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Classifying Dini's Theorem

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Abstract Dini's theorem says that compactness of the domain, a metric space, ensures the uniform convergence of every simply convergent monotone sequence of real-valued continuous functions whose limit is continuous. By showing that Dini's theorem is equivalent to Brouwer's fan theorem for detachable bars, we provide Dini's theorem with a classification in the recently established constructive reverse mathematics propagated by Ishihara. As a complement, Dini's theorem is proved to be equivalent to the analogue of the fan theorem, weak König's lemma, in the original classical setting of reverse mathematics started by Friedman and Simpson.

1 Introduction

Dini's theorem does not occur in the standard reference (Simpson [17]) for the program of reverse mathematics founded by Friedman and Simpson. We now undertake a classification of Dini's theorem within the constructive reverse mathematics put forward by Ishihara ([6], [7], [8], and [9]).[1] In particular, we work over the constructive mathematics initiated by Bishop ([1], [2], and [3]).

Bishop's theory can be seen as mathematics with intuitionistic logic in place of classical logic (Richman [15]), in which vein "classical" is sometimes used as a synonym for "using the law of excluded middle." Apart from the different choice of the underlying logic, one proceeds in Bishop's framework as in—the then-dubbed classical—customary mathematics.

Our principal objective is to establish Dini's theorem as an equivalent of Brouwer's fan theorem (for detachable bars). We first show that the latter is equivalent to Dini's theorem for functions on the Cantor space. Only then we prove that the fan theorem is equivalent to Dini's theorem on every compact metric space or, alternatively, on the unit interval.

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The implications which we assert from Remark 2.1 through Lemma 5.1 hold within $\mathrm{EL} + \mathrm{AC}_{00}$, that is, elementary analysis ([18], Chapter 3, Section 6) enriched with number-number choice. For the sake of an easier reading, we do not encode finite sequences into integers, which is a routine task in the present context. We sometimes refer to work done directly in EL [8] or in a subsystem thereof [9] and to constructions from [3] and [18] which one can carry over to $\mathrm{EL} + \mathrm{AC}_{00}$ without any difficulty.

In addition to invoking unique and dependent choice,[2] unrestricted use is made in Bishop's setting of induction over the natural numbers. The latter principle distinguishes, among other things, Bishop's framework from the corresponding formal system RCA_0 in Simpson's classical hierarchy, in which induction is restricted to Σ^0_1-formulas ([17], Remark I.8.9, Section I.12). To conclude the present paper, we show with Theorem 5.2 that Dini's theorem is equivalent, over RCA_0, to the so-called weak König's lemma—the counterpart of Brouwer's fan theorem in classical reverse mathematics.

2 Some Alternative Formulations of Dini's Theorem

We essentially follow Bishop's choice of definitions for compactness and continuity ([1], [2]). To start with, a metric space is said to be compact precisely when it is totally bounded and complete. In particular, every compact metric space is separable and can thus be represented in terms of binary sequences ([3], Chapter 5, Section 1; [18], Chapter 7, Section 4). Furthermore, a continuous mapping on a compact metric space is uniformly continuous according to Bishop's definition. To give a continuous function on a compact metric space is therefore the same as to give a uniformly continuous function on any dense subspace.

Throughout this note, let X be a metric space. We consider the following conclusion of Dini's theorem as a property of X.

DT_X *If a monotone sequence (f_n) of continuous functions on X converges simply to a continuous function f on X, then (f_n) converges uniformly to f.*

So Dini's theorem says that if X is compact, then DT_X holds. Unless specified otherwise, all functions occurring in this context are understood to be real-valued.

One arrives at equivalents of DT_X if 'monotone' is replaced either by 'increasing' (that is, $f_n \leqslant f_{n+1}$ for all n) or by 'decreasing' (that is, $f_n \geqslant f_{n+1}$ for all n); one may further assume that $f = 0$. In particular, DT_X is equivalent to its following specific form.

> *For every decreasing sequence (g_n) of nonnegative continuous functions on X, if (g_n) converges simply to 0, then (g_n) converges uniformly to 0.*

The latter equivalent has the virtue that it allows for carrying over Dini's theorem to mappings on X with values in an arbitrary metric space Y. Given mappings $g, h : X \to Y$, define $d(g, h) : X \to \mathbb{R}$ by assigning $d(g(x), h(x))$ to every $x \in X$. Also, if g and h are real-valued functions on X, write $g \leqslant h$ whenever $g(x) \leqslant h(x)$ for all $x \in X$, and likewise with $<$ in place of $\leqslant$.

Remark 2.1 DT_X is equivalent to the validity of the following statement for all metric spaces Y: if (f_n) is a sequence of continuous mappings $f_n : X \to Y$ that converges simply to a continuous mapping $f : X \to Y$ such that $d(f_{n+1}, f) \leqslant d(f_n, f)$ for every $n \in \mathbb{N}$, then (f_n) converges uniformly to f.

3 Dini's Theorem for Functions on the Cantor Space

As usual, let $\{0,1\}^{\mathbb{N}}$ denote the set of infinite binary sequences $\alpha, \beta, \ldots$, and let $\{0,1\}^*$ stand for the set of finite binary sequences $u, v, w, \ldots$. The letters $k, \ell, m, M, n, N, p, q, \ldots$ are understood as variables ranging over the set $\mathbb{N}$ of nonnegative integers.

If $u \in \{0,1\}^n$ for some n, then $|u| = n$ is the length of u. The nth finite initial segment $\overline{\alpha}n = (\alpha(0), \ldots, \alpha(n-1))$ of α has length n, which includes the case $n = 0$ of the empty sequence. Concatenation of sequences is denoted by juxtaposition, and $w \geqslant u$ means that $w = uv$ for some v; that is, w is an extension of u and u is a restriction of w.

We know that $\{0,1\}^{\mathbb{N}}$ is a compact metric space, the Cantor space, under the metric

$$d(\alpha, \beta) = \inf\{2^{-n} : \overline{\alpha}n = \overline{\beta}n\},$$

for which

$$d(\alpha, \beta) \leqslant 2^{-m} \iff \overline{\alpha}m = \overline{\beta}m\,.$$

So a function f on $\{0,1\}^{\mathbb{N}}$ is continuous precisely when for every k there is m such that

$$\overline{\alpha}m = \overline{\beta}m \implies |f(\alpha) - f(\beta)| \leqslant 2^{-k}$$

for all α and β.

The open balls of $\{0,1\}^{\mathbb{N}}$ are the subsets $\{\alpha : \alpha \in u\}$ with $u \in \{0,1\}^*$, where $\alpha \in u$ is written in place of $\overline{\alpha}\,|u| = u$. Moreover, $\{0,1\}^*$ is a countable dense subset of $\{0,1\}^{\mathbb{N}}$, where every u is identified with its trivial extension $u00\ldots$. For a more detailed treatment of all this we refer to [3], Chapter 5, Section 3; [5], Section 3.2; and [18], Chapter 4, Section 7.

To unwind Dini's theorem on the Cantor space, consider a decreasing sequence (g_n) of nonnegative continuous functions on $\{0,1\}^{\mathbb{N}}$. The implication

if (g_n) *converges simply to* 0*, then* (g_n) *converges uniformly to* 0,

crucial for the corresponding instance of $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$, is equivalent to

$$\forall k\, \forall \alpha\, \exists n \left(g_n(\alpha) < 2^{-k}\right) \implies \forall k\, \exists N\, \forall \alpha\, \exists n \leqslant N \left(g_n(\alpha) < 2^{-k}\right). \tag{1}$$

(Because we want to relate Dini's theorem to the fan theorem, we need to formulate this implication and their following equivalents in a way that at first glance may seem unnecessarily involved.) Furthermore, (1) is equivalent to

$$\forall k\, \forall \alpha\, \exists n\, \forall \beta \in \overline{\alpha}n \left(g_n(\beta) < 2^{-k}\right) \implies \forall k\, \exists N\, \forall \alpha\, \exists n \leqslant N\, \forall \beta \in \overline{\alpha}n \left(g_n(\beta) < 2^{-k}\right). \tag{2}$$

We now suppose that each g_n has a modulus of continuity; that is, there is a sequence (M_{nk}) of nonnegative integers with

$$\overline{\alpha}M_{nk} = \overline{\beta}M_{nk} \implies |g_n(\alpha) - g_n(\beta)| \leqslant 2^{-k} \tag{3}$$

for all α, β. By increasing the modulus if necessary, we can achieve that $M_{nk} \geqslant n$, and may thus set

$$W_k(u) = \left\{w : w \geqslant u \;\;\&\;\; |w| = M_{|u|k}\right\}$$

for every u and every k. For each $\beta \in u$ there is exactly one $w \in W_k(u)$ with $\beta \in w$, and we have $\left|g_{|u|}(\beta) - g_{|u|}(w)\right| \leqslant 2^{-k}$ for this $w = \overline{\beta} M_{|u|k}$. Hence (2) is equivalent to

$$\forall k \, \forall \alpha \, \exists n \; (\overline{\alpha} n \in U_k) \implies \forall k \, \exists N \, \forall \alpha \, \exists n \leqslant N \; (\overline{\alpha} n \in U_k) \tag{4}$$

with

$$U_k = \left\{u : \forall w \in W_k(u) \left(g_{|u|}(w) < 2^{-k}\right)\right\} .$$

Since $W_k(u)$ is a finite subset of $\{0, 1\}^*$, we have the simpler characterization

$$U_k = \left\{u : G_k(u) < 2^{-k}\right\}$$

with

$$G_k(u) = \max\left\{g_{|u|}(w) : w \in W_k(u)\right\} .$$

As $\{0, 1\}^*$ is dense in $\{0, 1\}^{\mathbb{N}}$, we may further suppose that each g_n is given by a sequence $r_n = (r_{n\ell})$ of functions on $\{0, 1\}^*$ with rational values. The intended meaning is that for every u the sequence $r_n(u) = (r_{n\ell}(u))$ of rational numbers represents the real number $g_n(u)$ with $|g_n(u) - r_{n\ell}(u)| < 2^{-\ell}$ for all ℓ. We thus require the presence of a sequence of functions $r_{n\ell} : \{0, 1\}^* \to \mathbb{Q}$ with

$$\left|r_{np}(u) - r_{nq}(u)\right| < 2^{-p} + 2^{-q} . \tag{5}$$

The conditions $g_n \geqslant 0$ and $g_{n+1} \geqslant g_n$ can then be put as

$$r_{n\ell}(u) \geqslant -2^{-\ell} \quad \text{and} \quad r_{n+1,2\ell}(u) \geqslant r_{n,2\ell}(u) - 2^{-\ell}, \tag{6}$$

respectively. To express that each g_n is continuous, we may assume that the modulus of continuity (M_{nk}) with $M_{nk} \geqslant n$ from (3) works also for r_n; that is,

$$\overline{u} M_{nk} = \overline{v} M_{nk} \implies \forall \ell \left(\left|r_{n,2\ell}(u) - r_{n,2\ell}(v)\right| < 2^{-k} + 2^{-\ell}\right) . \tag{7}$$

A decreasing sequence (g_n) of nonnegative continuous functions on $\{0, 1\}^{\mathbb{N}}$ can therefore be identified with

(∗) a sequence of functions $r_{n\ell} : \{0, 1\}^* \to \mathbb{Q}$ and a sequence of integers M_{nk} for which $M_{nk} \geqslant n$ and which satisfy the conditions (5), (6), and (7).[3]

In particular, the real number $G_k(u)$ is given by the sequence $(R_{k\ell}(u))$ of rational numbers

$$R_{k\ell}(u) = \max\left\{r_{|u|\ell}(w) : w \in W_k(u)\right\} .$$

Hence $G_k(u) < 2^{-k}$ means that $R_{k\ell}(u) + 2^{-\ell} < 2^{-k}$ for some ℓ, and $u \in U_k$ corresponds to $u \in A_k$ with

$$A_k = \left\{u : \exists \ell \left(R_{k\ell}(u) + 2^{-\ell} < 2^{-k}\right)\right\} .$$

In all, to assert $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$ means that all data of type (∗) satisfy

$$\forall k \, \forall \alpha \, \exists n \; (\overline{\alpha} n \in A_k) \implies \forall k \, \exists N \, \forall \alpha \, \exists n \leqslant N \; (\overline{\alpha} n \in A_k) . \tag{8}$$

Note that (8) is the counterpart of (4).

4 The Fan Theorem as an Equivalent of Dini's Theorem

A subset B of $\{0, 1\}^*$ is detachable if $u \in B$ is a decidable predicate of $u \in \{0, 1\}^*$; that is, for each u either $u \in B$ or else $u \notin B$. To give a detachable subset B of $\{0, 1\}^*$ is the same as to give its characteristic function $\chi_B : \{0, 1\}^* \to \{0, 1\}$ with $\chi_B(u) = 1$ precisely when $u \in B$.

Moreover, a subset B of $\{0, 1\}^*$ is a bar if for every α there is n with $\overline{\alpha}n \in B$, while a bar B is uniform if there exists N such that for every α there is $n \leqslant N$ with $\overline{\alpha}n \in B$. Brouwer's fan theorem for detachable bars reads as follows:

FT *Every detachable bar is uniform.*

Another way to put FT is to require the validity of the implication

$$\forall \alpha\, \exists n\, (\overline{\alpha}n \in B) \implies \exists N\, \forall \alpha\, \exists n \leqslant N\, (\overline{\alpha}n \in B) \tag{9}$$

from all detachable subsets B of $\{0, 1\}^*$. We refer to [8] for formal versions of the notion of a detachable subset of $\{0, 1\}^*$ and of the more specific notion which occurs next.

One arrives at an equivalent of FT by restricting it to the subsets B of $\{0, 1\}^*$ which are closed under extension; that is, if $u \in B$ and $w \geqslant u$, then $w \in B$. Every B satisfying this extra condition is a uniform bar precisely when there exists N such that $\overline{\alpha}N \in B$ for every α.

Lemma 4.1 ([8], Lemma 1) FT *is equivalent to the statement that the implication*

$$\forall \alpha\, \exists n\, (\overline{\alpha}n \in B) \implies \exists N \forall \alpha\, (\overline{\alpha}N \in B) \tag{10}$$

holds for all detachable subsets B of $\{0, 1\}^*$ which are closed under extension.

Note the difference between (10) and (9).

Proposition 4.2 FT *follows from* $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$.

Proof We use Lemma 4.1. Let B be a detachable subset of $\{0, 1\}^*$ that is closed under extension, and assume that B is a bar. For every n define $f_n : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}$ by setting $f_n(\alpha) = \chi_B(\overline{\alpha}n)$. Each f_n is continuous, because $f_n(\alpha)$ depends—for n fixed—only on $\overline{\alpha}n$. In addition, the sequence (f_n) is increasing and converges simply to 1. Hence the convergence is uniform, which is to say that B is a uniform bar. □

To show the reverse implication, FT needs to be extended to the subsets B of $\{0, 1\}^*$ which—in the terminology of [3]—are simply existential; that is, there is a sequence (C_ℓ) of detachable subsets of $\{0, 1\}^*$ such that $u \in B$ precisely when $u \in C_\ell$ for some ℓ. This condition is equivalent to the existence of a detachable subset C of $\{0, 1\}^* \times \mathbb{N}$ such that $u \in B$ if and only if $(u, \ell) \in C$ for some ℓ.

The following is contained in [9], Proposition 16.15. We do a proof without coding.

Lemma 4.3 FT *is equivalent to the statement that the implication*

$$\forall \alpha\, \exists n\, (\overline{\alpha}n \in B) \implies \exists N\, \forall \alpha\, \exists n \leqslant N\, (\overline{\alpha}n \in B) \tag{11}$$

holds for all simply existential subsets B of $\{0, 1\}^*$.

Proof Only one direction needs a proof. Let B be a simply existential subset of $\{0,1\}^*$, and pick a detachable subset C of $\{0,1\}^* \times \mathbb{N}$ for which $u \in B$ if and only if $(u,\ell) \in C$ for some ℓ. Set

$$D = \{u : \exists n, \ell \leqslant |u| \, ((\overline{u}n, \ell) \in C)\} \ ,$$

which is a detachable subset of $\{0,1\}^*$. If B is a bar, then so is D (because if $(\overline{\alpha}n, \ell) \in C$, then $\overline{\alpha}m \in D$ with $m = \max\{n, \ell\}$). On the other hand, if D is a uniform bar, then so is B (because if $\overline{\alpha}m \in D$ for some $m \leqslant N$, then $(\overline{\alpha}n, \ell) \in C$ for certain $n, \ell \leqslant m \leqslant N$). So if (9) holds with D in place of B, then (11) follows. □

As (11) and (9) are identical, FT can also be put as "every simply existential bar is uniform."

Proposition 4.4 FT *implies* $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$.

Proof Assume that we are given data of type (∗). To arrive at (8), it suffices to achieve

$$\forall \alpha \, \exists n \, (\overline{\alpha}n \in A_k) \implies \exists N \, \forall \alpha \, \exists n \leqslant N \, (\overline{\alpha}n \in A_k) \tag{12}$$

for arbitrary but fixed k. Since $R_{k\ell}(u)$ is a rational number, the condition $R_{k\ell}(u) + 2^{-\ell} < 2^{-k}$ is a decidable property of u for any given ℓ. Hence A_k is a simply existential subset of $\{0,1\}^*$, and Lemma 4.3 applies. □

Corollary 4.5 FT *and* $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$ *are equivalent.*

Since FT already follows from $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$ for functions with values in $\{0,1\}$ (see the proof of Proposition 4.2), Corollary 4.5 would still hold if one restricted $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$ to functions with values in $\{0,1\}$ or, more generally, in $\mathbb{N}$. The equivalence of FT and $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$ for functions with values in $\mathbb{N}$ has also been shown by Veldman [19].

Proposition 4.6 $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$ *entails* DT_X *for all compact metric spaces* X.

Proof Every compact metric space X is a continuous image of the Cantor space ([3], Chapter 5, Theorem 1.4; [18], Chapter 7, Corollary 4.4); that is, there is a continuous mapping from $\{0,1\}^{\mathbb{N}}$ onto X. Along any such mapping, DT_X can be deduced from $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$ in the obvious way. □

Proposition 4.7 $\mathrm{DT}_{[0,1]}$ *implies* FT.

Proof Let B be a bar. By [3], Chapter 6, Theorem 2.7, there is a continuous function h on $[0,1]$ with $h > 0$ such that $\inf h > 0$ if and only if B is uniform. Set $f = 1 - \min\{h, 1/2\}$, for which $0 < f < 1$. Since $\inf h > 0$ if and only if $\sup f < 1$, it suffices to show that f is bounded away from 1. To this end, set $f_n = 1 - f^n$ for every $n \in \mathbb{N}$. Note that f and all the f_n are continuous and that (f_n) is increasing and converges simply to 1. By hypothesis, (f_n) converges uniformly to 1; whence $f^n = 1 - f_n < 1/2$ and thus $f < \sqrt[n]{1/2}$ for some n. □

The idea underlying the foregoing proof stems from the recursive counterexample to Dini's theorem that Bridges [4] ascribes to Richman.[4]

Theorem 4.8 *The following four items are equivalent:* FT, $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$, DT_X *for all compact metric spaces* X, *and* $\mathrm{DT}_{[0,1]}$.

5 The Relation of Dini's Theorem to Weak König's Lemma

A (binary) tree is a detachable subset T of $\{0, 1\}^*$ which contains the empty sequence and which is closed under restriction; that is, if $w \in T$ and $u \leqslant w$, then $u \in T$. A tree T is infinite if for every n there is $u \in T$ with $|u| = n$, and an infinite path of T is an α with $\overline{\alpha}n \in T$ for all n.

Weak König's lemma in Simpson's terminology [17] is the following statement:

WKL *Every infinite tree has an infinite path.*

A tree T is infinite precisely when for every n there is α with $\overline{\alpha}n \in T$. Hence to postulate WKL amounts to require the validity of the implication

$$\forall n\, \exists \alpha\ (\overline{\alpha}n \in T) \implies \exists \alpha\, \forall n\ (\overline{\alpha}n \in T) \tag{13}$$

from all trees T.

It is known that WKL and FT are the classical contrapositives of each other. Ishihara has even proved that WKL implies FT over EL [8]. For the sake of completeness, we shed some light on the classical equivalence of WKL and FT, following [8].

A tree T is finite if there is N such that $|u| < N$ for every $u \in T$, and without infinite path if for every α there is n with $\overline{\alpha}n \notin T$. These are classically equivalent ways to express that a tree is not infinite and has no infinite path, respectively. The classical contrapositive of WKL can therefore be put as follows:

WKL$^\neg$ *Every tree without infinite path is finite.*

A tree T is finite if and only if there is N such that $\overline{\alpha}N \notin T$ for all α. To assert WKL$^\neg$ thus amounts to require the validity of the implication

$$\forall \alpha\, \exists n\ (\overline{\alpha}n \notin T) \implies \exists N\, \forall \alpha\ (\overline{\alpha}N \notin T) \tag{14}$$

from all trees T. Note that (13) and (14) are the classical contrapositives of each other.

Lemma 5.1 WKL$^\neg$ *and* FT *are equivalent.*

Proof Suppose that $\{0, 1\}^*$ is the disjoint union of two inhabited subsets B and T. This is the same as to give a pair of inhabited and detachable subsets B and T each of which is the complement of the other. In this situation, B is closed under extension precisely when T is closed under restriction (that is, T is a tree), which we assume from now on. Moreover, B is a bar if and only if T has no infinite path, and B is a uniform bar if and only if T is a finite tree. In other words, (14) is the same as (10) for any such choice of B and T. □

During the rest of this paper we work within the formal system RCA_0 from [17], whose notations and conventions we adopt. In particular, we switch from constructive to classical reverse mathematics.

Theorem 5.2 DT_X *for all compact metric spaces X is equivalent, over* RCA_0*, to* WKL.

Proof By combining Lemma 5.1 with Proposition 4.2, one can deduce WKL from $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$ in RCA_0. To verify that WKL implies DT_X for all compact X in RCA_0, we mimic the proof of [17], Theorem IV.2.2 as follows.

Let $X = \widehat{A}$ be a compact metric space. Suppose that (g_n) is a decreasing sequence of nonnegative continuous functions on X which converges simply to 0. Let $\varphi(n, a, r, m)$ be a Σ^0_1-formula which says that $a \in A$, $r \in \mathbb{Q}^+$, $m \in \mathbb{N}$, and that

(§) there are $b \in \mathbb{Q}$ and $s \in \mathbb{Q}^+$ with $b < 2^{-n-1}$ and $s < 2^{-n-1}$ such that $(a, r)\, g_m\, (b, s)$.

One can show that

(†) for every $x \in X$ and every n there are a, r, m with $\varphi(n, a, r, m)$ and $d(x, a) < r$.

By [17], Lemma II.3.7, there is a sequence $\langle (a_{ni}, r_{ni}, m_{ni}) : i, n \in \mathbb{N}\rangle$ such that $\varphi(n, a, r, m)$ if and only if $(a, r, m) = (a_{ni}, r_{ni}, m_{ni})$ for some i. By (†), $\langle\langle B(a_{ni}, r_{ni}) : i \in \mathbb{N}\rangle : n \in \mathbb{N}\rangle$ is a sequence of open coverings of X, which—according to [17], Theorem IV.1.6—gives rise to a sequence of finite subcoverings $\langle\langle B(a_{ni}, r_{ni}) : i \leqslant k_n\rangle : n \in \mathbb{N}\rangle$. If we now set $N_n = \max\{m_{ni} : i \leqslant k_n\}$ for every n, then $g_{N_n}(x) < 2^{-n}$ for every $x \in X$. In fact, for every n and every x there is $i \leqslant k_n$ with $x \in B(a_{ni}, r_{ni})$, for which $g_{m_{ni}}(x)$ belongs to the closure of $B(b, s)$ for some b and s as in (§) with $(n, a_{ni}, r_{ni}, m_{ni})$ in place of (n, a, r, m). Since, in particular, $b < 2^{-n-1}$ and $s < 2^{-n-1}$, we have $g_{N_n}(x) \leqslant g_{m_{ni}}(x) \leqslant b + s < 2^{-n}$ as required. In other words, (g_n) converges uniformly to 0. □

Kohlenbach [12] deduces DT_X for $X = [0, 1]^n$ with $n \in \mathbb{N}$ from a strong principle of uniform boundedness, which he extracts from a generalization of WKL to higher types [11].

6 Discussion

Bishop's concept of a continuous function on a compact metric space includes a modulus of uniform continuity, whose existence is guaranteed anyhow in the presence of countable choice for natural numbers. On the other hand, "it is interesting to note that 'any continuous function which arises in practice' can be proved in RCA_0 to have a modulus," while "in general its existence is not provable in RCA_0" ([17], Remark IV.2.8). In the present paper, the additional information given by a modulus was only needed on our way from FT to $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$, whereas no modulus occurred at any other place—let alone during the corresponding argument, which we gave in RCA_0, that DT_X for every compact X follows from WKL.

We anyway hold Brouwer's fan theorem for conceptually more appropriate than weak König's lemma to classify uniformity theorems such as Dini's. While FT provides us with a uniform bound—a single natural number—in a way similar to $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$, the conclusion of WKL consists of the existence of an object of a different nature: an infinite sequence. More specifically, the logical form (8) of $\mathrm{DT}_{\{0,1\}^{\mathbb{N}}}$ corresponds rather to (9) than to (13). One cannot make this distinction unless one moves from classical to constructive reverse mathematics.

Notes

1. Related work has been done in parallel by Loeb [13] and Veldman [19].

2. Richman ([16], [14]) has initiated a constructive mathematics without countable choice.

3. Needless to say, these sequences can be put as functions $r : \mathbb{N} \times \mathbb{N} \times \{0, 1\}^* \to \mathbb{Q}$ and $M : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $r(n, \ell, u) = r_{n\ell}(u)$ and $M(n, k) = M_{nk}$—or, by means of an appropriate coding of finite sequences, as functions of type $\mathbb{N} \to \mathbb{N}$.

4. Kamo [10] has nonetheless come up with an effective version of Dini's theorem in computable analysis.

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