of natural numbers to the desired objects" (p. 43). Subspreads of spreads are then defined in terms of subtrees of trees.

Next, van Atten proceeds to the definition of a bar: "If B is a bar for a spread M, this means that each of the infinite choice sequences in the underlying tree of the spread (call it U) has a finite initial segment which is an element of B:

$$\forall \alpha \in U \exists n (\overline{\alpha} n \in B).$$

The Bar Theorem will involve the (intuitionistic) well-ordering of bars, but the definition of bar allows for the possibility of redundant elements (/nodes) in the bar, and the statement of the theorem will require the notion of a *thin* bar, in which redundancies have been eliminated: using the ordering reflected in the underlying tree, we require that $b \in B \wedge a < b \rightarrow a \notin B$ (p. 44). Finally, in addition, we must require that the original bar is decidable: "of any node we should be able to tell in finite time whether it belongs to the bar or not" (ibid.). Then the bar theorem may be stated as "if B is a decidable bar, then it contains a well-ordered thin bar" (p. 45).

As the statement of the theorem relies on a notion of intuitionistic well-ordering, van Atten proceeds to show why the classical notion of well-ordering is unsuitable in an intuitionistic context using a weak-counterexample argument. The intuitionistic notion of well-ordering, which mimics Cantor's earlier 1883 approach to well-ordering, is then supplied. This proceeds in terms of an induction basis, that is, all one-element species are declared well-ordered, and then a recursive definition is supplied for the well-ordering of larger spreads by an induction step which includes two types of generating operations (again in direct imitation of Cantor).

Before proceeding to the proof of the Bar Theorem, I would like to make completely explicit an argument van Atten presents concerning the decidability of bars; although this does not bear directly on the proof of the theorem, the argument-form will reappear in the proof. Here, in particular, van Atten asserts that if "one thinks of bars that are implicitly determined by the continuity principle, as Brouwer does in the two proofs mentioned," then the decidability criterion is satisfied. But what is a bar "implicitly determined" by the continuity principle? As van Atten puts it, given a spread M and a function assigning a natural number β to each element of M, "the continuity principle then says that for every sequence a number n can be found such that you need only the first n choices in the sequence to calculate the number β that the function assigns to it" (p. 44); the close relation to the weak-continuity principle considered above should be obvious. Next, we use this function and the given value of n to "generate" a bar: "Given an initial segment of a choice sequence—which corresponds to a node in the tree—determine n for that sequence; the segment belongs to the bar exactly if its length is equal to or greater than n and this we can decide" (pp. 44–45). Now, suppose that we consider a choice sequence α in the spread; this sequence is associated with a value n' with respect to the numerical value β' which the function assigns to it, and we know by construction that for all $n \geq n', \overline{\alpha}n \in B$. In other words, here the bar consists of *all* nodes that coincide with or follow the respective choices of the value n as we proceed out along the tree (note that these values need not correspond to the first appearances of the associated fixed value β, and see further discussion of this point below). Hence, a bar "implicitly determined by the continuity principle" is decidable, but the converse is not clear. Although Brouwer apparently assumed that the bars in question in the proof of the Bar Theorem were determined by the continuity principle, van Atten supplies a proof given the weaker property of bar-decidability, thus effectively supplying a generalization of Brouwer's original theorem.

To construct a well-ordered thin bar we must eliminate all predecessors and retain a residual set of nodes which remains a bar and which can be well-ordered; this is what the proof of the bar theorem accomplishes. Since the statement of the Bar Theorem is an implication $P \Rightarrow Q$, where P is the statement "B is a decidable bar," and Q is the statement "B contains a well-ordered thin bar," what we must see is that any proof of P may be transformed into a proof of Q. But in order to do this we must have some sense of the "information" that might be embodied in *any* proof of P; it is for this reason that Brouwer is forced into the position of needing to have some account of the *general* nature of a proof of P; here the analogies to strategies of reasoning in Hilbert-style metamathematics reinforce van Atten's earlier point that in his distinction between levels of mathematical activity Brouwer anticipated the mathematics/metamathematics distinctions of Hilbert, Bernays, and Tarski (p. 14). But here, we may say, the "distinction" emerges at an even more primitive level, namely, at the level of the distinction between direct construction and proof as an "indirect" construction. In the case of the proof of a conditional, the "indirect construction" of the proof must itself proceed over yet another "indirect construction," namely, an arbitrarily supplied proof of the antecedent of the conditional.

Van Atten breaks the proof of the Bar Theorem into two parts. In the first, we might say, we organize the information which we see must be contained in the supplied proof of the antecedent. As van Atten remarks, in an intuitionistic proof of an implication, "one usually doesn't need much more information about a proof of the antecedent beyond the fact that, in the case of a conjunction, for example, one indeed has a proof of each conjunct"; for the Bar Theorem, however, "Brouwer analyses what a proof of the antecedent could be like in great detail" (p. 47). In van Atten's presentation, this process proceeds by reflecting on the tree-structure of the underlying spread M and then "transferring" this tree-theoretic information explicitly to a tree-structure associated with the proof of the antecedent itself; it is in terms of this latter tree-structure that van Atten couches the notion of *canonical proof*. In the second half of the proof of the Bar Theorem, van Atten then uses this canonical proof of the antecedent to build up a well-ordered thin bar by recursion. Here we are engaged in a process whereby the information in the tree-structure associated with a canonical proof is "carried over" to the tree-structure associated with the elements of a relevant well-ordered thin bar and then use an inductive process to "paste together" smaller well-ordered thin bars into larger ones progressively until we have arrived at a well-ordered thin bar for the entire spread.

As van Atten admits, this second part of the proof is passed over by Brouwer in short order in a brief footnote to his paper, and van Atten also notes that "it has been questioned whether Brouwer's full proof of the bar theorem really is more evident than the principle formulated in this footnote" (p. 51), not citing, however, any particular source (but see remarks below). Arguably the extent to which van Atten is both able to and indeed required to flesh out the "details" of Brouwer's procedure turns on the particular conception of canonical proof which van Atten advocates. From the perspective of one (like myself) who would insist on the problematic extent to which intuitionism remains strongly reliant on appeals to the continuity principle or its near neighbors, a passing remark van Atten makes points to the way in which the crux of the Bar Theorem turns on the distinction between a particular pair of necessary and sufficient conditions:

> We saw that in the setting in which Brouwer proves the bar theorem we have a spread M and a function or algorithm that assigns to every choice sequence in M a natural number β. Because of the continuity principle, this implicitly defines a thin bar μ_1 in the underlying tree. A sufficient condition for a node in M to have the property that any choice sequence passing through it will be assigned the same number β is that this node, or an ascendant of it, was obtained by the correlation law from a node in μ_1. But this is not a necessary condition: perhaps one can find, once a sufficient number of such assignments of numbers β to choice sequences have been determined, a node above the bar (i.e., in σ)[4] such that its correlated node in M also has that property [that any choice sequence passing through it will be assigned the same number β]. (pp. 48–49)

In other words, if we are in a position to rely on the continuity principle to construct a well-ordered thin bar "implicitly," this will not necessarily be a thin bar which possesses the property that no antecedent nodes in the underlying tree are such that the relevant functional value can be fixed in terms of this antecedent node; our algorithm for fixing such values will in general not be "efficient." The case is analogous if we merely assume that the bar B is decidable. Here, instead of a function assigning values to nodes in the underlying tree we have instead an algorithm which serves as warrant for the proposition: $\forall\alpha \in M \exists n(\overline{\alpha} \in B)$, which is simply the defining condition for a bar already given above.[5] Here again, however, we must not assume that the supplied algorithm is efficient, so that, as van Atten puts it, "there can be a huge gap between having a proof that there is a bar and knowing exactly what the bar looks like" (p. 48). In considerable part, the proof of the Bar Theorem is supplied precisely in order to secure the *irrelevance* of such an epistemic gap. But more importantly, what this analysis shows is the extent to which the definition of a bar stands in proxy relation to the continuity principle itself. The proof of the Bar Theorem does not supply such deep insight into intuitionistic proof by any accident: upon reflection it is clear that the Bar Theorem is central precisely because the definition of bar stands in such close relation to the intuitionistic appeal to various closely related principles of continuity.

In conclusion, I would like to make some remarks about the morals van Atten draws concerning canonical proof in the context of his presentation of the Bar Theorem. Van Atten associates these canonical proofs very closely with the underlying trees which are drawn from the steps of the proofs in question. But to the extent that van Atten takes these underlying trees to represent the proofs in question, he has substituted for the original notion of proof as an indirect construction the notion of an underlying mathematical object, namely, the tree in question, which is in some sense (closely) associated with this indirect construction. This requires the notion of "levels" of mathematical activity as described previously (pp. 13–14, alluded to above), a notion which embeds logico-linguistic description inside of "higher-order" mathematical activity and so presupposes some account of the logico-discursive nature of such higher-order activities (apparently including, as we shall see later, a need to defend the intersubjective availability of these activities). And it is presumably in these terms that van Atten is thinking when he remarks that the well-ordered structure of the trees associated (or identified?) with the relevant canonical proofs "is at the basis of Brouwer's proof of the bar theorem" (p. 52). Prima facie, this may seem to put the cart before the horse, but van Atten's running argument, as I understand it, is that we must understand such a complicated proof as that of the Bar Theorem as relying in a complicated way on appeals not just to anteceding proofs but indeed to the canonical form with which these proofs are supplied by rational consideration of the proofs in question. What is hard to understand is just how such rational consideration is itself to be understood along *intuitionist* lines; this, however, points in the direction of issues that van Atten will discuss in his phenomenological interpretation of intuitionism. The relevance of this interpretation to the set of issues posed by the presentation of the proof of the Bar Theorem is indicated when van Atten insists that his presentation is more fundamental than those of Heyting, Kleene, Troelstra, and Dummett. These latter authors adopt Brouwer's perspective in the above-mentioned footnote and formalize the commitments expressed therein as a conjunction of four conditions of which, as Kleene in particular remarks, "we are unconvinced that any known substitute is more fundamental and intuitive" (cited, p. 59). Van Atten disagrees, insisting that a presentation of the proof which proceeds instead in terms of canonical proofs is more fundamental "in the sense that it makes the role of intentionality in proofs explicit" (p. 59). This issue, the role of intentionality in proofs, is implicit in van Atten's interpretation of the notion of canonical proof and can be made completely explicit by appeal to Husserl's notion of the noetic-noematic correlation, already invoked by van Atten in Chapter 3 in the context of arguments for the weak continuity principle. In this earlier context, the noetic-noematic correlation was invoked as a way of addressing the notion of "informal rigor" associated with the constructive power of the creative subject, and so it is appropriate to turn next to Chapter 5, in which van Atten discusses the creative subject and so-called creative subject arguments in detail.

4 The Creating Subject

Although, as van Atten recognizes, there is a legitimate sense in which all intuitionistic mathematics could be construed as involving 'creating subject arguments', this term is usually restricted to arguments which appeal to certain 'creating subject principles'. Van Atten focuses in particular on the principle "from perpetual ignorance to negation" (PIN), which he formulates as

$$\neg\exists n E_n p \to \neg p,$$

where '$E_n p$' stands for "the subject has experienced p at time n." Van Atten argues for this proposition as a logical inference: "if, of a given proposition p, the creating subject comes to know that it will never prove it, [then] that is a sufficient ground for the subject to conclude that p is false" (p. 64). On the proof-interpretation of intuitionism, which I have suggested above broadly dovetails with van Atten's proposed "intention-fulfillment" interpretation so far as the construal of proof is concerned, this implication can be glossed by saying that any time we have a proof that there is no time at which the subject will have experienced p, we are warranting the transformation of this proof into a proof that p implies a contradiction. In my own thinking about this principle I have found this proof-interpretation construal helpful to keep in mind, as it serves in particular as a reminder that the truth of the principle is itself tensed in the sense that we are warranting the truth of it now; the proof-interpretation helps to make it clear what our commitment to the logical implication embedded in this principle commits us to.

As van Atten goes on to note, from the classical perspective the principle is "surprising." It might be better to say there is no widely accepted classical argument for the principle, since, on the one hand, the inconsistency of the principle with classical mathematics would also require an argument, while on the other hand the principle has consequences which may surprise the otherwise unflappable intuitionist as well, some of which van Atten goes on to point out. Be this all as it may, the creating subject arguments are clearly at the heart of van Atten's particular portrayal of intuitionism, as is clear from the opening passages of his preface, cited above. This should not be surprising, for the creative subject arguments embody the most powerful intuitionistic appeal to the central role played by the mathematical subject, even allowing for a unification of the two fundamental acts of intuitionism by appeal to the insight that "the structure of the creating subject's activity turns out to be the same as that of a choice sequence" (p. 69). As such, the centrality of the creating subject also exposes the need for a fully developed account of this subject, which van Atten will go on to sketch in terms of an appeal to the phenomenology of Husserl.

What is perhaps most striking about van Atten's presentation on further reflection is that in fact the creative subject arguments which he discusses do not directly involve a proof that a proof of a particular proposition p will never be forthcoming. Indeed, what would such a proof look like, other than, that is, a proof that p entails a contradiction? Rather, it seems, the controversial flavor of "creating subject arguments" in the proper, that is, narrow, sense stems largely from the way in which they reason about *general* consequences of a situation of perpetual ignorance. In this regard, it is difficult if not impossible to tell without referring back to the original papers of Brouwer whether creating subject arguments remain at this general level, but such seems to be likely given that the example which van Atten cites involves the strengthening of a weak counterexample into a strong counterexample in such a way that a universal quantifier is involved. Hence, for example, on the basis of a creating subject argument, Brouwer is able to strengthen the claim that there exists a number x that (currently) is different from 0 without our being able to prove that it lies apart from 0, that is, we cannot prove $\exists n(\mid x \mid > 2^{-n})$ to the claim $\neg\forall x \in R(x \neq 0 \rightarrow x\#0)$, where '$x\#0$' stands for "$x$ lies apart from 0."

The structural identity between the construction of a choice sequence and the activity of the creating subject, and hence between the first and second acts (and, presumably, beyond, if there were to be any acts beyond the second act) is rendered explicit in terms of the schema of so-called Kripke Axioms (the term was introduced by Myhill). With respect to any proposition p, the activity of the creating subject can be "logged" in a choice sequence $\alpha(n)$ which records the value 0 so long as the subject has created no proof of p, and 1 thereafter. Kripke's schema is the axiom schema which guarantees the existence of such an α for each proposition p: $\exists\alpha(p \leftrightarrow \exists x(\alpha(x) = 1))$. In addition to formalizing the unification of the creating subject and the mathematical structure of choice sequences, this axiom scheme entails PIN and so generates all the counterexamples considered. As van Atten also points out, it has the potential advantage that it doesn't mention either the creating subject or the time-dependence explicitly: "some people feel uncomfortable with them" (p. 69). But as we have seen, both are certainly an integral part of the proposed justification for the axiom schema.

Van Atten concludes the chapter on creating subject arguments by presenting a paradox, due to Troelstra, into which the creating subject may fall if s/he allows for the construction of choice sequences involving impredicative definitions. Van Atten's proposed resolution, along the lines of earlier discussions in the book as well, is to distinguish a hierarchy of mathematical activities and so block the possibility of the offending definition. I will not consider this issue further here beyond pointing out that the notion of impredicativity is even more delicate in an intuitionistic context due to the tensed nature of mathematical truth.[6]

The natural bridge from the concerns with creating subject arguments in Chapter 5 to the focus on intersubjectivity in Chapter 6 lies in what we might call the metaphysical status of the creating subject: is this subject to be construed as a psychological subject in some sense, or is the creating subject of intuitionistic mathematics more akin to the philosophical notion of a transcendental, hence nonpsychological, nonempirical subject? Presenting documentary evidence, van Atten shows that Brouwer came to distinguish himself sharply and explicitly from any construal of the creating subject as empirically psychological, hence subject to the contingencies of forgetfulness, boredom, and specific limitation in time, that is, death. In response to arguments proposed by van Dantzig, Brouwer made it perfectly clear in correspondence that he could ascribe no importance to the contingent limitations of empirical, human subjects mentioned above. On the other hand, Brouwer remained firmly committed to the notion of the subject as intrinsically temporal throughout his career—how could he not?! What is needed, then, is an account of the creating subject as temporal but nonempirical, and a number of commentators before van Atten have taken Brouwer's own favorable references (in other contexts) to Kant as a motivation for characterizing the Brouwerian subject as transcendental in Kant's sense. Following an unelaborated suggestion by J. Roberts, van Atten argues instead for a Husserlian construal of the creating subject as transcendental.

There are many regards in which a construal of Brouwer's subject as a Husserlian transcendental subject is attractive; let me note immediately, however, that van Atten only proposes to identify the Brouwerian and Husserlian subjects "as far as mathematics is concerned" (p. 80). Perhaps not least among these attractions is the dovetailing of the strong autonomy of the transcendental subject in Husserl with a defense of the intersubjective access of subjects to one another, for this would perhaps open a road for responding to those many critics of Brouwer's enterprise who have objected to his philosophical commitment to solipsism. Brouwer, indeed, promoted the mathematical subject as solipsistic in particularly aggressive terms early in his career. On the other hand, some have identified the later development of creating subject arguments as the "most" solipsistic dimension of the intuitionist enterprise,[7] and if Husserl's transcendental phenomenology could provide a defense against solipsism in the *pejorative* sense (i.e., one which blocks intersubjective access) while defending this arguably most radical strain of the intuitionist program, the warrant for pursuing such an interpretation would be quite strong. Indeed, as a reconstruction (see [2], p. 1), I think the warrant for such an interpretation, *is* quite strong, though there also seem to me difficulties in the project which van Atten either understresses or fails to acknowledge.

Let me begin, however, by listing the points upon which I stand in complete agreement with van Atten, beginning with the distinctions he draws between Kant's position and Brouwer's. First, Kant identifies mathematical objects with the subjective conditions for the possibility of our knowledge of experience; on Brouwer's account mathematical objects are "real" and entirely independent of the conditions of empirical reality. Second, Kant's account of the objectivity of mathematical objects presupposes the objectivity of (general) logic, which Kant takes "simply as given"; van Atten's further claim, however, that Brouwer, like Husserl, "recognizes that logic, too, is an accomplishment of the subject," while not, I think, incorrect, is potentially misleading and will require further comment below. Third, and in line with the external status of general logic, Kant's determination of the transcendental subject proceeds externally in terms of a deduction, albeit a transcendental one. But manifestly the (mathematical) determination of the Brouwerian creating subject cannot depend on deductive determination. Finally, unlike Brouwer's creative subject, Kant's transcendental subject is located in the world. On all four counts I believe van Atten is correct to object to a Kantian interpretation of Brouwer's creating subject.[8]

Van Atten's positive identification of Brouwer and Husserl, to which I now proceed, while offering potential fruits, also causes greater problems. According to van Atten, Husserl's construal of the transcendental subject is "congenial to intuitionism" (p. 77), in large part because it is constituted in accordance with the fundamental intuition of time. This is, of course, not "clock," or physical time, but rather that temporal awareness which "consists in the awareness present in every intentional act, that other acts have preceded it and that others will follow it" (ibid.). Van Atten cites a particular interesting passage from Brouwer's 1948 essay, "Consciousness, philosophy and mathematics," which provides particularly compelling support in this regard. Here Brouwer speaks of the "deepest home" of consciousness, in which there is an oscillation "between stillness and sensation." This sensation allows the "initial phenomenon of the said transition" which is "a *move of time*" (cited, ibid.). Here there seems a particularly compelling dovetailing of the positions of Brouwer and Husserl.

My concern, then, is not with the suggestion of a deep affinity between Brouwer and Husserl, but rather with the particular way in which van Atten construes this affinity. In particular, there are certain regards in which the fundamental intuition of time is developed by Brouwer and Husserl in disparate ways, or so I will argue. These points bear most heavily on two aspects of van Atten's interpretation, and so I will consider them each in turn. The first involves the support for a defense of Brouwerian mathematics as intersubjective by way of affiliation with Husserl's development of the intersubjective status of the transcendental ego. On this point, van Atten remarks,

> in the notion of a transcendental subject are implied aspects of subjectivity that are the same for everyone precisely in virtue of each being a subject, and that in no way depend on the empirical. If mathematics can be founded on some of these aspects, then an account of intersubjectivity is within reach. If we construe the creating subject this way, then intersubjectivity is not a problem for, but rather a consequence of, the notion of the creating subject. While mathematics is ultimately traced back to subjectivity, this happens in a way that is necessarily the same for every subject, as mathematics then only depends on aspects that all subjects share simply because they are *subjects*. ("Intuitionistic mathematics is inner architecture"). (pp. 80–81; the last parenthetical quotation comes from the same 1948 paper quoted above)

I think the general line of argument here is clear, but there are finer aspects of it that are less so. First, it bears remarking that while the claim that "in the notion of a transcendental subject are implied aspects of subjectivity precisely in virtue of each being a subject" is relatively unobjectionable, the further claim that they are the same for *each* subject depends, at a minimum, on a common source of intuitive evidence. No more than Kant can Husserl claim that the uniformity of the transcendental subject is independent of the particular source of temporal intuition possessed by the subject; the difference lies rather in the status of the procedure by which this transcendental subject is established (along with other assumptions that Kant must make and Husserl need not). Although (for Brouwer) this does not depend on empirical intuition (like Kant he takes the status of the fundamental intuition of time to be nonempirical), it does (apparently) depend on the contingent (empirical) fact of our existence. But arguably it is at just this level that the issue of solipsism emerges in its most aggressive form. (I decline to interpret Husserl on this point as his position is, at least to my mind, extremely delicate.) About the rest of the paragraph cited, it is necessary only to underline its *conditional* status: the strategy for defending intuitionistic mathematics as intersubjective is predicated on establishing the *identity* of the subject with respect to the structure of the transcendental ego.

A second, and related, set of concerns arises when van Atten affiliates Brouwer's claim that "to find the deepest level of consciousness, we have to abandon logic" with the assertion that "like Husserl but unlike Kant, Brouwer recognizes that logic, too, is an accomplishment of the subject" (p. 79). The problem here is that I see no good reason to think that logic is an "accomplishment of the subject" in the same ways for Brouwer and Husserl, respectively. Famously, Husserl discloses a stratum of "categorial intuition" in the *Logical Investigations* that indeed seems anathematic to the sense in which we must "abandon logic" for Brouwer, since the status of intuition prohibits it being the case that for Husserl this source of categorial evidence should be *derived* from an antecedent recourse to temporal intuition (or any other intuition). Although Husserl's position certainly continues to develop throughout his later career, I do not discern in these developments any point which would contravene the sharp distinction between Brouwer and Husserl I have indicated on this point. Indeed, by parity, it would seem that if we must understand Husserl to abandon logic in the deepest home of consciousness then we would need to ask him to abandon time as well. But as is manifest throughout Husserl's career it is from the intuition of time that the intentional structure of the subject-object relation is constituted, as van Atten himself recognizes. Van Atten cannot have it both ways, and this causes serious problems for his proposal. I think it also points out the need to adopt a looser fitting "Husserlian" interpretation of Brouwer, one in which this salient difference is explicitly respected. But doing so also potentially threatens using Husserl's exposition of the notion of intersubjectivity to save Brouwer from the threat of solipsism: this would be the case, in particular, if Husserl's defense of intersubjectivity must necessarily appeal to the status of shared logical intuitions.

At the end of the book van Atten does concede that there may be a limited fit between intuitionism and Husserl's phenomenology even so far as the mathematical subject is concerned, but he locates this in terms of Husserl's phenomenology providing too "general" an account. Noting that "among the motives that led Husserl to develop phenomenology was the desire to devise a philosophical account of *classical* mathematics," van Atten proposes that "one way to negotiate this limitation is to suggest that the phenomenological considerations I have mentioned so far, while perhaps able to supply intuitionism as well as classical mathematics with a coherent interpretation, are too general to distinguish between the two" (p. 84). He then goes on to suggest that we might differentiate between "levels of evidence and to say that intuitionism is the mathematics of a class of objects that are given to us with a particularly high degree of evidence" (p. 84). I see no explicit support for this suggestion in Husserl, and indeed there are strong reasons to think that Husserl's enterprise is incompatible with it. In particular, as Roger Schmitt has noted, in the mathematical domain Husserl remained strongly committed to the Law of Excluded Middle throughout his life ([8], p. 61). It is difficult to conceive that Husserl would have attributed this law a lesser degree of evidence in some sense, at least in the mathematical domain, where he views it as necessary. One might argue that Husserl was simply mistaken in this regard, but that is something else again, and I do not find van Atten making any such claim in this book.[9]

Rather, it seems much more likely that Husserl's thoughts about mathematical intuitionism would have run along the lines of what he has to say about "intuitionism" in the Third Volume of *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy*:

> *Intuitionism*, therefore, reacted with complete justification against the one-sided surrender of mankind to the expansion of the sciences as techniques of thought. What matters is to put an end to the plight, grown intolerable,

> of reason, which amidst all the riches of its theoretical possessions sees its proper aim, world understanding, insight into truth, recede ever farther in the distance. But, of course, Intuitionism must not degenerate into mysticism instead of approaching sober tasks that are forthcoming from the situation described. ([7], p. 83)

Although the "Intuitionism" referred to here is presumably a philosophical program antecedent to Brouwer's (there is no reason to think that Husserl had any knowledge of Brouwer's program in 1912, when *Ideas* was composed),[10] I think Husserl would have likely seen Brouwer's mathematical program as threatening to degenerate into a form of mysticism, but *not* because of Brouwer's own avowed mystical philosophy. Rather, I think Husserl would have found any position which failed to secure the logical ideals required (according to Husserl) for the rational pursuits he advocates in the above paragraph in such a danger. Although the source of Husserlian evidence lies in *intuition*, the *goal* of the Husserlian enterprise lies in *reason*. In Brouwer's intuitionism, on the other hand, reasoning can only stand in a secondary relation to the primary level of construction, no matter now secure such reasoning may become. And this, I claim, indicates a fundamental distinction between Brouwer's and Husserl's respective enterprises both in structure and in the values which they respectively promote.

5 Conclusion

Van Atten has done the mathematical and philosophical communities a tremendous service by offering an introduction to the thought of Brouwer which is both accessible and ambitious in its scope. In addition, it has the further merit of pushing the interpretation of Brouwer's program in a direction which avoids many of the morasses into which debates about intuitionism have too often fallen over the years. But ultimately I think there is more warrant for some of the concerns about Brouwer's intuitionism than van Atten is willing to acknowledge. This is the consequence of the laudable enterprise in which he is engaged: to fly the banner of intuitionism by defending it as strenuously as is philosophically possible. For van Atten this takes the form of a Husserlian "reconstruction" of Brouwer's program. Yet what may perhaps emerge, over time, is a recognition that the program van Atten supports—despite, or indeed even because of, its departure in some regards from Brouwer's views strictly conceived—may be valuable on its own terms. In defending Brouwer's intuitionism, van Atten may in fact have offered us a first sketch of an "intentionalist mathematics" that will, in time, distinguish itself further from intuitionism, while acknowledging its roots in Brouwer, Husserl, and potentially other sources as well.

Notes

1. In [2], p. 90, van Atten maintains the distinction between, as he calls it, "process" and "object," but maintains that mathematics is *about* the objects. For reasons that will become clear shortly, I think this a significant misconstrual of Brouwer's position. Perhaps, ultimately, van Atten would not disagree, at least strenuously, since he insists on an intentionalist reconstruction of Brouwer's intuitionism in which I would agree that the centrality of objects is defensible.

2. Van Atten's revisionism is in this, as in other, regards much more explicit in the earlier text [2].

3. Here, see also [2], p. 68.

4. σ is the set of "unsecured" elements relative to the spread M, that is, "the elements that are admissible but that have not yet hit the bar" (p. 48).

5. On p. 48, van Atten declares that a proof of this proposition gives us a proof that B is a *thin* bar, but this is obviously a typographical error; he must mean that it gives us a proof that B is a *decidable* bar.

6. In fact, it is actually not the *tensed* nature of truth itself but rather the *dynamic* nature of truth in intuitionism that generates the relevant novelty; see [2], pp. 32–34. However, since this is what people usually seem to *mean* when they speak of tensed truth, I will retain the common usage here.

7. "Brouwer's 'creative subject' arguments may be viewed as an extreme expression of Brouwer's solipsistic view of mathematics" ([9], p. 236).

8. On one point I disagree with van Atten's construal of Kant: he asserts that for Kant "subjective time consciousness presupposes objective consciousness" (p. 83). But in fact it is only the *determination* of subjective time consciousness which presupposes objective consciousness. This does not, however, bear greatly on van Atten's larger agenda.

9. In [2], p. 13, van Atten insists that "Husserl's logic is not necessarily classical," and that the Law of Excluded Middle "in many cases is an idealization." Hence, van Atten continues, "a region of objects may be such that it does not allow for PEM [i.e., Law of Excluded Middle]." But van Atten's point is weak in the current context, since Husserl manifestly did take the Law of Excluded Middle to be *necessary* in the mathematical domain. In this earlier context, however, van Atten *is* in a position to argue that Husserl was mistaken about PEM, since he thinks Husserl was mistaken to take all mathematical objects to be omnitemporal and hence nondynamic. See [2], pp. 68–69.

10. For references to Brouwer in Husserl's manuscripts and correspondence, see [2], p. 61, 70ff. What Husserl *does* actually say about intuitionism I understand differently than van Atten does, but he does not supply this passage in *On Brouwer*; see [2], pp. 60–61. Van Atten takes Husserl to be suggesting that phenomenology of mathematics should not "follow the lead of any particular foundational program" because it "should study the core meaning of mathematics instead" ([2], p. 62); van Atten rightly objects that this is at odds with the phenomenological project. I, however, read Husserl as providing *strategic* counsel in this passage: that it is not "wise" to orient phenomenological investigation in this way. This dovetails with other passages van Atten cites in which Husserl repeatedly counsels against engaging in arguments with mathematicians because of the extensive technical prerequisites involved in entering into such arguments, where one small mistake can topple the entire issue ([2], pp. 70–71).

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O. Bradley Bassler
Department of Philosophy
University of Georgia
109 Peabody Hall
Athens GA 30602-1627
bbassler@uga.edu