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More Fragments of Language

Ian Pratt-Hartmann and Allan Third

Abstract By a *fragment* of a natural language, we understand a collection of sentences forming a naturally delineated subset of that language and equipped with a semantics commanding the general assent of its native speakers. By the *semantic complexity* of such a fragment, we understand the computational complexity of deciding whether any given set of sentences in that fragment represents a logically possible situation. In earlier papers by the first author, the semantic complexity of various fragments of English involving at most *transitive* verbs was investigated. The present paper considers various fragments of English involving *ditransitive* verbs and determines their semantic complexity.

1 Motivation

What logical resources do the various constructions of natural languages put at their speakers' disposal? What, for example, can we say using relative clauses, or pronouns, or passives, that we could not have said without them? What additional expressive power is provided by such mechanisms as quantifier rescoping, plural quantification, tense and aspect, and temporal or spatial prepositions?

One approach to this issue is to define fragments of natural languages involving these linguistic constructions—severally or in combination—and to determine the semantic complexity of these fragments. By a *fragment* of a natural language, we understand a collection of sentences forming a naturally delineated subset of that language and equipped with a truth-conditional semantics commanding the general assent of its native speakers. By the *semantic complexity* of such a fragment, we understand the computational complexity of deciding whether any given set of sentences in that fragment represents a logically possible situation. On this approach, the logical power of the linguistic constructions under investigation is measured by their effects on the cost of determining entailments within the fragments of language in which they feature.

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This general approach was adopted by the first author in two earlier papers (Pratt-Hartmann [9], [10]) where, in particular, the logical power of relative clauses and anaphora was investigated in fragments of English involving at most *transitive* verbs. The present paper discusses the rather different complexity-theoretic landscape that emerges when ditransitive verbs are added to these fragments. In order to make this paper more self-contained, the basic concepts and results of the earlier papers are summarized below.

2 Background

Henceforth, we take a fragment of English to be a set of English sentences defined by a semantically annotated context-free grammar, possibly with additional movement rules (discussed below). The context-free grammar comprises three components: the *syntax*, the *formal lexicon*, and the *content lexicon*. The syntax deals with the expansion of nonterminal categories; the formal lexicon lists the terminals which contribute only logical constants; and the content lexicon lists the terminals which contribute nonlogical constants. For example, the following set of productions yields, to a good approximation, the language of the traditional syllogistic.

Syntax	**Formal lexicon**
$\mathrm{IP}/\varphi(\psi) \to \mathrm{NP}/\varphi, \mathrm{I}'/\psi$	$\mathrm{Det}/\lambda p \lambda q[\exists x(p(x) \wedge q(x))] \to$ some
$\mathrm{I}'/\varphi \to$ is a N'/φ	$\mathrm{Det}/\lambda p \lambda q[\forall x(p(x) \to q(x))] \to$ every
$\mathrm{I}'/\neg\varphi \to$ is not a N'/φ	$\mathrm{Det}/\lambda p \lambda q[\forall x(p(x) \to \neg q(x))] \to$ no
$\mathrm{NP}/\varphi \to \mathrm{PropN}/\varphi$	
$\mathrm{NP}/\varphi(\psi) \to \mathrm{Det}/\varphi, \mathrm{N}'/\psi$	
$\mathrm{N}'/\varphi \to \mathrm{N}/\varphi$	

Content lexicon	
$\mathrm{N}/\lambda x[\mathrm{man}(x)] \to$ man	$\mathrm{PropN}/\lambda p[p(\mathrm{socrates})] \to$ Socrates
$\mathrm{N}/\lambda x[\mathrm{mortal}(x)] \to$ mortal	$\mathrm{PropN}/\lambda p[p(\mathrm{diogenes})] \to$ Diogenes
$\vdots$	$\vdots$

These productions generate a set of sentences by successive expansion of the nonterminals in the usual way. Since the primary form-determining element they feature is the copula, we refer both to the above set of productions and to the set of sentences they generate as the fragment *Cop*. No harm need come of this deliberate ambiguity.

The expressions of higher-order logic to the right of the obliques indicate how the semantic value of each phrase depends on the semantic values of its immediate constituents, where $\varphi(\psi)$ indicates the result of applying the function φ to the argument ψ. Thus, a fragment of English, in our sense, not only defines a subset of English sentences, but also assigns to any sentence in that subset one or more formulas of first-order logic representing its possible meanings. The tree in Figure 1 illustrates how this assignment works in practice. The content-lexicon, comprising the open word-classes of common and proper nouns, is assumed to be open-ended. Thus, Cop is more properly thought of as a *family* of languages, each member of which corresponds to a choice of content-lexicon. However, to avoid cumbersome formulations, we speak of "the fragment Cop" to refer to the union of all these languages (or of their corresponding sets of productions).

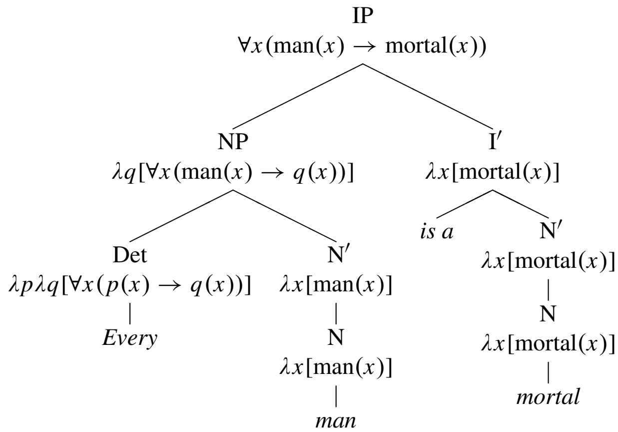

Figure 1 Sentence-generation in the fragment Cop

Calculations such as that of Figure 1 translate the following argument as shown.

Every man is a mortal	$\forall x(\text{man}(x) \rightarrow \text{mortal}(x))$
Socrates is a man	$\text{man}(\text{socrates})$
Socrates is a mortal	$\text{mortal}(\text{socrates})$

Such translations allow familiar semantic concepts to be transferred from first-order logic to the fragment of English in question in the obvious way. In particular, a set of sentences E can be said to *entail* a sentence e if the formulas to which E is translated entail the formula to which e is translated in the usual sense of first-order logic; likewise, a set of sentences E can be said to be *satisfiable* if the set of formulas to which E is translated is satisfiable in the usual sense of first-order logic. For fragments equipped with sentence negation (as are all the fragments considered below), entailment and satisfiability are dual notions in the familiar sense.

Define the *size* of an English sentence to be the number of words it contains; likewise, define the size of a set E of sentences, denoted $\|E\|$, to be the sum of the sizes of its members. Using this concept of size, we can formulate complexity-theoretic questions concerning fragments of English. In particular, the computational complexity of the satisfiability problem for an English fragment is the number of steps of computation required to determine algorithmically whether a given finite set E of sentences in that fragment is satisfiable, expressed as a function of $\|E\|$.

A quick check confirms that every sentence of Cop translates into a formula of one of the forms

$$\pm p_1(c), \quad \exists x_1(p_1(x_1) \wedge \pm p_2(x_1)), \quad \forall x_1(p_1(x_1) \rightarrow \pm p_2(x_1)), \tag{1}$$

where c is an individual constant and p_i ($1 \leq i \leq 2$) are unary predicates. The following observation ([10], Theorem 1) then follows very easily.

Observation 2.1 The problem of determining the satisfiability of a set of sentences in Cop is in PTIME.

No surprises here: the syllogistic is tractable. But the question then arises: what happens to this complexity result as we expand the fragment of English under consideration?

3 Fragments of English without Relative Clauses

The task of this section is to extend the fragment Cop with productions handling transitive and ditransitive verbs. Let TV denote the following collection of productions.

Syntax	Formal lexicon
$\mathrm{I}'/\varphi \rightarrow \mathrm{VP}/\varphi$	Neg $\rightarrow$ does not
$\mathrm{I}'/\varphi \rightarrow \mathrm{NegP}/\varphi$	
$\mathrm{NegP}/\neg\varphi \rightarrow \mathrm{Neg}, \mathrm{VP}/\varphi$	**Content Lexicon**
$\mathrm{VP}/\varphi(\psi) \rightarrow \mathrm{TV}/\varphi, \mathrm{NP}/\psi$	
	$\mathrm{TV}/\lambda s \lambda x[s(\lambda y[\mathrm{admire}(x, y)])] \rightarrow$ admires
	$\mathrm{TV}/\lambda s \lambda x[s(\lambda y[\mathrm{despise}(x, y)])] \rightarrow$ despises
	⋮

For the sake of clarity, we have suppressed the issue of verb-inflections as well as that of negative polarity determiners, since they are easily seen not to affect the results reported below. In the sequel, we will silently correct any such syntactic shortcomings as required.

Similarly, let DTV denote the following set of productions.

Syntax

$\mathrm{VP}/(\varphi(\psi))(\pi) \rightarrow \mathrm{DTV}/\varphi, \mathrm{NP}/\psi$, to, NP/π

Content Lexicon

$\mathrm{DTV}/\lambda s \lambda t \lambda x[s(\lambda y[t(\lambda z[\mathrm{recommend}(x, y, z)])])] \rightarrow$ recommends

⋮

Augmenting the productions of Cop with those of TV and DTV, we obtain a new fragment of English, Cop+TV+DTV, with semantics computed in the same way. Thus, for example, Cop+TV+DTV contains the following sentence and translates it to the indicated first-order formula.

No stoic recommends every sceptic to some cynic
$$\forall x(\mathrm{stoic}(x) \rightarrow \neg\forall y(\mathrm{sceptic}(y) \rightarrow \exists z(\mathrm{cynic}(z) \wedge \mathrm{recommend}(x, y, z)))). \tag{2}$$

For the sake of simplicity, we have employed productions which determine relative scopes of quantifiers in a very specific way: subjects outscope direct objects, which in turn outscope indirect objects. This restriction simplifies the presentation of the semantics and saves us from having to worry about scope ambiguities. Of course, there is no reason in principle why fragments with different scoping regimes cannot be treated using essentially the same techniques as those employed here. However, we show below that all such fragments have the same complexity as the one presented here.

Having defined the fragment Cop+TV+DTV, we now show that it has a tractable satisfiability problem. In the sequel, we regularly employ familiar terminology and techniques from the theorem-proving literature. In particular, we take for granted the notions of clausal form and the conversion of first-order formulas to clausal form. If Γ is a set of clauses, we write $|\Gamma|$ to denote the number of clauses in Γ and $\|\Gamma\|$ to denote the total number of symbols occurring in Γ. If X is any expression (term, atom, literal, clause), we write $\mathrm{Vars}(X)$ for the set of variables occurring in X. Clauses are, of course, read as being implicitly universally quantified. In particular, a *model* of a clause C is a model of its universal closure $\forall x_1 \ldots \forall x_n C$, where $\{x_1, \ldots, x_n\} = \mathrm{Vars}(C)$. A clause is *negative* if all its literals are negative. A clause is *Horn* if it involves at most one positive literal. An expression is *functional* if it involves at least one function-symbol. For any literal L, let us say that L is a *unary* literal if the predicate occurring in L is a unary predicate. Finally, we assume familiarity with the standard apparatus and terminology of A-ordered resolution theorem-proving (for an introduction, see, e.g., Leitsch [6]).

Let e be a sentence in the fragment Cop+TV+DTV. Let φ be the formula which results from taking the translation of e produced by the grammar of Cop+TV+DTV and then moving any negations inward so that they apply only to atomic formulas. Then φ will be of one of the forms

$$\begin{array}{ll}
\pm p_1(c) & L_0 \\
\exists x_1(p_1(x_1) \wedge \pm p_2(x_1)) & Q_1x_1(p_1(x_1), L_1) \\
\forall x_1(p_1(x_1) \rightarrow \pm p_2(x_1)) & Q_1x_1(p_1(x_1), Q_2x_2(p_2(x_2), L_2)) \\
 & Q_1x_1(p_1(x_1), \\
 & \quad Q_2x_2(p_2(x_2), Q_3x_3(p_3(x_3), L_3))),
\end{array} \tag{3}$$

where c is an individual constant, p_i $(1 \leq i \leq 3)$ is a unary predicate, L_i $(0 \leq i \leq 3)$ is a nonfunctional, nonunary literal involving *exactly* the variables $\{x_1, \ldots, x_i\}$ (so that L_0 is ground) and $Q_ix_i(\varphi, \psi)$ $(1 \leq i \leq 3)$ is either $\exists x_i(\varphi \wedge \psi)$ or $\forall x_i(\varphi \rightarrow \psi)$. The L_i may include individual constants.

Now let φ be Skolemized and converted into clausal form in the standard way. The resulting clauses will all be of the forms

$$\begin{array}{ll}
\pm p_1(c) & L_0 \\
\neg p_1(x_1) \vee \pm p_2(x_1) & \neg p_1(x_1) \vee L_1 \\
\neg p_1(x_1) \vee p_2(f(x_1)) & \neg p_1(x_1) \vee \neg p_2(x_2) \vee L_2 \\
 & \neg p_1(x_1) \vee \neg p_2(x_2) \vee \neg p_3(x_3) \vee L_3 \\
 & \neg p_1(x_1) \vee \neg p_2(x_2) \vee p_3(g(x_1, x_2)),
\end{array} \tag{4}$$

where c is an individual constant, p_i $(1 \leq i \leq 3)$ is a unary predicate, L_i $(0 \leq i \leq 3)$ is a nonunary literal involving exactly the variables $\{x_1, \ldots, x_i\}$, and f, g are (Skolem) function-symbols of the indicated arities. Here the L_i may include individual constants and function-symbols.

Any clause C having one of the forms (4) evidently satisfies all the following properties:

P1 C has at most one nonunary literal;
P2 if C has a nonunary literal, then every unary literal of C has the form $\neg p(x)$, where x is a variable and p a unary predicate;
Q1 C is Horn;
Q2 if C has a positive unary literal, C is of one of the forms

$$p(c), \quad \neg p(x) \vee q(x), \quad \neg p(x) \vee o(f(x)), \quad \neg p(x) \vee \neg q(y) \vee o(g(x, y)) \tag{5}$$

where p, q, and o are unary predicates, c is an individual constant, and f and g are (Skolem) function-symbols.

These properties of clauses arise directly from properties of the English sentence-forms in Cop+TV+DTV. In particular, every Cop+TV+DTV sentence has at most one main verb, and all predicates contributed by common nouns are unary. Property **P1** gives rise to the following lemma, which we use repeatedly below. To state this lemma, we introduce some (nonstandard) terminology. Let C' and C'' be two clauses which resolve to form a clause C. We call C a *nonunary resolvent* of C' and C'' if the eliminated literal of C' (and hence also of C'') in this resolution is nonunary. If Γ is a set of clauses, the *nonunary derived set* of Γ is the set of all clauses C which are nonunary resolvents of some pair of clauses in Γ.

Lemma 3.1 *Let Γ be a set of clauses each of which has at most one nonunary literal. Let Γ_1 be the set of clauses in Γ having only unary literals; let Γ_2 be the set of clauses in Γ having exactly one nonunary literal. Now let Γ'_2 be the nonunary derived set of Γ_2, and let $\Gamma' = \Gamma_1 \cup \Gamma'_2$. Then Γ has a model if and only if Γ' has.*

Proof The only-if direction is immediate, since Γ entails Γ'. For the if direction, suppose Γ has no model. Define the partial order $\prec^*$ on the set of atoms by

$$A \prec^* A' \text{ iff } A \text{ is unary and } A' \text{ is nonunary.}$$

Trivially, $\prec^*$ is well-founded and invariant under substitutions and, thus, is an A-ordering. By the completeness theorem for A-ordered resolution, there is a derivation $\mathbb{D}$ of $\perp$ from Γ using $\prec^*$-ordered resolution and factoring. (Think of $\mathbb{D}$ as a tree of inference steps with leaves in Γ and root $\perp$.) Since $\prec^*$ ranks nonunary literals above unary literals, any resolutions in $\mathbb{D}$ which eliminate nonunary literals lie at the leaves of $\mathbb{D}$. Removing these leaves will leave us with a derivation of $\perp$ from clauses in Γ', whence Γ' has no model. $\square$

Theorem 3.2 *The problem of determining the satisfiability of a set of sentences in* Cop+TV+DTV *is in* PTIME*.*

Proof Let E be a set of sentences in Cop+TV+DTV. Let Φ be the set of formulas obtained by taking the first-order translations of E and moving negations inward. Let Γ be the set of clauses obtained by converting Φ to clausal form in the standard way. Certainly, Γ can be computed from E in polynomial time, and we have already observed that every $C \in \Gamma$ satisfies properties **P1**–**P2** and **Q1**–**Q2**. By **P1**, let Γ' be the set of clauses formed by eliminating nonunary literals from Γ as specified in Lemma 3.1. It is easy to check that every $C \in \Gamma'$ satisfies properties **Q1**–**Q2**. Certainly, $\|\Gamma'\|$ is bounded by a polynomial function of $\|\Gamma\|$. Thus it suffices to show that we can determine in polynomial time whether the set of clauses Γ' has a model.

Denote the signature of Γ' by $\Sigma = (K, F, P)$, where K is the set of constants, F the set of function-symbols (unary and binary), and P the set of unary predicates. Consider any $f \in F$. Since f was introduced by Skolemization of formulas of forms (3), there exists exactly one clause $C \in \Gamma$ such that C contains a positive unary literal and f appears in C. It follows from **P1** and **P2** that Γ' contains exactly one such clause too; let us denote this clause by C_f. By **Q2**, we see that the clause C_f can be written as $\gamma_f(\bar{x}) \vee o_f(f(\bar{x}))$, where $\gamma_f(\bar{x})$ is a nonfunctional clause and o_f a unary predicate; in particular, o_f and γ_f are determined by f. We introduce the following notation. If $p, p' \in P$, we write $p \Rightarrow p'$ if there exist $p_0, \ldots, p_n \in P$ such that $p = p_0$, $p' = p_n$, and $\neg p_i(x) \vee p_{i+1}(x) \in \Gamma'$ for all i $(0 \leq i < n)$. By **Q1**, Γ' is a set of Horn clauses; so let $\mathfrak{H}$ be the structure over the Herbrand universe of Σ whose diagram is generated by applying hyperresolution to the clauses in Γ' to exhaustion in the usual way. It is well known that if Γ' has any model, then $\mathfrak{H} \models \Gamma'$ (i.e., $\mathfrak{H}$ is the "least true" model of Γ'). In fact, if Γ'' is any other set of Horn clauses with the same nonnegative clauses as Γ' and Γ'' has any model, then $\mathfrak{H} \models \Gamma''$. It follows from **Q2** that, for all $p \in P$, all $f \in F$, and all tuples of ground terms $\bar{t}$ over Σ (of the same arity as f),

$$\mathfrak{H} \models p(f(\bar{t})) \text{ iff } \mathfrak{H} \not\models \gamma_f(\bar{t}) \text{ and } o_f \Rightarrow p. \tag{6}$$

This leads to the following observation. Let $f \in F$, $p_1, \ldots, p_m \in P$, and let $\bar{t}_1, \ldots, \bar{t}_m$ be tuples of ground terms over Σ, all of the same arity as f. If, for all j $(1 \leq j \leq m)$, $\mathfrak{H} \models p_j(f(\bar{t}_j))$, then there is a common tuple $\bar{t}$ such that, for all j $(1 \leq j \leq m)$, $\mathfrak{H} \models p_j(f(\bar{t}))$. To see this, put $\bar{t} = \bar{t}_j$ for *any* j such that $1 \leq j \leq m$, and apply (6).

Our next step is to eliminate the function-symbols occurring in the negative clauses of Γ'. Suppose that $C = L_1 \vee \cdots \vee L_n$ is a negative clause of Γ' containing a function-symbol. Thus, for some i $(1 \leq i \leq n)$, we have $L_i = \neg p(f(\bar{u}))$, where $p \in P$, $f \in F$, and $\bar{u}$ is a tuple of terms. Define the clause C' by

$$C' = L_1 \vee \cdots \vee L_{i-1} \vee \gamma_f(\bar{u}) \vee L_{i+i} \vee \cdots \vee L_n, \tag{7}$$

and define the clause set Γ'' by

$$\Gamma'' = \begin{cases} (\Gamma' \setminus \{C\}) \cup \{C'\} \text{ if } o_f \Rightarrow p \\ \Gamma' \setminus \{C\} \text{ otherwise.} \end{cases}$$

Thus, Γ'' has fewer occurrences of function-symbols in negative clauses than Γ'. We claim that Γ'' has a model if and only if Γ' has. For suppose that Γ' has a model, so that $\mathfrak{H} \models \Gamma'$. We show that, if $o_f \Rightarrow p$, then $\mathfrak{H} \models C'$. For contradiction, suppose that $C'\theta$ is a ground instance of C' such that $\mathfrak{H} \not\models C'\theta$. Then $\mathfrak{H} \not\models L_j\theta$ for all j $(1 \leq j \leq n)$ such that $j \neq i$, and $\mathfrak{H} \not\models \gamma_f(\bar{u})\theta$. Since $o_f \Rightarrow p$, by (6), $\mathfrak{H} \models p(f(\bar{u}))\theta$ so that $\mathfrak{H} \not\models C\theta$, contradicting $\mathfrak{H} \models \Gamma'$. Hence, $\mathfrak{H} \models \Gamma''$. Conversely, suppose that Γ'' has a model, so that again $\mathfrak{H} \models \Gamma''$ (for Γ' and Γ'' have the same nonnegative clauses). We show that $\mathfrak{H} \models C$. For contradiction, suppose that $C\theta$ is a ground instance of C such that $\mathfrak{H} \not\models C\theta$. Then $\mathfrak{H} \not\models L_j\theta$ for all j $(1 \leq j \leq n)$ and $\mathfrak{H} \models p(f(\bar{u}))\theta$, so that, by (6), $\mathfrak{H} \not\models \gamma_f(\bar{u})\theta$ and $o_f \Rightarrow p$. But then $\mathfrak{H} \not\models C'\theta$ and $C' \in \Gamma''$, contradicting $\mathfrak{H} \models \Gamma''$. Thus, Γ' has a model if and only if Γ'' has, as claimed. Proceeding in this way, we obtain a collection of clauses Δ such that Δ has a model if and only if Γ' has and such that no function-symbols appear in the negative clauses of Δ. Finally, we see by **Q2** that any clause γ_f contains only one occurrence of each of its free variables. Hence the clauses (7) used in the construction of Δ do not involve the duplication of material, whence that construction proceeds in polynomial time.

Our final step is to eliminate the remaining function-symbols from Δ. Let Σ^0 be the signature $(K \cup F, \varnothing, P)$, with individual constants $K \cup F$, unary predicates P, and no function-symbols. That is, Σ^0 is the same as Σ, except that the function-symbols have been rebranded as individual constants. For every $C \in \Delta$, let C^0 be the result of replacing any literal of the form $p(f(\bar{u}))$, where $p \in P$ and $f \in F$, by the ground literal (over Σ^0) $p(f)$, and let $\Delta^0 = \{C^0 \mid C \in \Delta\}$. Thus, Δ^0 is a set of Horn clauses over the signature Σ^0. We claim that Δ^0 has a model if and only if Δ has. The only-if direction is simple: given any structure $\mathfrak{A}^0$ interpreting Σ^0, we can convert $\mathfrak{A}^0$ to a structure $\mathfrak{A}$ interpreting Σ over the same domain by setting, for each function-symbol $f \in F$, $f^{\mathfrak{A}}(\bar{a}) = f^{\mathfrak{A}^0}$, where $\bar{a}$ is any tuple of the appropriate arity. That is, the function-symbols in F are interpreted as the obvious constant functions. It is then immediate that, if $\mathfrak{A}^0 \models \Delta^0$, then $\mathfrak{A} \models \Delta$.

Conversely, suppose Δ has a model so that $\mathfrak{H} \models \Delta$. Define a structure $\mathfrak{H}^0$ over the domain $H^0 = K \cup F$ by setting, for all $c \in C$, $f \in F$ and $p \in P$:

$$\begin{array}{rcl} c^{\mathfrak{H}^0} & = & c \\ f^{\mathfrak{H}^0} & = & f \\ p^{\mathfrak{H}^0} & = & \{c \in K \mid \mathfrak{H} \models p(c)\} \cup \\ & & \{f \in F \mid \mathfrak{H} \models p(f(\bar{t})) \text{ for some ground tuple } \bar{t}\}. \end{array}$$

We claim that $\mathfrak{H}^0 \models \Delta^0$. To see this, we consider the nonnegative clauses and the negative clauses of Δ^0 separately. The former are easily dealt with. Suppose $C = D^0$ is a nonnegative clause in Δ^0. By **Q2**, D has one of the forms (5), and we can argue by cases that $\mathfrak{H}^0 \models C$. For illustration, suppose that D is $\neg p(x) \vee o(f(x))$, so that C is $\neg p(x) \vee o(f)$. If $\mathfrak{H}^0 \not\models C$, then either $\mathfrak{H}^0 \models p(c)$ for some $c \in K$ or $\mathfrak{H}^0 \models p(g)$ for some $g \in F$. Either way, by construction of $\mathfrak{H}^0$, $\mathfrak{H} \models p(t)$ for some closed term t over Σ, whence $\mathfrak{H} \models o(f(t))$, whence, again by construction of $\mathfrak{H}^0$, $\mathfrak{H}^0 \models o(f)$, contradicting the supposition that $\mathfrak{H}^0 \not\models C$. The other cases in (5) are dealt with similarly or are even simpler. It remains to deal with the negative clauses of Δ^0. Suppose that C is a negative clause in Δ^0. Since Δ has no occurrences of function-symbols in negative clauses (for we removed all such occurrences in the construction of Δ), it follows from the definition of Δ^0 that $C \in \Delta$ and C is nonfunctional. By reordering the literals in C, we may write $C = \delta_1(x_1) \vee \cdots \vee \delta_n(x_n) \vee \epsilon$, where ϵ is a ground clause, the $x_1, \ldots, x_n$ are distinct variables and, for each i $(1 \leq i \leq n)$, $\delta_i(x_i) = \neg p_{i1}(x_i) \vee \cdots \vee \neg p_{im_i}(x_i)$. If $\mathfrak{H}^0 \not\models C$ then, $\mathfrak{H}^0 \not\models \epsilon$ and, for all i $(1 \leq i \leq n)$, we have either $\mathfrak{H}^0 \not\models \delta_i(c_i)$ for some $c_i \in K$ or $\mathfrak{H}^0 \not\models \delta_i(f_i)$ for some $f_i \in F$. In the former case, by construction of $\mathfrak{H}^0$, $\mathfrak{H} \not\models \delta_i(c_i)$. In the latter case, by construction of $\mathfrak{H}^0$, there are tuples $\bar{t}_{i1}, \ldots, \bar{t}_{im_i}$ of ground terms over Σ such that, for all j $(1 \leq j \leq m_i)$, $\mathfrak{H} \models p_{ij}(f_i(\bar{t}_{ij}))$. But we observed above that, in that case, there exists a common ground tuple $\bar{t}_i$ such that $\mathfrak{H} \models p_{ij}(f_i(\bar{t}_i))$ for all j $(1 \leq j \leq m_i)$—that is, $\mathfrak{H} \not\models \delta_i(f_i(\bar{t}_i))$. Either way, for all i $(1 \leq i \leq n)$, there exists a ground term t_i over Σ such that $\mathfrak{H} \not\models \delta_i(t_i)$. Finally, since $\mathfrak{H}^0 \not\models \epsilon$, we have, by the construction of $\mathfrak{H}^0$, $\mathfrak{H} \not\models \epsilon$. Therefore, since the $x_1, \ldots, x_n$ are distinct, $\mathfrak{H} \not\models C$, contradicting $\mathfrak{H} \models \Delta$.

Thus, we have shown that Δ has a model if and only if Δ^0 has. Since Δ^0 is a set of function-free Horn clauses, the question of whether Δ^0 has a model can be answered in time bounded by a polynomial function of $\|\Delta^0\|$ and, hence, of $\|\Delta\|$. This completes the proof. □

We illustrate the procedure of the above proof with a concrete example. The following is a valid argument in Cop+TV(+DTV).

Every philosopher despises some cynic
Every gentleman is a philosopher
Every cynic is a man
Every man is a human
Socrates is a gentleman.

Some gentleman despises some human.

Let Φ be the set of formulas consisting of the first-order translations of the premises and the negation of the first-order translation of the conclusion. The validity or otherwise of the argument is of course equivalent to the unsatisfiability or otherwise of Φ. Converting Φ into clausal form and Skolemizing, we obtain a set Γ consisting of the clauses

$$\begin{array}{ll} \neg p(x) \vee c(f(x)) & \neg p(x) \vee d(x, f(x)) \\ \neg g(x) \vee p(x) & \neg c(x) \vee m(x) \\ \neg m(x) \vee h(x) & \neg g(x) \vee \neg h(y) \vee \neg d(x, y) \\ g(s), & \end{array} \tag{8}$$

where p is the unary predicate corresponding to philosopher, c corresponds to cynic, and so on, and f is a Skolem-function. Resolving away the nonunary literals of Γ yields the set Γ' consisting of the clauses

$$\begin{array}{ll} \neg p(x) \vee c(f(x)) & \neg g(x) \vee p(x) \\ \neg c(x) \vee m(x) & \neg m(x) \vee h(x) \\ g(s) & \neg p(x) \vee \neg g(x) \vee \neg h(f(x)). \end{array} \tag{9}$$

By Lemma 3.1, Γ has a model if and only if Γ' has.

We now eliminate the single Skolem-function f from Γ', beginning with occurrences of f in negative clauses. Only one negative clause in Γ' contains f, namely, $C = \neg p(x) \vee \neg g(x) \vee \neg h(f(x))$. In the notation of the proof, we have $o_f = c$ and $\gamma_f(x) = \neg p(x)$; moreover, it is easily checked that $c \Rightarrow h$. We can thus eliminate f from negative clauses of Γ' by replacing C with $C' = \neg p(x) \vee \neg g(x) \vee \neg p(x)$. Thus, Γ' has a model if and only if Δ has, where Δ is the set of clauses

$$\begin{array}{ll} \neg p(x) \vee c(f(x)) & \neg g(x) \vee p(x) \\ \neg c(x) \vee m(x) & \neg m(x) \vee h(x) \\ g(s) & \neg p(x) \vee \neg g(x) \vee \neg p(x). \end{array} \tag{10}$$

The final step is to eliminate f from the nonnegative clauses in Δ. There is precisely one such clause, namely, $C_f = \neg p(x) \vee c(f(x))$. Let $C_f^0 = \neg p(x) \vee c(f)$, treating the symbol f as a constant rather than a function symbol, and replace C_f with C_f^0 in Δ to obtain the set Δ^0 consisting of the nonfunctional Horn clauses

$$\begin{array}{ll} \neg p(x) \vee c(f) & \neg g(x) \vee p(x) \\ \neg c(x) \vee m(x) & \neg m(x) \vee h(x) \\ g(s) & \neg p(x) \vee \neg g(x) \vee \neg p(x). \end{array} \tag{11}$$

Resolution and factoring can then be applied to Δ^0 to derive $\perp$, thus showing the validity of the original argument.

Actually, our proof establishes a little more than we claimed. We mentioned above that the grammar of Cop+TV+DTV makes rather specific decisions about the relative scopes of quantifiers, and so it is natural to ask whether the satisfiability problem for this fragment would remain in PTIME if these decisions were made differently, or indeed if quantifier rescoping were allowed. The above proof shows that the answer is yes. For the only effect of rescoping on the translations of sentences in Cop+TV+DTV is to reorder the arguments in the literals L_1, L_2, and L_3 in the forms (3). Yet the proof of Theorem 3.2 made no reference to the order of arguments in these literals. Hence, for present purposes, there is no point in complicating our grammar in respect of quantifier scoping: whatever those complications, the satisfiability problem for the resulting fragment will remain in PTIME.

4 Fragments with Relative Clauses

In this section, we show that, in the presence of relative clauses, the addition of transitive and ditransitive verbs successively increases the complexity class of the satisfiability problem. The following productions suffice to generate relative clauses.

Syntax	**Formal Lexicon**
$\mathrm{N}'/\varphi(\psi) \to \mathrm{N}/\psi,\ \mathrm{CP}/\varphi$	$\mathrm{C} \to$
$\mathrm{CP}/\varphi(\psi) \to \mathrm{CSpec}_t/\varphi,\ \mathrm{C}'_t/\psi$	$\mathrm{RelPro} \to$ who
$\mathrm{C}'_t/\lambda t[\varphi] \to \mathrm{C},\ \mathrm{IP}/\varphi$	$\mathrm{RelPro} \to$ which
$\mathrm{NP}/\varphi \to \mathrm{RelPro}/\varphi$	
$\mathrm{CSpec}_t/\lambda q \lambda p \lambda x[p(x) \wedge q(x)] \to$	

In addition, we assume that, following generation of an IP by these productions, relative pronouns are subject to wh-movement to produce the observed word-order. For our purposes, we may take the wh-movement rule to require (i) the empty position CSpec_t must be filled by movement of a RelPro from within the IP which forms its right-sister (i.e., which it c-commands); (ii) every RelPro must move to some such CSpec_t position; (iii) every RelPro moving to CSpec_t leaves behind a (new) trace t, which contributes the semantic value $\lambda p[p(t)]$. We denote by Rel the collection of productions above, together with the rule of wh-movement. For the sake of clarity, we have ignored the issue of agreement of relative pronouns with their antecedents—animate or inanimate. By combining these rules variously with the sets of productions Cop, TV, and DTV, we obtain, for example, the fragments Cop+Rel, Cop+Rel+TV, and Cop+Rel+TV+DTV.

The semantic information with which the above rules are augmented can then be understood as for our previous fragments, with meanings computed *after* wh-movement. Figure 2 illustrates a typical derivation in Cop+Rel, with the arrow indicating wh-movement in the obvious way.

Calculations such as that of Figure 2 show that, for example, Cop+Rel generates all the sentences featured in the evidently valid argument

Every philosopher who is not a stoic is a cynic
Every stoic is a man
Every cynic is a man

Every philosopher is a man,

while Cop+Rel+TV+DTV generates all the sentences featured in the (less evidently) valid argument

Every sceptic recommends every sceptic to every cynic
No sceptic recommends any stoic who hates some cynic to any philosopher
Diogenes is a cynic whom every sceptic hates
Every cynic is a philosopher

No stoic is a sceptic.

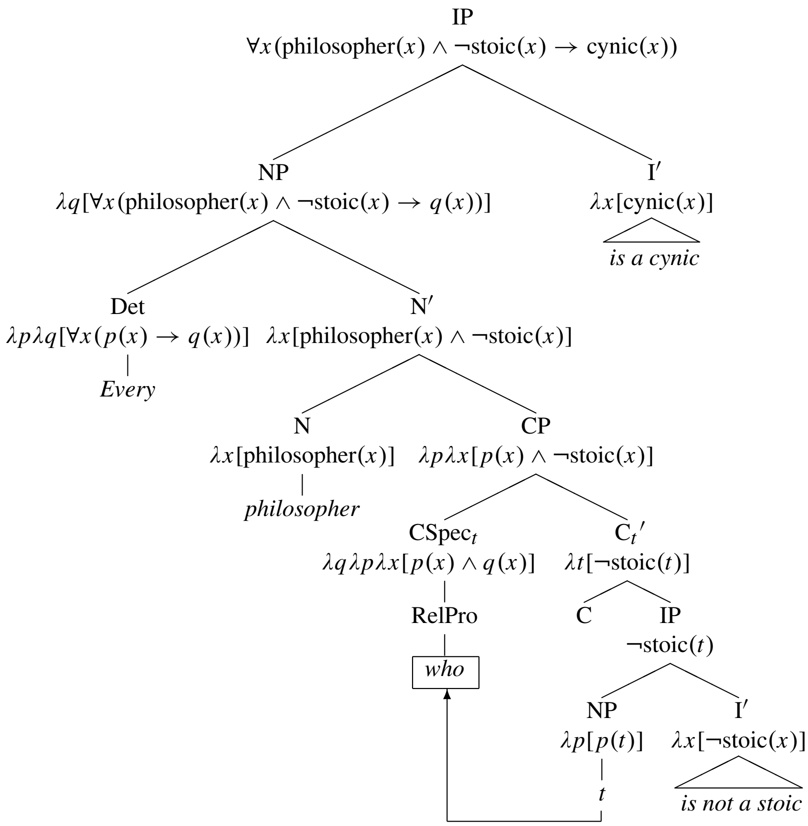

Figure 2 Typical phrase-structure in the fragment Cop+Rel ([10], p. 213, Figure 1)

(Note that we have corrected the determiner some to its negative-polarity counterpart any.) The resulting logical translations produced by the given semantics are exactly what one would expect and need not be spelled out here.

Having defined fragments Cop+Rel, Cop+Rel+TV, and Cop+Rel+TV+DTV, we now turn to analyzing their complexity.

Theorem 4.1 ([10], Theorem 2) *The problem of determining the satisfiability of a set of sentences in* Cop+Rel *is* NP-*complete.*

Theorem 4.2 ([10], Theorem 3) *The problem of determining the satisfiability of a set of sentences in* Cop+Rel+TV *is* EXPTIME-*complete.*

Theorem 4.1 is straightforward and need not concern us further in this paper. Theorem 4.2, by contrast, is harder, and since the techniques involved will prove valuable in the sequel, we repeat the proof here.

Before embarking on this proof, consider a typical sentence recognized by Cop+Rel+TV:

No sceptic likes any stoic who hates some cynic. (12)

Applying the semantics of Cop+Rel+TV and moving the negation inward in the usual way produces the logical translation:

$$\forall x(\text{sceptic}(x) \rightarrow \forall y(\text{stoic}(y) \wedge \exists z(\text{cynic}(z) \wedge \text{hate}(y, z)) \rightarrow \neg\text{like}(x, y))). \quad (13)$$

Suppose now that we introduce the unary predicate p to stand for the property of not hating any cynic, and the unary predicate q to stand for the property of being a stoic who hates some cynic; then we may replace (13) with the following formulas:

$$\begin{array}{l} \forall y(p(y) \rightarrow \forall z(\text{cynic}(z) \wedge \neg\text{hate}(y, z))) \\ \forall y(\text{stoic}(y) \wedge \neg p(y) \rightarrow q(y)) \\ \forall x(\text{sceptic}(x) \rightarrow \forall y(q(y) \rightarrow \neg\text{like}(x, y))). \end{array} \quad (14)$$

Evidently, the formulas (14) together imply (13); conversely, any structure satisfying (13) can be expanded (by interpreting the new predicates p and q as indicated) to a structure satisfying (14). In effect, the N$'$ stoic who hates some cynic in (12) has been "defined out" using the new predicates p and q.

More generally, let E be any collection of Cop+Rel+TV-sentences. By successively defining out relative clauses as described above, the translations of E may be equisatisfiably transformed, in polynomial time, into a set of formulas of the forms

$$\begin{array}{ll} \pm p_1(c) & L_0 \\ \exists x_1(p_1(x_1) \wedge \pm p_2(x_1)) & Q_1 x_1(p_1(x_1), L_1) \\ \forall x_1(p_1(x_1) \rightarrow \pm p_2(x_1)) & Q_1 x_1(p_1(x_1), Q_2 x_2(p_2(x_2), L_2)) \\ \forall x_1(p_1(x_1) \wedge \pm p_2(x_1) \rightarrow p_3(x_1)), & \end{array} \quad (15)$$

and thence, also in polynomial time, into a set of clauses of the forms

$$\begin{array}{ll} \pm p_1(c) & L_0 \\ \neg p_1(x_1) \vee \pm p_2(x_1) & \neg p_1(x_1) \vee L_1 \\ \neg p_1(x_1) \vee p_2(f(x_1)) & \neg p_1(x_1) \vee \neg p_2(x_2) \vee L_2 \\ \neg p_1(x_1) \vee \pm p_2(x_1) \vee p_3(x_1), & \end{array} \quad (16)$$

using the same notation as before. Evidently, properties **P1–P2** continue to hold for clauses of the forms (16). By contrast, **Q1–Q2** no longer hold; this is the fault of the relative clauses, of course.

In the sequel, we take the *depth* of an atom A, denoted $d(A)$, to be the maximal functional depth of any term in that atom, counting nonfunctional terms as having depth 1. Thus, $d(p(c)) = d(p(x)) = 1$, $d(r(x, f(x))) = 2$, $d(p(g(x, f(x)))) = 3$, and so on. If $x \in \text{Vars}(A)$, we define the *depth of x in A*, denoted $d(x, A)$ to be the functional depth of the deepest occurrence of x in A, defined similarly. Thus, $d(x, (p(x))) = 1$, $d(x, r(x, f(x))) = 2$, $d(x, p(g(x, f(x)))) = 3$, and so on. With these preliminaries behind us, we turn now to the complexity of satisfiability in Cop+Rel+TV.

Lemma 4.3 *The problem of determining the satisfiability of a set of sentences in* Cop+Rel+TV *is in* EXPTIME.

Proof Let Φ be the first-order translations of some set of Cop+Rel+TV-sentences E. By the foregoing remarks, we may transform Φ in polynomial time into a set of clauses Γ of the forms (16) such that Φ is satisfiable if and only if Γ has a model. By **P1**, we can apply Lemma 3.1 to construct, in polynomial time, a clause set Γ' involving only unary literals such that Γ' has a model if and only if Γ has. Clearly, $|\Gamma'| \leq |\Gamma|^2$. Since all function-symbols in Γ are Skolem-functions, the depth of every clause in Γ is at most 2; moreover, it is simple to check that this is also true of every clause in Γ'.

Since the signature of Γ involves no function-symbols of arity greater than 1, any clause $C \in \Gamma'$ containing distinct variables x and y may be written as a disjunction $C_0 \vee C_1 \vee C_2$, where C_0 is ground, C_1 has no ground literals and involves only the variable x, and C_2 has no ground literals and involves only the variable y. Thus, for any structure $\mathfrak{A}$, $\mathfrak{A} \models C$ if and only if $\mathfrak{A} \models C_0$ or $\mathfrak{A} \models C_1$ or $\mathfrak{A} \models C_2$. Now let Δ be the result of replacing any such $C \in \Gamma'$ by one of the corresponding clauses C_0, C_1, or C_2. Since $|\Gamma'| \leq |\Gamma|^2$, there are at most exponentially many possibilities for Δ, and it is obvious that Γ' has a model if and only if some such set Δ has. That is, we can construct, in time bounded by an exponential function of $\|\Gamma'\|$, a set of sets of clauses $\mathbf{K}$ such that Γ' has a model if and only if some clause set in $\mathbf{K}$ has, and such that, for every $\Delta \in \mathbf{K}$ and every $C \in \Delta$, $d(C) \leq 2$, and one of the following conditions holds:

N1 C is ground;
N2 for every literal L of C, $\mathrm{Vars}(L) = \mathrm{Vars}(C) = \{x\}$ for some variable x.

Let us call a clause satisfying either **N1** or **N2** *normal*.

Define the ordering $\prec^d$ on atoms by

$$A \prec^d A' \text{ iff } \quad \begin{array}{l} d(A) < d(A'), \mathrm{Vars}(A) \subseteq \mathrm{Vars}(A'), \text{ and} \\ d(x, A) < d(x, A') \text{ for all } x \in \mathrm{Vars}(A). \end{array}$$

It is well known that $\prec^d$ is an A-ordering. (For details, see [6], p. 218ff.) Moreover, $\prec^d$-ordered resolution and factoring preserves the property of normality as defined above and does not increase the depth of normal clauses. Hence, saturation of any set of normal clauses with a fixed depth bound under $\prec^d$-ordered resolution and factoring can be computed in exponential time. This completes the proof that the satisfiability problem for Cop+Rel+TV is in EXPTIME. □

Lemma 4.4 *The problem of determining the satisfiability of a set of sentences in* Cop+Rel+TV *is* EXPTIME-*hard.*

Proof from [10], p. 217 Recall that the logic K^U is the modal logic K together with an additional universal modality U, whose semantics are given by

$$\models_w U\varphi \text{ if and only if } \models_{w'} \varphi \text{ for all worlds } w'.$$

The satisfiability problem for K^U is EXPTIME-hard. (The proof is an easy adaptation of the corresponding result for propositional dynamic logic; see, e.g., Harel et al. [5], p. 216ff.) It suffices, therefore, to reduce this problem to satisfiability in Cop+Rel+TV. Let φ be a formula of K^U. For convenience, we take V to be the dual modality to U. Let there be content lexicon entries specifying that the symbols Es and Rs are transitive verbs, that the symbol element is a noun, and also that, for every proper or improper subformula ψ of φ, the symbol A_ψ is a noun. Now define, for each such ψ a set of Cop+Rel+TV-sentences T_ψ inductively as follows.

$$T_p = \emptyset \text{ if } p \text{ is atomic.}$$

$$T_{\psi\wedge\pi} = T_\psi \cup T_\pi \cup \left\{ \begin{array}{l} \textsf{Every } \mathsf{A}_\psi \textsf{ which is an } \mathsf{A}_\pi \textsf{ is an } \mathsf{A}_{\psi\wedge\pi}, \\ \textsf{Every } \mathsf{A}_{\psi\wedge\pi} \textsf{ is an } \mathsf{A}_\psi, \\ \textsf{Every } \mathsf{A}_{\psi\wedge\pi} \textsf{ is an } \mathsf{A}_\pi. \end{array} \right\}$$

$$T_{\neg\psi} = T_\psi \cup \left\{ \begin{array}{l} \textsf{Every element which is not an } \mathsf{A}_\psi \textsf{ is an } \mathsf{A}_{\neg\psi}, \\ \textsf{No } \mathsf{A}_\psi \textsf{ is an } \mathsf{A}_{\neg\psi}. \end{array} \right\}$$

$$T_{\diamond\psi} = T_\psi \cup \left\{ \begin{array}{l} \textsf{Every element which Rs some } \mathsf{A}_\psi \textsf{ is an } \mathsf{A}_{\diamond\psi}, \\ \textsf{Every } \mathsf{A}_{\diamond\psi} \textsf{ Rs some } \mathsf{A}_\psi. \end{array} \right\}$$

$$T_{V\varphi} = T_\psi \cup \left\{ \begin{array}{l} \textsf{Every element which Es some } \mathsf{A}_\psi \textsf{ is an } \mathsf{A}_{V\psi}, \\ \textsf{Every } \mathsf{A}_{V\psi} \textsf{ Es some } \mathsf{A}_\psi. \end{array} \right\}$$

Now let S_φ be the collection of Cop+Rel+TV-sentences

$$\{\textsf{Every } \mathsf{A}_\psi \textsf{ is an element} \mid \psi \text{ a subformula of } \varphi\} \cup$$
$$\{\textsf{Some } \mathsf{A}_\varphi \textsf{ is an } \mathsf{A}_\varphi, \textsf{ Every element Es every element}\}.$$

It is routine to show that φ is satisfiable if and only if $T_\varphi \cup S_\varphi$ is satisfiable. □

Lemmas 4.3 and 4.4 establish Theorem 4.2.

It may not have escaped the reader's attention that the grammar of Cop+Rel+TV fails to prohibit center-embedded sentences:

Every ship which some sailor who some dog likes owns is a wreck

$$\begin{array}{l} \forall x(\text{ship}(x)\wedge \\ \quad \exists y(\text{sailor}(y)\wedge \\ \qquad \exists z(\text{dog}(z) \wedge \text{like}(z,y)) \wedge \text{own}(y,x)) \to \text{wreck}(x)). \end{array}$$

Such sentences are certainly unnatural and arguably ungrammatical. The question therefore arises as to whether a more sophisticated grammar for a fragment of English involving relative clauses and transitive verbs—one which, for example, filters out center-embedded sentences—would still yield an EXPTIME-complete satisfiability problem.

The answer is that it would. Trivially, restricting the fragment cannot affect the upper complexity bound of its satisfiability problem, so we need only worry about establishing EXPTIME-hardness. But all of the sentences in the set $T_\varphi \cup S_\varphi$ featured in the proof of Lemma 4.4 are grammatically unobjectionable and, in particular, do not exhibit center-embedding. (In fact, they do not involve multiple relative clauses at all.) It follows that no linguistically motivated tightening of the fragment Cop+Rel+TV could possibly invalidate Lemma 4.4 or, therefore, Theorem 4.2. As an aside, we remark that none of the sentences in $T_\varphi \cup S_\varphi$ involves object-relative clauses. Thus, eliminating these from the fragment would not render it tractable either.

We are now ready to tackle the more difficult problem of the complexity of fragments involving both relative clauses and ditransitive verbs. As a preliminary, consider a typical sentence of Cop+Rel+TV+DTV:

No sceptic recommends any stoic who hates some cynic to any philosopher.

Applying the semantics of Cop+Rel+TV+DTV and moving negations inward, we obtain

$$\forall x(\text{sceptic}(x) \to \forall y(\text{stoic}(y) \wedge \exists z(\text{cynic}(z) \wedge \text{hate}(y, z)) \to \\ \forall w(\text{phil}(w) \to \neg\text{recommend}(x, y, w)))). \quad (17)$$

Just as with the fragment Cop+Rel+TV, so too with Cop+Rel+TV+DTV, we can "define out" relative clauses by introducing new unary predicates. For example, the formula (17) is equisatisfiable with the formulas

$$\forall y(p(y) \to \forall z(\text{cynic}(z) \to \neg\text{hate}(y, z)))$$
$$\forall y(\text{stoic}(y) \wedge \neg p(y) \to q(y))$$
$$\forall x(\text{sceptic}(x) \to \forall y(q(y) \to \forall w(\text{phil}(w) \to \neg\text{recommend}(x, y, w))))$$

by a similar argument to that which allowed us to transform (13) equisatisfiably into (14).

More generally, let E be any collection of Cop+Rel+TV+DTV-sentences and let Φ be their first-order translations. By repeatedly defining out relative clauses as described above, Φ may be equisatisfiably transformed, in polynomial time, into a set of formulas of the forms

$$\begin{array}{ll} \pm p_1(c) & L_0 \\ \exists x_1(p_1(x_1) \wedge \pm p_2(x_1)) & Q_1 x_1(p_1(x_1), L_1) \\ \forall x_1(p_1(x_1) \to \pm p_2(x_1)) & Q_1 x_1(p_1(x_1), Q_2 x_2(p_2(x_2), L_2)) \\ \forall x_1(p_1(x_1) \wedge \pm p_2(x_1) & Q_1 x_1(p_1(x_1), Q_2 x_2(p_2(x_2), \\ \qquad\qquad \to p_3(x_1)) & \qquad\qquad Q_3 x_3(p_3(x_3), L_3))), \end{array} \quad (18)$$

and thence, also in polynomial time, into a set of clauses of the forms

$$\begin{array}{ll} \pm p_1(c) & L_0 \\ \neg p_1(x_1) \vee \pm p_2(x_1) & \neg p_1(x_1) \vee L_1 \\ \neg p_1(x_1) \vee \pm p_2(x_1) \vee p_3(x_1) & \neg p_1(x_1) \vee \neg p_2(x_2) \vee L_2 \\ \neg p_1(x_1) \vee p_2(f(x_1)) & \neg p_1(x_1) \vee \neg p_2(x_2) \vee \neg p_3(x_3) \vee L_3 \\ & \neg p_1(x_1) \vee \neg p_2(x_2) \vee p_3(g(x_1, x_2)), \end{array} \quad (19)$$

using the same notation as before. It is easy to verify that clauses of the forms (19) satisfy properties **P1**–**P2** but not **Q1**–**Q2**.

If C is any clause, denote by **P3** the property

P3 every variable in C is an argument of every nonunary literal in C.

Not every clause satisfies this property. For example, if $C = \neg p(x) \vee \neg q(f(x), c)$, we have $x \in \text{Vars}(C)$, but x is not an *argument* of the nonunary literal $\neg q(f(x), c)$. Nevertheless, we claim that any clause C arising from the translation of a Cop+Rel+TV+DTV-sentence (after defining out relative clauses) does satisfy **P3**. To see why, observe that if C contains a nonunary literal L, then L corresponds to a transitive or ditransitive verb (possibly embedded inside one or more CPs). But it is easy to see that any variable x in C arises from an NP which is an argument of that verb, whence x is an argument of L. We use this fact in the next lemma.

Lemma 4.5 *The problem of determining the satisfiability of a set of sentences in* Cop+Rel+TV+DTV *is in* NEXPTIME.

Proof Let Φ be the first-order translations of some set of Cop+Rel+TV+DTV-sentences E. By the foregoing remarks, we may transform Φ in polynomial time into a set of clauses Γ of the forms (19) such that Φ is satisfiable if and only if Γ has a model.

By **P1**, we resolve away all nonunary literals to obtain the clause set Γ' as specified in Lemma 3.1. Again, it is routine to verify that every $C \in \Gamma'$ satisfies **P1–P2** and that the functional depth of any clause in Γ' is bounded by a constant K (in fact, $K = 3$ suffices).

We claim that, for every $C \in \Gamma'$, if C contains a literal with distinct variables x and y, then (i) that literal is of the form $\pm r(f(x, y))$ and (ii) all other literals of C are of the forms $\neg p(x)$ or $\neg p(y)$ for various unary predicates p. Note that Γ' contains only unary predicates, so suppose $C \in \Gamma'$ and τ is a term in C containing distinct variables x and y. Thus, τ certainly contains at least one binary function-symbol f. If C is in both Γ' and Γ, then by scanning the forms (19), C is of the form $\neg p_1(x_1) \vee \neg p_2(x_2) \vee p_3(f(x_1, x_2))$. If, on the other hand, $C \in \Gamma' \setminus \Gamma$, let D, D' be the clauses in Γ which resolve to form C, let L, L' be the nonunary literals of D and D', respectively, and let θ be the substitution involved in the resolution. Without loss of generality, suppose that f occurs in D, and hence, by **P2**, in L. Since f is a Skolem function, L has an argument $f(u_1, u_2)$, with $u_1, u_2 \in \text{Vars}(D)$, whence, by **P3**, the arguments of L are $u_1, u_2, f(u_1, u_2)$ (in some order). Since f is a Skolem function, L' cannot contain f, so that, by the requirements of unification, the arguments of L' are τ_1, τ_2, υ (in the corresponding order), where υ is a variable and τ_1, τ_2 are terms not involving υ. But since $\tau = f(u_1, u_2)\theta = f(u_1\theta, u_2\theta) = f(\tau_1\theta, \tau_2\theta)$ involves two variables, τ_1 and τ_2 between them involve at least two variables, which, as we have said, must be distinct from υ. Hence, by **P3**, all three arguments of L' are variables. By inspection of the forms (19), we see that D must be of the form $\neg p_1(x) \vee \neg p_2(y) \vee \pm r(x, y, f(x, y))$ (disregarding the order of arguments in the final literal), and D' of the form $\neg p'_1(x') \vee \neg p'_2(y') \vee \neg p'_3(z) \vee \mp r(x', y', z)$. Therefore, C is of the form

$$\neg p_1(x) \vee \neg p_2(y) \vee \neg p'_1(x) \vee \neg p'_2(y) \vee \neg p'_3(f(x, y)).$$

This proves the claim.

Since Γ' contains only unary predicates, all clauses which do not contain any term with two distinct variables can be split into variable-disjoint components exactly as for Lemma 4.3. Thus, we can construct, in time bounded by an exponential function of $\|\Gamma'\|$, a set of sets of clauses $\mathbf{K}$ such that Γ' has a model if and only if some clause set in $\mathbf{K}$ has, and such that, for every $\Delta \in \mathbf{K}$ and every $C \in \Delta$, one of the following conditions holds:

N1 C is ground;
N2 for every literal L of C, $\text{Vars}(L) = \text{Vars}(C) = \{x\}$ for some x;
N3 C is of the form

$$\pm p_1(x) \vee \cdots \vee \pm p_k(x) \vee \pm q_1(y) \vee \cdots \vee \pm q_l(y) \vee \pm r_1(f(x, y)) \vee \cdots \vee \pm r_m(f(x, y)), \quad (20)$$

where $k \geq 0$, $l \geq 0$, and $m \geq 1$. (In fact, $m = 1$, but no matter.) Let us call a clause satisfying any of these three conditions *semi-normal*. (Thus, a semi-normal clause is one which is normal, as defined in Lemma 4.3, or which satisfies **N3**.)

Consider again $\prec^d$-ordered resolution. Suppose C, C' are semi-normal clauses which $\prec_d$-resolve to form a clause C''. We claim that C'' is either itself semi-normal or the disjunction of two normal clauses C_1 and C_2 such that $\text{Vars}(C_1) \cap \text{Vars}(C_2) = \varnothing$.

This is established by considering all nine possibilities determined by which of the conditions **N1**–**N3** C and C' satisfy. We present only the case where C' satisfies **N2** and C satisfies **N3**. (The other cases are all more straightforward.) Write $C = D(x, y) \vee L(x, y)$, and $C' = D(x') \vee L'(x')$, where $L(x, y)$ is the resolved-upon literal of C and $L'(x')$ the resolved-upon literal of C'. From the ordering $\prec_d$, $L(x, y)$ is of the form $\pm r(f(x, y))$, whence $L'(x')$ is either of the form $\mp r(f(\tau_1, \tau_2))$, for some terms τ_1 and τ_2, or of the form $\mp r(x')$. If $L'(x')$ is of the form $\mp r(f(\tau_1, \tau_2))$, then $C'' = D(\tau_1, \tau_2) \vee D'(x')$ and hence is either a clause satisfying **N2** or else (in case C'' has ground literals) the disjunction of a clause satisfying **N1** and a clause satisfying **N2**. If $L'(x')$ is of the form $\mp r(x')$, then the order $\prec^d$ ensures that C' is nonfunctional, so that $C'' = D(x, y) \vee D'(f(x, y))$ is either a clause satisfying **N3** or else (in case there are no occurrences of f in C'') the disjunction of two clauses satisfying **N2**.

We are now in a position to state the procedure for determining whether any clause set $\Delta \in \mathbf{K}$ has a model. Let Δ_1 be the set of clauses in Δ having at most one variable and Δ_2 the set of clauses in Δ having two variables. The procedure is as follows.

1. Guess a set Θ of semi-normal clauses over the signature of Δ, with functional depth bounded by the constant K (=3). Let Θ_1 be the set of clauses in Θ having at most one variable, and Θ_2 the set of clauses in Θ having two variables.
2. Saturate the set of normal clauses $\Theta_1 \cup \Delta_1$ under $\prec_d$-ordered resolution, and let the result be Δ_1^∞. (As noted above, this process terminates in exponential time.) If $\perp \in \Delta_1^\infty$, exit with failure.
3. If any clauses in $\Delta_2 \cup \Theta_2$ $\prec^d$ resolve to form a clause which is not subsumed by any clause of Θ, exit with failure.
4. If any clause in $\Delta_2 \cup \Theta_2$ $\prec^d$ resolves with any clause in Δ_1^∞ to form a clause which is not subsumed by any clause of Θ, exit with failure.
5. Exit with success.

Any run of this procedure certainly terminates in time bounded by an exponential function of $\|\Delta\|$. It suffices then to show that there is a successfully terminating run if and only if Δ has a model.

Suppose that Δ has no model. However Θ is chosen in Step 1, there is certainly a $\prec_d$-ordered deduction $\mathbb{D}$ of $\perp$ from the set of clauses $\Delta \cup \Theta$. Without loss of generality, assume $\mathbb{D}$ is smallest, so that, in particular, no steps of resolution result in any clause C'' which is subsumed by a clause in Θ. If $\mathbb{D}$ involves no clauses with two variables, we have $\perp \in \Delta_1^\infty$, so that the procedure fails at Step 2. On the other hand, if $\mathbb{D}$ does involve a resolution-step featuring a clause with two variables, consider the *first* such resolution step executed in $\mathbb{D}$. Suppose that, in this resolution step, C and C' resolve to form a clause C''. By exchanging C and C' if necessary, we have $C \in (\Delta_2 \cup \Theta_2)$ and either $C' \in (\Delta_2 \cup \Theta_2)$ or $C' \in \Delta_1^\infty$. Moreover, by the minimality of $\mathbb{D}$, C'' is not subsumed by any clause in Θ. If $C' \in (\Delta_2 \cup \Theta_2)$, then the procedure fails at Step 3; if $C' \in \Delta_1^\infty$, then it fails at Step 4.

Suppose, conversely, that $\mathfrak{A} \models \Delta$, and consider the run of the procedure where the set Θ chosen in Step 1 is the set of all and only the semi-normal clauses (over the relevant signature) of functional depth bounded by K (=3) which are true in $\mathfrak{A}$. Certainly, $\perp \notin \Delta_1^\infty$, so that this run of the procedure does not fail at Step 2. Suppose $C, C' \in \Delta_2 \cup \Theta_2$ resolve to form a clause C''. Obviously, $\mathfrak{A} \models C''$. Moreover, we have shown above that C'' is either itself semi-normal or else the disjunction $C_1 \vee C_2$ of two variable-disjoint normal clauses. In the former case, $C'' \in \Theta$; and in the latter case, the variable disjointness of C_1 and C_2 ensures that either $\mathfrak{A} \models C_1$ or $\mathfrak{A} \models C_2$ whence either $C_1 \in \Theta$ or $C_2 \in \Theta$. Thus, C'' is subsumed by a clause in Θ, and this run of the procedure does not fail at Step 3. A similar argument shows that it does not fail at Step 4 either. Therefore, it terminates with success. □

For the matching lower bound, recall that the *two-variable fragment* $\mathcal{L}^2$ is the set of function-free formulas of first-order logic featuring at most two variables. The satisfiability problem for $\mathcal{L}^2$ is known to be NEXPTIME-hard, a result which can be established by encoding exponential tiling problems using $\mathcal{L}_2$-formulas. For a detailed explanation, see, for example, Börger et al. [2], p. 253ff. By taking a certain amount of care with the encodings, this result can be strengthened slightly, as follows.

Lemma 4.6 *The problem of determining whether a set of clauses of the form*

$$\begin{array}{ll} \neg p_{i1}(x) \vee \neg p_{i2}(y) \vee \neg p_{i3}(x, y) & (1 \leq i \leq n_1) \\ \neg q_{i1}(x, y) \vee \neg q_{i2}(x, y) \vee \neg q_{i3}(x, y) & (1 \leq i \leq n_2) \\ s_{i1}(x) \vee s_{i2}(x) & (1 \leq i \leq n_3) \\ t_{i1}(x, y) \vee t_{i2}(x, y) & (1 \leq i \leq n_4) \\ \neg r(x, f(x)), & \end{array} \tag{21}$$

has a model, where $n_1, \ldots, n_4$ are nonnegative integers, the (subscripted) p, q, r, s, and t are predicates of the indicated arities (not necessarily distinct), and f is a function-symbol, is NEXPTIME*-hard.*

Proof Routine (but tedious) massaging of the clauses given in [2], p. 253ff. □

Now we can establish a lower complexity bound for Cop+Rel+TV+DTV.

Lemma 4.7 *The problem of determining the satisfiability of a set of sentences in* Cop+Rel+TV+DTV *is* NEXPTIME*-hard.*

Proof We reduce the problem of whether a set of clauses of the form given in Lemma 4.6 has a model to the satisfiability problem for Cop+Rel+TV+DTV. Let Γ be such a set of clauses then. For every binary predicate p appearing in Γ, let p^+ be a new unary predicate, and additionally, for each i, $(1 \leq i \leq n_2)$ let q^+_{i12} be a new unary predicate. Finally, let n and o be new unary predicates, c_0 a new individual constant, $\oplus$ a new binary function-symbol (written with infix notation), and d a new ternary predicate. Now let Δ be the clause set

$$\begin{array}{ll} \neg n(x) \vee \neg p_{i1}(x) \vee \neg n(y) \vee \neg p_{i2}(y) \vee \neg p^+_{i3}(z) \vee d(x, y, z) & (1 \leq i \leq n_1) \\ \neg q^+_{i12}(z) \vee \neg q^+_{i3}(z) \vee n(z) & (1 \leq i \leq n_2) \\ \neg q^+_{i1}(z) \vee \neg q^+_{i2}(z) \vee q^+_{i12}(z) & (1 \leq i \leq n_2) \\ \neg n(x) \vee s_{i1}(x) \vee s_{i2}(x) & (1 \leq i \leq n_3) \\ \neg o(z) \vee t^+_{i1}(z) \vee t^+_{i2}(z) & (1 \leq i \leq n_4) \end{array}$$

$$\begin{array}{ll} \neg n(x) \vee \neg r^+(z) \vee d(x, f(x), z) & \neg n(x) \vee n(f(x)) \\ \neg n(x) \vee \neg n(y) \vee \neg d(x, y, x \oplus y) & \neg n(x) \vee \neg n(y) \vee o(x \oplus y) \\ \neg n(x) \vee \neg n(y) \vee \neg n(z) \vee d(x, y, z) & \neg n(x) \vee \neg o(x) \\ n(c_0). & \end{array}$$

We claim that Γ has a model if and only if Δ has. For suppose $\mathfrak{A} \models \Gamma$. We assume, without loss of generality, that $A \cap A^2 = \emptyset$. Define a structure $\mathfrak{B}$ as follows. Let $B = A \cup A^2$. If p is any unary predicate in Γ, let $p^{\mathfrak{B}} = p^{\mathfrak{A}}$. Let $n^{\mathfrak{B}} = A$ and $o^{\mathfrak{B}} = A^2$. If p is any binary predicate in Γ, let $p^{+\mathfrak{B}} = p^{\mathfrak{A}}$ (note that $p^{\mathfrak{A}} \subseteq A^2 \subseteq B$). Let $c_0^{\mathfrak{B}}$ be any element of A. For all $a \in A$, define $f^{\mathfrak{B}}(a) = f^{\mathfrak{A}}(a)$ and extend $f^{\mathfrak{B}}$ to the whole of B arbitrarily. For all $a, a' \in A$, let $a \oplus^{\mathfrak{B}} a' = \langle a, a' \rangle$ and extend $\oplus^{\mathfrak{B}}$ to the whole of B^2 arbitrarily. Finally, set $q_{i12}^{+\mathfrak{B}} = q_{i1}^{+\mathfrak{B}} \cap q_{i2}^{+\mathfrak{B}}$ for each i $(1 \leq i \leq n_2)$ and set $d^{\mathfrak{B}} = \{\langle a, b, c \rangle \mid a, b \in A, \text{ and } c \neq \langle a, b \rangle\}$. It is routine to check that $\mathfrak{B} \models \Delta$. Conversely, suppose $\mathfrak{B}$ is any structure such that $\mathfrak{B} \models \Delta$. Define a structure $\mathfrak{A}$ as follows. Let $A = n^{\mathfrak{B}}$. If p is any unary predicate in Γ, let $p^{\mathfrak{A}} = p^{\mathfrak{B}} \cap A$. If p is any binary predicate in Γ, let $p^{\mathfrak{A}} = \{\langle a, a' \rangle \in A^2 \mid a \oplus^{\mathfrak{B}} a' \in p^{+\mathfrak{B}}\}$. Define $f^{\mathfrak{A}}$ to be the restriction of $f^{\mathfrak{B}}$ to A. We note that, since Δ contains the clause $n(c_0)$, we have $A \neq \emptyset$, and since Δ contains the clause $\neg n(x) \vee n(f(x))$, $f^{\mathfrak{A}}$ is properly defined. It is routine to check that $\mathfrak{A} \models \Gamma$.

Finally, let n, o and the subscripted letters p, p^+, q^+, r^+, s, and t^+ be common nouns corresponding to the unary predicates of Δ in the obvious way. Let d be a ditransitive verb. Let E be the set of sentences

Every p_{i1} which is an n ds every p_{i2} which is an n to every p_{i3}^+	$(1 \leq i \leq n_1)$	
Every q_{i12}^+ which is a q_{i3}^+ is an n	$(1 \leq i \leq n_2)$	
Every q_{i1}^+ which is a q_{i2}^+ is a q_{i12}^+	$(1 \leq i \leq n_2)$	
Every n which is not an s_{i1} is an s_{i2}	$(1 \leq i \leq n_3)$	(22)
Every o which is not a t_{i1}^+ is a t_{i2}^+	$(1 \leq i \leq n_4)$	
Every n ds some n to every r^+		
No n ds any n to every o		
Every n ds every n to every n		
No n is an o		
Some n is an n.		

Thus, E translates into formulas whose clausal forms are, up to renaming of Skolem functions, the set of clauses Δ. But then E is satisfiable if and only if Δ has a model, which in turn holds if and only if Γ has a model. The NEXPTIME-hardness of Cop+Rel+TV+DTV follows. □

Lemmas 4.5 and 4.7 give us the following.

Theorem 4.8 *The problem of determining the satisfiability of a set of sentences in* Cop+Rel+TV+DTV *is* NEXPTIME-*complete.*

Again, we remark that the sentences (22) used in the proof of Lemma 4.7 are unimpeachably grammatical. It follows that no linguistically motivated tightening of the fragment Cop+Rel+TV+DTV could possibly invalidate Lemma 4.7 or, therefore, Theorem 4.8. Hence, the linguistic simplifications we have made in the definition of Cop+Rel+TV+DTV are harmless; removing them would simply clutter the presentation without changing the final complexity result.

5 Anaphora

This section investigates the effect of anaphora on the semantic complexity of fragments featuring ditransitive verbs. Consider the following productions.

Syntax	**Formal Lexicon**	
NP → Reflexive	Reflexive → itself (himself/herself)	(23)
NP → Pronoun	Pronoun → it (he/she/him/her)	

For simplicity, we shall always take pronouns and reflexives to have antecedents in the sentences in which they occur. That is to say: all anaphora is intrasentential. We further assume the selection of such antecedents to be subject to the usual rules of binding theory, which we need not rehearse here. For the sake of brevity, we have suppressed the semantic annotations for the above productions, which involve somewhat tedious complications of no concern to the present paper. In the sequel, we assume a formal semantics which provides the generally agreed translations. The interested reader is invited to consult [9] for details.

One semantic issue, however, does require clarification before we proceed. When added to the fragments Cop+Rel+TV and Cop+Rel+TV+DTV, the productions (23) generate sentences featuring anaphoric ambiguities. Thus, for example, in

Every sceptic who admires a cynic despises every stoic who hates him, (24)

the pronoun may take as antecedent either the NP headed by sceptic or the NP headed by cynic. (The NP headed by stoic is not available as a pronoun antecedent here.) These two indexation patterns correspond, respectively, to the first-order translations

$$\forall x(\text{sceptic}(x) \wedge \exists y(\text{cynic}(y) \wedge \text{admire}(x, y)) \rightarrow \forall z(\text{stoic}(z) \wedge \text{hate}(z, x) \rightarrow \text{despise}(x, z))) \quad (25)$$

$$\forall x \forall y(\text{sceptic}(x) \wedge \text{cynic}(y) \wedge \text{admire}(x, y) \rightarrow \forall z(\text{stoic}(z) \wedge \text{hate}(z, y) \rightarrow \text{despise}(x, z))). \quad (26)$$

In defining fragments of English equipped with anaphora, therefore, we must decide how to treat ambiguities.

Two options present themselves. The first is to adopt a general method of resolving ambiguities by artificial stipulation; the second is to decorate nouns and pronouns in these sentences with indices specifying which pronouns take which NPs as antecedents. Considering the former option, let the rules RA (RA for restricted anaphora) denote the above productions for pronouns and reflexives equipped with suitable semantics, together with the artificial stipulation that *pronouns must take their closest allowed antecedents*. Here, *closest* means "closest measured along edges of the phrase-structure" and *allowed* means "allowed by the principles of binding theory." (We ignore case and gender agreement.) Figure 3 illustrates this restriction for sentence (24), which lies in the fragment Cop+Rel+TV+RA. Evidently, the NP headed by sceptic is closer, in the relevant sense, to the pronoun him than is the NP headed by cynic. Since co-indexing the pronoun with the NP headed by sceptic corresponds to the sentence-meaning captured by formula (25), this is the formula to which the semantics of Cop+Rel+TV+RA map the sentence (24).

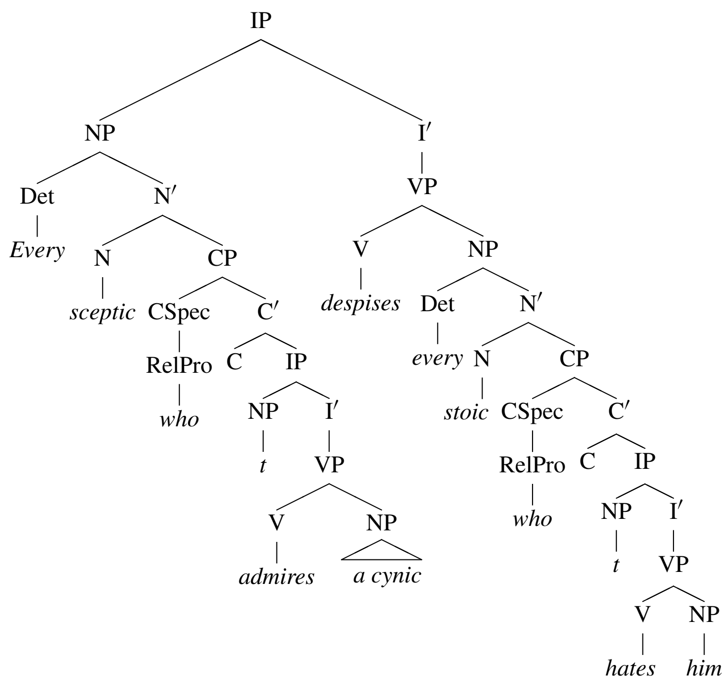

Figure 3 Sentence generation in the fragment Cop+Rel+TV+RA ([10], p. 219, Figure 2)

The fragment Cop+Rel+TV+RA corresponds closely to the two-variable fragment $\mathcal{L}^2$, which, as we mentioned above, has a NEXPTIME-complete satisfiability problem.

Theorem 5.1 ([9], Corollaries 1 and 2) *The problem of determining the satisfiability of a set of sentences in* Cop+Rel+TV+RA *is* NEXPTIME*-complete.*

Turning our attention now to the latter option for dealing with anaphoric ambiguity, let the rules GA (GA for general anaphora) denote the above productions for pronouns and reflexives, where anaphoric antecedents are indicated by co-indexing in the usual way, subject only to the rules of binding theory. (Again, we assume a semantics which yields the generally accepted translations of indexed sentences.) Thus, in fragments involving GA, the meaningful expressions are not sentences, but rather sentences with NP-indices explicitly given. In particular, sentence (24) corresponds to two essentially distinct indexed sentences of Cop+Rel+TV+GA, depending on which NP the pronoun takes as its antecedent. One of these indexed sentences translates to the formula (25), the other, to the formula (26).

Theorem 5.2 ([9], Theorem 5) *The problem of determining the satisfiability of a set of (*NP*-indexed) sentences in* Cop+Rel+TV+GA *is undecidable.*

The main task of this section is to investigate the effect of adding anaphora to fragments of English involving *ditransitive* verbs. In particular, we consider the fragment Cop+Rel+TV+DTV+RA formed by adding the above productions for pronouns and reflexives to the fragment Cop+Rel+TV+DTV, subject to the artificial stipulation that pronouns take their closest allowed antecedents in the sentence in which they occur (in the sense explained above). We continue to suppress the formal presentation of the semantics for this fragment, since the ensuing argument relies only on claims about the logical translations of various sentences which are not open to serious doubt.

For example, the following sentence is in Cop+Rel+TV+DTV+RA.

Every stoic who despises a cynic recommends him to every sceptic whom he fears. (27)

Here we have two pronouns. The rules of binding theory force the pronoun him to take the NP headed by cynic as its antecedent. (Remember, all anaphoric resolution in this fragment is intrasentential by stipulation.) By contrast, the same rules of binding theory allow the pronoun he to take either of two possible antecedents: the NP headed by stoic or the NP headed by cynic. However, since the former is the closer (in the phrase-structure), this is the antecedent which he must take. Hence, the meaning of the sentence (27) in Cop+Rel+TV+DTV+RA is given by the first-order translation

$$\forall x \forall y(\text{stoic}(x) \wedge \text{cynic}(y) \wedge \text{despise}(x, y) \rightarrow \\ \forall z(\text{sceptic}(z) \wedge \text{fear}(x, z) \rightarrow \text{recommend}(x, y, z))). \quad (28)$$

To obtain a lower complexity bound for Cop+Rel+TV+DTV+RA, we review some basic material concerning undecidable problems. Recall that an *unbounded tiling problem* is a triple (C, H, V), where C is a finite set, and H, V are binary relations over C. We call the elements of C *colors*, and we call H and V the *horizontal constraints* and the *vertical constraints*, respectively. A *solution* for (C, H, V) is a function $T : \mathbb{N}^2 \rightarrow C$ such that, for all $i, j \in \mathbb{N}$, $\langle T(i, j), T(i+1, j)\rangle \in H$, and $\langle T(i, j), T(i, j+1)\rangle \in V$. We sometimes refer to such a T as a *tiling*. Intuitively, the elements of C represent types of unit square tile which must be arranged so as to fill the entire upper right quadrant of the plane. Each tile-type is identified by its *color*, and the constraints H (respectively, V) list which colors are allowed to go to the right of (respectively, above) which others. It is well known that determining whether a given unbounded tiling problem has a solution is undecidable (see, e.g. [2], Section 3.1.1).

Theorem 5.3 *The satisfiability problem for* Cop+Rel+TV+DTV+RA *is undecidable.*

Proof We reduce the unbounded tiling problem (C, H, V) to the problem of determining the satisfiability of a set of sentences E in Cop+Rel+TV+DTV+RA. Write C as $\{c_1, \ldots, c_N\}$, with $N \geq 2$. Our sentences employ the following content lexicon.

N/$\lambda x[o(x)] \rightarrow$ one
N/$\lambda x[c_i(x)] \rightarrow$ cee$_i$ $(1 \leq i \leq N)$
N/$\lambda x[d_i(x)] \rightarrow$ dee$_i$ $(1 \leq i \leq N)$

TV/$\lambda s \lambda x[s(\lambda y[f(x, y)])] \rightarrow$ effs
TV/$\lambda s \lambda x[s(\lambda y[g(x, y)])] \rightarrow$ gees
TV/$\lambda s \lambda x[s(\lambda y[d(x, y)])] \rightarrow$ dees

DTV/$\lambda s \lambda t \lambda x[s(\lambda y[t(\lambda z[p(x, y, z)])])] \rightarrow$ peas
DTV/$\lambda s \lambda t \lambda x[s(\lambda y[t(\lambda z[q(x, y, z)])])] \rightarrow$ queues.

For ease of reading, we write every one, some one in the more familiar way as everyone and someone, respectively. Let E be the union of three sets of sentences E_1, E_2, and E_3, defined as follows. The sentences E_1 state that the cee$_i$s partition the nonempty set of people corresponding to the colors in C:

Someone is a dee$_1$ Everyone is a dee$_1$

No cee$_i$ is a cee$_j$ $(1 \leq i < j \leq N)$

Every dee$_i$ who is not a cee$_i$ is a dee$_{i+1}$ $(1 \leq i < N)$

Every dee$_N$ is a cee$_N$.

The sentences E_2 correspond to the horizontal and vertical constraints:

No cee$_i$ effs a cee$_j$ $(1 \leq i \leq N, 1 \leq j \leq N, \langle c_i, c_j \rangle \notin H)$

No cee$_i$ gees a cee$_j$ $(1 \leq i \leq N, 1 \leq j \leq N, \langle c_i, c_j \rangle \notin V)$.

And the sentences E_3 are responsible for manufacturing a grid:

Everyone who dees someone peas him to everyone whom he effs
Everyone gees everyone who someone peas to him
Everyone who dees someone queues him to everyone whom he gees
Everyone effs everyone who someone queues to him
Everyone dees someone Everyone effs someone Everyone gees someone.

Let us consider how the sentences in E_3 function. The first sentence has the same form as (27) and therefore translates to

$$\forall x \forall y (o(x) \wedge o(y) \wedge d(x, y) \rightarrow \forall z (o(z) \wedge f(x, z) \rightarrow p(x, y, z))). \tag{29}$$

(Note: this translation respects the restriction that pronouns take their closest allowed antecedents.) Moreover, the second sentence in this group translates unambiguously to

$$\forall x \forall y (o(x) \wedge o(y) \wedge \exists z (o(z) \wedge p(z, y, x)) \rightarrow g(x, y)). \tag{30}$$

Together, (29) and (30) entail

$$\forall x \forall y \forall z (o(x) \wedge o(y) \wedge o(z) \wedge d(x, y) \wedge f(x, z) \rightarrow g(z, y)). \tag{31}$$

Likewise, the third and fourth sentences of this group translate to formulas entailing

$$\forall x \forall y \forall z (o(x) \wedge o(y) \wedge o(z) \wedge d(x, y) \wedge g(x, z) \rightarrow f(z, y)). \tag{32}$$

The entailments (31) and (32) are depicted in Figure 4. Finally, the remaining three sentences in E_3 evidently translate to

$\forall x (o(x) \rightarrow \exists y (o(y) \wedge d(x, y)))$ $\forall x (o(x) \rightarrow \exists y (o(y) \wedge f(x, y)))$

$\forall x (o(x) \rightarrow \exists y (o(y) \wedge g(x, y)))$.

Figure 4 Grid entailments of sentences of E

Let Φ be the set of first-order translations of the sentences in E. We show that Φ is satisfiable if and only if (C, H, V) has a tiling. For the if-direction, suppose $T : \mathbb{N}^2 \to C$ is a tiling; we construct a model $\mathfrak{A}$ over the domain $A = \mathbb{N}^2$ as follows.

$$
\begin{aligned}
c_k^{\mathfrak{A}} &= \{(i, j) \in A \mid T(i, j) = c_k\} && (1 \leq k \leq N)\\
d_k^{\mathfrak{A}} &= \bigcup_{k \leq l \leq N} c_l^{\mathfrak{A}} && (1 \leq k \leq N)\\
o^{\mathfrak{A}} &= A\\
f^{\mathfrak{A}} &= \{\langle (i, j), (i+1, j)\rangle | i, j \in \mathbb{N}\}\\
g^{\mathfrak{A}} &= \{\langle (i, j), (i, j+1)\rangle | i, j \in \mathbb{N}\}\\
d^{\mathfrak{A}} &= \{\langle (i, j), (i+1, j+1)\rangle | i, j \in \mathbb{N}\}\\
p^{\mathfrak{A}} &= \{\langle (i, j), (i+1, j+1), (i+1, j)\rangle | i, j \in \mathbb{N}\}\\
q^{\mathfrak{A}} &= \{\langle (i, j), (i+1, j+1), (i, j+1)\rangle | i, j \in \mathbb{N}\}.
\end{aligned}
$$

It is routine to verify that $\mathfrak{A} \models \Phi$.

Conversely, suppose $\mathfrak{A} \models \Phi$; we construct a tiling T for (C, H, V). The first step is to define a mapping $\alpha : \mathbb{N}^2 \to A$ such that, for all $i, j \in \mathbb{N}$,

$$
\begin{aligned}
&\langle \alpha(i, j), \alpha(i+1, j)\rangle \in f^{\mathfrak{A}}\\
&\langle \alpha(i, j), \alpha(i, j+1)\rangle \in g^{\mathfrak{A}}.
\end{aligned}
\tag{33}
$$

Given the first sentence in E_1, $o^{\mathfrak{A}}$ is certainly nonempty, so choose $a \in o^{\mathfrak{A}}$ and set $\alpha(0, 0) = a$. If $\alpha(i, 0)$ has been defined but $\alpha(i+1, 0)$ has not, choose $a' \in o^{\mathfrak{A}}$ such that $\mathfrak{A} \models f[\alpha(i, 0), a']$ and set $\alpha(i+1, 0) = a'$. Similarly, if $\alpha(0, j)$ has been defined but $\alpha(0, j+1)$ has not, choose $a' \in o^{\mathfrak{A}}$ such that $\mathfrak{A} \models g[\alpha(0, j), a']$ and set $\alpha(0, j+1) = a'$. Finally, if $\alpha(i, j)$ has been defined but $\alpha(i+1, j+1)$ has not, choose $a' \in o^{\mathfrak{A}}$ such that $\mathfrak{A} \models d[\alpha(i, j), a']$ and set $\alpha(i+1, j+1) = a'$. The sentences in E_3 ensure that these choices are possible, and α is thus defined over the whole of $\mathbb{N}^2$. A double induction using the formulas (31) and (32) shows that (33) hold for all $i, j \in \mathbb{N}$. Our tiling $T : \mathbb{N}^2 \to C$ is then defined by $T(i, j) = c_i$, where $\mathfrak{A} \models c_i[\alpha(i, j)]$. The sentences E_1 ensure that T is well defined. The sentences E_2 together with (33) ensure that T respects the vertical and horizontal constraints. □

6 Discussion

In this paper, we have investigated the computational complexity of determining the satisfiability of sets of sentences in various simple, yet linguistically natural, fragments of English. Table 1 summarizes our results, which extend those reported in [10] by including fragments involving ditransitive verbs.

In recent decades, great strides have been made in locating decidable fragments of first-order logic and determining the computational complexity of their satisfiability problems. Familiar examples are the various classical decidable prefix classes ([2], Ch. 1), the two-variable fragment (Mortimer [8]) the guarded fragment (Andréka et al. [1]) and (curiously neglected) Quine's Fluted fragment (see, e.g., Purdy [14]). By contrast, little has been published on the corresponding problem for fragments of natural languages. The best-known example is McAllester and Givan [7], where a

Fragment	**Complexity**
Cop+TV+DTV	PTIME
Cop+Rel	NP-complete
Cop+Rel+TV	EXPTIME-complete
Cop+Rel+DTV	NEXPTIME-complete
Cop+Rel+TV+RA	NEXPTIME-complete
Cop+Rel+TV+GA	undecidable
Cop+Rel+TV+DTV+RA	undecidable

Table 1 Summary of English fragments and their complexity

formal language with quantificational mechanisms resembling those of natural language is defined and shown to have (in favorable cases) a tractable satisfiability problem. However, the fit between McAllister and Givan's language and any fragment of natural language is very loose.

Some attempts to characterize the logic of natural language fragments have sought to avoid translation into first-order logic. Thus, for example, Fitch [3] proposed the use of combinatory logic; Suppes [16], relation algebra; Purdy ([11], [12], and [13]), his own "natural logic"; and Fyodorov et al. [4], an "order-based" calculus. The motivation for these approaches seems to be the belief that, when reasoning with information expressed in natural language, these formalisms make for better efficiency than does the syntax of first-order logic. The results reported here should not be taken as lending support to this belief, since the satisfiability problem for a given fragment of natural language is defined independently of the formalism used to determine satisfiability. True, the notion of satisfiability in a natural language fragment depends on a semantics, and we have indeed used first-order logic to give truth-conditions of sentences in the fragments we studied. But once having determined *which* sets of sentences (of the relevant fragments) count as satisfiable—and our assignments of truth-conditions could hardly be described as controversial—the complexity-theoretic problem of determining satisfiability makes no further reference to any particular form of representation for sentence-meanings. The satisfiability problems are as easy or as hard as they are independently of any representation system.

The question naturally arises as to whether fragments of natural language such as those identified in this paper correspond in some way to any of the familiar fragments of first-order logic mentioned above. The answer seems to be no. For example, the first-order translations of sentences in the English fragment Cop+TV+DTV are not in any classical decidable prefix fragment, are not in the two-variable fragment, and are not guarded. Sentences of Cop+TV+DTV and indeed of Cop+Rel+TV do translate to formulas in a slightly extended version of the fluted fragment. However, this observation does not yield a tight complexity bound in either case: Purdy [15] shows that deciding satisfiability in this fragment is NEXPTIME-complete. Likewise, the fragment Cop+Rel+TV+DTV lies outside Purdy's fragment, yet is still in NEXPTIME. These fragments, it seems, are new. Of course, this novelty is unsurprising: we cannot expect fragments owing their salience to the syntactic regime of the Russell-Whitehead notation to coincide with those defined in terms of collections of grammatical constructions in English.

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School of Computer Science
University of Manchester
Manchester M13 9PL
UNITED KINGDOM
ian.pratt-hartmann@cs.manchester.ac.uk

The School of Computing
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM
thirda@comp.leeds.ac.uk