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Reflexive Intermediate Propositional Logics

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Abstract Which intermediate propositional logics can prove their own completeness? I call a logic reflexive if a second-order metatheory of arithmetic created from the logic is sufficient to prove the completeness of the original logic. Given the collection of intermediate propositional logics, I prove that the reflexive logics are exactly those that are at least as strong as testability logic, that is, intuitionistic logic plus the scheme $\neg\varphi \vee \neg\neg\varphi$. I show that this result holds regardless of whether Tarskian or Kripke semantics is used in the definition of completeness. I also show that the operation of creating a second-order metatheory is injective, thereby insuring that I am actually considering each logic independently.

1 Introduction

1.1 History and goals

Definition 1.1 (Countermodel completeness) For any set Γ of formulas, if $\Gamma \nvdash \bot$ then $\exists M (M \vDash \Gamma)$.

It has been known since the 1920s that a classical metatheory can prove the countermodel completeness of classical propositional logic (Bernays [1] and Post [14], independently). Yet a strictly intuitionistic metatheory cannot do the same for its propositional case (McCarty [13], Kreisel [8]). In fact, in McCarty [11], one finds a proof that a strictly intuitionistic metatheory cannot prove countermodel completeness for intuitionistic propositional logic even when the set Γ of Definition 1.1 is restricted to be subfinite. (Subfinite sets are subsets of finite sets; intuitionistically, subfinite does not imply finite.)

Thus we have an important distinction between intuitionistic and classical logic: Classical logic can be used to prove countermodel completeness for its own propositional flavor, but intuitionistic logic cannot. This is the motivation for the work of this

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paper. It causes us to ask what other logics are in the neighborhood of classical and intuitionistic logic and which of them can prove their own completeness theorems.

The logics that are at least as strong as intuitionistic logic (Int) and at most as strong as classical logic (Cl) are called the intermediate logics. The main purpose of this paper is to answer the question, "Which intermediate propositional logics can prove their own completeness theorems?" Let us begin with some necessary definitions.

1.2 Definitions and concepts

Definition 1.2 (*Cpl*) I write $Cpl(L)$ to abbreviate "the logic L is standard-countermodel-complete," expressed by the formal second-order statement

$$\forall\Gamma((\Gamma \nvdash_L \bot) \to \exists M(M \vDash \Gamma)).$$

Since this paper discusses many different logics, I subscript the derivability symbol to remove ambiguity; thus $\vdash_L$ means "provable in the logic L." Definition 1.2 formalizes Definition 1.1 so that I can reason about completeness formally. This is necessary because I aim to use second-order metatheories to prove completeness theorems for some propositional logics.

It is natural at this point to ask which semantics should be used in the formal investigation of reflexivity. That is, in Definition 1.2, what sort of object must M be? Since the syntactic symbol $\vdash$ has a different meaning for each logic, one might expect that the set of models under consideration would also change with L. However, I use Tarskian, or truth-functional, semantics for every logic and later prove that this does not harm the results. For example, Section 4.1 proves that the classification of which logics are reflexive is independent of whether Tarskian or Kripke semantics is used.

Definition 1.3 (Truth function) A truth function M is a function from the set of propositional formulas to the powerset of $\{ 0 \}$ satisfying the truth conditions below. I write $M \vDash \varphi$ for $0 \in M(\varphi)$.

1. $M \vDash \neg\varphi \iff M \nvDash \varphi$.
2. $M \vDash \varphi \wedge \psi \iff M \vDash \varphi$ and $M \vDash \psi$.
3. $M \vDash \varphi \vee \psi \iff M \vDash \varphi$ or $M \vDash \psi$.
4. $M \vDash \varphi \to \psi \iff$ if $M \vDash \varphi$ then $M \vDash \psi$.
5. $M \nvDash \bot$.

This notation extends to $M \nvDash \varphi$ and $M \vDash \Delta$ in the usual ways. The language for propositional logic is defined in Definition 1.6.

The reason that it is safe to use the same semantics (the truth functions of Definition 1.3) with every logic is that the behavior of the truth functions themselves changes based on the logic that is reasoning about them. For instance, a classical metatheory can prove that for any truth function M and any formula A, $(M \vDash A) \vee (M \nvDash A)$, but a strictly intuitionistic metatheory cannot prove this. Thus the behavior of the truth functions actually depends on the metatheory, obviating the need for any other semantics. This tactic has been used in [11] and Leivant [10], and I will discuss it further in Section 4.1.

Definition 1.4 (Reflexive) A logic L is reflexive if and only if a simple metatheory built from L can prove the countermodel completeness theorem for L.

This paper classifies the intermediate propositional logics by finding a simple criterion equivalent to reflexivity (Corollary 3.9). I give a more formal version of Definition 1.4 in Definition 3.1 when I prove which intermediate logics are reflexive.

It is worth noting that countermodel completeness is only one formulation of completeness. Other notable formulations are weak and strong formula completeness, but for each of these, the question of reflexivity has a less interesting answer. Consider first weak formula completeness, the statement that for every formula φ, if φ is not provable, then it is not valid. (Here the syntax and semantics being compared are contained within the words "valid" and "provable.") Intuitionistic logic can prove its own weak formula completeness if the semantics in question is Kripke semantics. Thus formulating reflexivity in these terms would make all intermediate logics reflexive.

Alternatively, consider strong formula completeness, the statement that for every formula φ, if φ is valid then it is provable. Although this is the contrapositive of weak formula completeness, we must consider them as distinct notions because nonclassical logics do not equate contrapositives. I show in Theorem 1.5 below that classical logic is the only intermediate logic which proves its own strong formula completeness.

Thus when using either of these alternate formulations of completeness, the question of which logics are reflexive has a somewhat all-or-nothing answer (i.e., "only classical logic" or "all intermediate logics"). We will see that in the case of countermodel completeness, the answer is more interesting.

A deeper discussion of the various formulations of completeness and the history of their investigation in intuitionistic logic can be found in McCarty [12], which surveys Kreisel [6], [7], [8], and [9].

Theorem 1.5 *Consider a nonclassical intermediate logic L which is reflexive, as in Definition 1.4. No metatheory built on L can prove the strong formula completeness for L.*

Proof Assume toward a contradiction that a metatheory built from L can prove the strong formula completeness of L. Then a metatheory built on classical logic, being stronger, could also show strong formula completeness for L. But that classical metatheory could then prove both "φ is valid if and only if $\vdash_L \varphi$" and "φ is valid if and only if $\vdash_{\mathrm{Cl}} \varphi$." This would then allow such a classical metatheory to conclude that $\vdash_L \varphi$ if and only if $\vdash_{\mathrm{Cl}} \varphi$, a statement which is false. Since classical logic is sound, it cannot deduce false statements, and thus we have reached the desired contradiction. □

Proving which intermediate logics are reflexive means coming up with simple necessary and sufficient conditions for reflexivity. In order to use a logic to reason about itself, one needs to make a metatheory from the logic. That is, when I ask whether intuitionistic logic can prove completeness for its own propositional fragment, I'm really asking whether a standard intuitionistic metatheory that is capable of reasoning about propositional logic can carry out the completeness proof for a particular propositional logic, the intuitionistic one. I use the term "metatheory" quite often in this way throughout this paper. One might say that the work of this paper is therefore done in a metametatheory, because the reasoning is done in ordinary English prose using classical rules about various subclassical logics *and* metatheories.

Section 2.1 shows how to create a metatheory from a propositional logic, and Section 2.3 shows that the operation of doing so is injective, so that each propositional logic has its own unique metatheory for reasoning about the original propositional logic. Section 2.3 also discusses why such a property is highly desirable. Section 3 then proves that a necessary and sufficient condition for being a reflexive intermediate propositional logic is the condition of being at least as strong as testability logic.

1.3 Notation and terminology Because Kripke semantics is used in this paper, I take a moment now to state how I use it. I use standard Kripke models, consisting of a partially ordered set called the frame, each element of which is a node. If K is a Kripke model and its frame is the poset P, I say that K is "on frame P." In some contexts, the term "frame" and "poset" may get used interchangeably, but this should not cause any ambiguity.

Associated with each node in the frame is a classical model called a world, and between any two nodes α, β in the poset, if $\alpha \leq \beta$ then there exists a transition function from α to β. Transition functions in Kripke models preserve satisfaction of atomic formulas, and the composition of the transition from α to β with that from β to γ must be the transition from α to γ. If $\alpha < \beta$ and no other node lies between them, I say that β is an upper neighbor of α. If a node has no other nodes strictly larger than it in the frame, it is called a leaf.

I use the standard notation $\alpha \Vdash \varphi$ to indicate that the node α forces the formula φ when it is clear to which Kripke model α belongs. If more than one model is under discussion, I will subscript the forcing symbol, $\alpha \Vdash_K \varphi$, to disambiguate.

I use the notation $\varphi[x := t]$ to mean the formula φ with each free occurrence of the variable x replaced by the term t. I use the notation $[\![t]\!]$ to mean the interpretation of the term t, and I superscript it $[\![t]\!]^M$ to mean the interpretation of the term t in the classical model M. I write $M \vDash \varphi[a_1, \ldots, a_n]$ to mean that M satisfies the formula φ when any of the variables $x_1, \ldots, x_n$ appearing free in φ is interpreted to refer to the corresponding element a_i of the universe of M. That is, x_i is interpreted by a_i for all $i \in \{ 1, \ldots, n \}$. One can use $\varphi[\overline{a}]$ as a shorthand for $\varphi[a_1, \ldots, a_n]$.

I often create subfinite sets using notation $\{ A \mid B \}$, where A is an expression not dependent on B. For example, one may write $\{ 0 \mid \varphi \}$, meaning the set that contains 0 if φ holds and that contains nothing other than 0. In this manner, one can express conveniently sets whose exact cardinality is somewhat unclear intuitionistically. For instance, although $\{ 0 \mid p \vee \neg p \}$ can be shown intuitionistically to be nonempty, it cannot be shown in general to have cardinality 1.

1.4 Intermediate propositional logics

Definition 1.6 (Prop) The language I use for propositional logic is

$$\text{Prop} = \{ \wedge, \vee, \rightarrow, \bot \} \cup \{ p_i \mid i \in \mathbb{N} \}.$$

For any formulas φ and ψ of the language Prop, the notation $\varphi \leftrightarrow \psi$ is a convenience standing for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and the notation $\neg\varphi$ is a convenience standing for $\varphi \rightarrow \bot$.

Definition 1.7 (Propositional logic) Let us call a set L of formulas of Prop a propositional logic if it is closed under deduction, that is, under detachment. That is, whenever

$$\varphi_1 \in L, \ldots, \varphi_n \in L, \text{ and } (\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \psi \in L,$$

then we also have $\psi \in L$.

Definition 1.8 ($\mathcal{I}_{\mathbf{Prop}}$) The collection $\mathcal{I}_{\text{Prop}}$ of intermediate propositional logics is the collection of all L satisfying each of the following conditions.

1. A simple second-order intuitionistic metatheory can prove that L is a propositional logic as in Definition 1.7.
2. A simple second-order intuitionistic metatheory can prove that $\text{Int} \subseteq L \subseteq \text{Cl}$.
3. There is a set S of posets such that L is sound and complete over the set of Kripke models one can build on frames in S.

Allow me to explain some of the potentially unexpected aspects of this definition. Although conditions 1 and 2 are vague, the specific second-order metatheory I have in mind is *Meta*(Int), which I define below in Definition 2.7. For now, it is enough to note that $\mathcal{I}_{\text{Prop}}$ contains only logics L that are definable in a simple second-order language and for which proving closure and $\text{Int} \subseteq L \subseteq \text{Cl}$ requires no nonintuitionistic principles.

Condition 3 helps us guarantee that $\mathcal{I}_{\text{Prop}}$ contains logics that do not have unusual behavior, for example, treating one propositional letter differently than all the rest. Although I have not been so restrictive as to require the *finite* model property, condition 3 gives us a semantical grounding with enough power to accomplish the arguments of Section 2.

One can add structure to $\mathcal{I}_{\text{Prop}}$ by arranging the logics in a lattice. One way to do this was investigated in Rautenberg [15]. However, the material in this paper does not require introducing such a structure.

The collection $\mathcal{I}_{\text{Prop}}$ is uncountable and contains infinite ascending and descending chains. Results of this form are recorded in Gabbay [3], Chapter 4, §3. The intermediate logic that plays the lead role in this paper is testability logic.

Definition 1.9 (Principle of testability) The principle of testability is the propositional formula $(\neg\varphi) \vee (\neg\neg\varphi)$, a classical tautology that is not intuitionistically valid. Here φ is an arbitrary formula of Prop.

Definition 1.10 (Testability logic) Testability logic, Test, is the logic obtained by adding to intuitionistic logic the principle of testability as an axiom schema.

2 Metatheories

2.1 The *Meta* operator I now proceed to create a second-order metatheory from each logic in $\mathcal{I}_{\text{Prop}}$. In Section 3, I build on these foundations to determine which second-order metatheories can prove completeness for their underlying propositional logics.

Given that each metatheory needs to be able to reason about propositional logic, it is necessary that the metatheories be second-order. The statement of completeness, for example, quantifies over sets of formulas and over propositional structures (truth functions). To be able even to express completeness, therefore, one needs a second-order language.

I describe a process for converting a propositional logic into a second-order metatheory, and I write *Meta*(L) for the result of applying such a procedure to L. The next few definitions, and most notably Definition 2.7, accomplish this. I begin with a definition of the first- and second-order languages FO and SO and two definitions that create higher-order formulas from formulas of Prop.

Definition 2.1 (FO) The language I use for first-order arithmetic is

$$\text{FO} = \{ \forall, \exists, \wedge, \vee, \rightarrow, \bot, =, 0, S \} \cup \left\{ f_i^n \mid i \in \mathbb{N} \right\}.$$

Here 0 and S are the symbols for the zero and successor functions for arithmetic, and f_i^n is the function symbol for the ith primitive recursive function in a list of all primitive recursive functions, and it has arity n. The same conventions for $\leftrightarrow$ and $\neg$ apply here as in Prop, Definition 1.6. It is assumed that we also have countably many first-order variables, which I call things like x and y.

Definition 2.2 (SO) The language I use for second-order arithmetic is

$$\text{SO} = \text{FO} \cup \left\{ C_i^n \mid i \in \mathbb{N} \right\}.$$

Here, C_i^n is a second-order constant of arity n. It is assumed that we also have countably many second-order variables, which I call things like A and B. It will be evident from the case of the variable whether quantification is over first- or second-order objects (e.g., $\forall x$ vs. $\forall A$).

Definition 2.3 (Substitution φ_ψ) Let φ be a formula of Prop which employs at most the propositional letters $p_1, \ldots, p_n$. Let $\psi_1, \psi_2, \ldots$ be a finite or infinite sequence of formulas of SO which has at least length n. One can create a formula φ_ψ of SO by simultaneously replacing each p_i with ψ_i, for all $i \in \{1, \ldots, n\}$.

Definition 2.4 (Substitution Φ_ψ) Given an infinite sequence $\psi = \langle \psi_i \mid i \in \mathbb{N} \rangle$ of formulas of SO and a set Φ of propositional formulas, define $\Phi_\psi = \left\{ \varphi_\psi \mid \varphi \in \Phi \right\}$.

I use the notation $\psi = \langle \psi_i \mid i \in \mathbb{N} \rangle$ for infinite sequences of formulas several times throughout this paper. The symbol ψ then refers to the sequence as a whole, as a function from $\mathbb{N}$ to the set of formulas of SO, and the individual formulas of the sequence are referred to via subscripting, ψ_i.

Definitions 2.3 and 2.4 enable me to create second-order flavors of each logic in $\mathcal{L}_{\text{Prop}}$. The intuition being codified in this section is that second-order classical logic should have all the rules of classical propositional logic, plus rules for quantification, equality, and comprehension. So also, other second-order logics can be formed from their propositional counterparts by addition of those same rules.

This is the first of two steps that end with creating second-order metatheories. Because I wish eventually to use second-order metatheories to reason about propositional logics, my second step adds sufficient axioms so that the metatheory can reason about propositional logic.

Definition 2.5 (Substitution operator, s) For any set Φ of formulas from Prop, I write $s(\Phi)$ to mean all possible second-order substitutions of the formulas in Φ. Specifically, if Ψ is the set of all formulas from SO, then

$$s(\Phi) = \bigcup_{\psi \in \Psi^{\mathbb{N}}} \Phi_\psi.$$

Here, $\Psi^{\mathbb{N}}$ refers to all functions from $\mathbb{N}$ to Ψ, that is, all sequences of formulas from Ψ; this makes a function $\psi \in \Psi^{\mathbb{N}}$ an appropriate parameter for the substitution Φ_ψ, as in Definition 2.4.

Definition 2.6 (Heyting Arithmetic, HAS) I write HAS for the set of axioms of second-order Heyting Arithmetic, in the ordinary language for second-order arithmetic, listed here. The variables x, y are first-order (natural number variables), and the symbol φ ranges over all second-order formulas containing no occurrences of the second-order variable A.

1. $\forall x(S(x) \neq 0)$.
2. $\forall x \forall y(S(x) = S(y) \to x = y)$.
3. $[\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(S(x)))] \to \forall x \varphi(x)$.
4. For each recursive function f_i^n mentioned in Definition 2.1, include here its defining equations. For example, if f_9^2 were addition, the following two equations should be included in HAS.

   $$\forall x(f_9^2(0, x) = x) \qquad \forall x \forall y(f_9^2(Sx, y) = S(f_9^2(x, y)))$$

5. Comprehension axiom:

   $$\exists A \forall x_1, \ldots, \forall x_n(\varphi(x_1, \ldots, x_n) \leftrightarrow A(x_1, \ldots, x_n)).$$

Later I want to define some sets recursively using the axioms of HAS, and so it is important to point out here that a recursively defined sequence of sets can be formed using the comprehension axiom from HAS. For instance, the existence of a set A satisfying the requirements

$$A(0, x_1, \ldots, x_n) \leftrightarrow \varphi_{\text{base}}(x_1, \ldots, x_n)$$

and

$$\forall m[A(S(m), x_1, \ldots, x_n) \leftrightarrow \varphi_{\text{ind}}(x_1, \ldots, x_n)],$$

where φ_{base} and φ_{ind} are both second-order formulas with at most $x_1, \ldots, x_n$ free, can be created by instantiating the HAS comprehension axiom as follows.

$$\begin{aligned} \exists A \forall y \forall x_1 \cdots \forall x_n \big[\forall B \big((B(0, x_1, \ldots, x_n) &\leftrightarrow \varphi_{\text{base}}(x_1, \ldots, x_n) \\ \wedge \forall m(B(S(m), x_1, \ldots, x_n) &\leftrightarrow \varphi_{\text{ind}}(x_1, \ldots, x_n))) \\ &\to B(y, x_1, \ldots, x_n) \big) \\ \leftrightarrow A(y, x_1, \ldots, x_n) \big] . \end{aligned}$$

Definition 2.7 (*Meta*) Given a logic $L \in \mathcal{I}_{\text{Prop}}$, create a second-order theory $Meta(L)$ by taking the deductive closure of $s(L) \cup \text{HAS}$ under second-order intuitionistic derivability. I use the usual closure notation $\overline{s(L) \cup \text{HAS}}$ to express intuitionistic deductive closure.

I call this second-order theory $Meta(L)$ because it is not only a theory built from L, but also I use it as a metatheory to reason about propositional logics like L.

Definition 2.8 ($\mathcal{I}_{Meta}$) Let $\mathcal{I}_{Meta}$ be the collection of metatheories generated from intermediate logics, $\{ Meta(L) \mid L \in \mathcal{I}_{\text{Prop}} \}$.

2.2 Capabilities of *Meta*-theories I have just defined how to create a metatheory $Meta(L)$ from each propositional logic $L \in \mathcal{I}_{\text{Prop}}$. In Section 3, I use these metatheories to reason about propositional logics. Such work requires several fundamental facts about the capabilities of each $Meta(L)$, and so this section is dedicated to laying the groundwork by proving those fundamental facts.

In order prove anything about how the metatheories reason about propositional logics like Int and Cl, we must first define the symbol $\vdash$ in the language SO which the metatheories use.

Definition 2.9 (Derivability $\vdash$ in SO) In SO, the notation $\Gamma \vdash_L \varphi$ is shorthand for the expression

$$\exists n\ \exists\gamma_1, \ldots, \exists\gamma_n \in \Gamma\ ((\gamma_1 \wedge \cdots \wedge \gamma_n \to \varphi) \in L).$$

Here, Γ is a second-order variable.

Strictly speaking, even this definition is not a formula of SO. I should instead write something like

$$\exists n\ \exists\gamma\ \ (\mathit{Sequence}(\gamma) \wedge \mathit{Length}(\gamma) = n \wedge \forall i\ < n((\gamma)_i \in \Gamma) \wedge \cdots),$$

but it is clear that this *could* be done, and since the meaning of the earlier notation is clear, I use it because it is more convenient in proofs relying on this definition.

Note that embodied in Definition 2.9 is the finiteness of proofs. Furthermore, it enables a fairly straightforward demonstration of the fact that *Meta*(Int) is strong enough to show the deduction theorem for each $L \in \mathcal{I}_{\text{Prop}}$ (Theorem 2.13).

It is also important to note that the metatheory *Meta*(Int) clarifies conditions 1 and 2 from Definition 1.8. Those conditions required that every $L \in \mathcal{I}_{\text{Prop}}$ be such that *Meta*(Int) $\vdash$ (L is deductively closed) and *Meta*(Int) $\vdash$ Int $\subseteq L \subseteq$ Cl. What remains for me to clarify is that one can express all of these notions in the language SO. That is, can one express the terms Int and Cl and the deductive closure relation in SO? I show this informally.

Using Gödel numbering to encode formulas as natural numbers, one could express notions such as "the set of all propositional formulas of the form $(\varphi \wedge \psi) \to \varphi$." In this way one could assemble all instances of the Hilbert-style axioms for Int into a single term of SO. For a Hilbert-style axiomatization of Int, see, for example, pp. 26–27 of [3]. Then one can define deductive closure under modus ponens recursively as follows.

Definition 2.10 (Deductive closure under MP in SO) Define the relations D_i and DC in SO as follows.

$$D_0(A) = A.$$

$$D_{n+1}(A) = D_n(A) \cup \{\, \varphi \mid \exists\psi \in D_n(A) \text{ such that } (\psi \to \varphi) \in D_n(A)\,\}.$$

$$DC(A) = \cup_{n\in\mathbb{N}} D_n(A).$$

Definition 2.11 (The terms Int and Cl in SO) Define the SO-term Int to be DC applied to the axiom set for Int. Let TND stand for the set of all propositional instances of the schema tertium non datur, $\varphi \vee \neg\varphi$, and define the SO-term Cl to be the application of DC to the union of TND and the Int axioms.

Now we come to our first observation regarding the capabilities of *Meta*(Int) with respect to both the SO-terms Int and Cl of Definition 2.11 and the SO-relation $\vdash$ of Definition 2.9. Although this result is somewhat basic, it is important to attend to these details because this theorem is fairly foundational as we continue.

Theorem 2.12 *For any formula φ of the language* Prop*,*

$$\mathit{Meta}(\text{Int}) \vdash (\vdash_{\text{Int}} \varphi)\ \ \mathit{iff}\ \ \vdash_{\text{Int}} \varphi,$$

and

$$\mathit{Meta}(\text{Int}) \vdash (\vdash_{\text{Cl}} \varphi)\ \ \mathit{iff}\ \ \vdash_{\text{Cl}} \varphi.$$

Proof The forward direction of these implications is simply soundness for second-order Heyting Arithmetic. The reverse direction claims that if φ is derivable, then *Meta*(Int) knows as much. I consider only the case for Int; the case for Cl is analogous.

Given that φ is derivable, we can conclude from the Hilbert-style axiomatization of Int mentioned above that there exists a finite-length derivation $\delta_1, \ldots, \delta_m$ such that $\delta_m = \varphi$ and each δ_i is either an instance of an Int-axiom or is the result of applying modus ponens to $\delta_1, \ldots, \delta_{i-1}$. We simply need to show that *Meta*(Int) is capable of showing that Int can perform the derivation.

If we let IA stand for the SO term defining the set of Int-axioms, one can give an inductive proof that for each i, $\mathit{Meta}(\mathrm{Int}) \vdash (\delta_i \in D_i(\mathrm{IA}))$. Here, D_i is the ith phase of the deductive closure operator defined in Definition 2.10.

In the case when δ_i is an instance of an Int-axiom, *Meta*(Int) simply needs to be able to exhibit the small finite substitution function that maps metavariables to formulas. For instance, that $(x = y) \vee \neg(x = y)$ is an instance of $P \vee \neg P$ is witnessed by the function $\{ (P, (x = y)) \}$. The language SO is sufficient for expressing such small finite functions, and *Meta*(Int) is sufficient for performing substitutions and verifying equality of formulas. Thus $\mathit{Meta}(\mathrm{Int}) \vdash (\delta_i \in \mathrm{IA} \subset D_i(\mathrm{IA}))$.

In the case when δ_i follows from $\delta_1, \ldots, \delta_{i-1}$ by modus ponens, the recursive portion of Definition 2.10 places $\delta_i \in D_i(\mathrm{IA})$, and in order to verify this, *Meta*(Int) needs only to be able to reason about the basic set theory notation used in Definition 2.10. This completes the inductive proof, and thus

$$\mathit{Meta}(\mathrm{Int}) \vdash (\varphi = \delta_m \in D_m(\mathrm{IA}) \subset DC(\mathrm{IA}) = \mathrm{Int}). \qquad \square$$

Continuing to build a repertoire of essential facts about the metatheories, I now use the above result to show that *Meta*(Int) (and hence every *Meta*(L)) can prove the deduction theorem for any logic $L \in \mathcal{L}_{\mathrm{Prop}}$.

Theorem 2.13 (Deduction Theorem in the metatheory) *For any logic $L \in \mathcal{L}_{\mathrm{Prop}}$, Meta*(Int) *is sufficient to prove the Deduction Theorem for L.*

Proof The Deduction Theorem for L states that if $\Gamma, \varphi \vdash_L \psi$, then $\Gamma \vdash_L (\varphi \to \psi)$. I abbreviate "the Deduction Theorem holds for L" by DT(L). Working in *Meta*(Int), I proceed to show DT(L) as follows.

Assume $\Gamma, \varphi \vdash_L \psi$, and therefore,

$$\exists n\, \exists \gamma_1, \ldots, \exists \gamma_n \in \Gamma\ ((\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \to \psi) \in L).$$

By Theorem 2.12, *Meta*(Int) is then able to point out that

$$(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \to \psi) \to (\gamma_1 \wedge \cdots \wedge \gamma_n \to (\varphi \to \psi))$$

is an intuitionistic theorem and, therefore, is a member of L because *Meta*(Int) knows $L \supseteq \mathrm{Int}$ by the definition of $\mathcal{L}_{\mathrm{Prop}}$ in Definition 1.8. From the same definition, *Meta*(Int) also knows that L is closed under deduction, and thus we have that $(\gamma_1 \wedge \cdots \wedge \gamma_n \to (\varphi \to \psi))$ is also in L. According to the definition of $\vdash_L$, this gives us $\Gamma \vdash_L (\varphi \to \psi)$. $\square$

Corollary 2.14 *Since Meta*(Int) $\vdash$ DT(L), *by weakening we have that Meta*(L) $\vdash$ DT(L) *also.* $\square$

Using the deduction theorem in *Meta*(L), I can now expand Theorem 2.12 as follows.

Corollary 2.15 *For any formula φ of the language* Prop*,*

$$Meta(\text{Int}) \vdash (\psi_1, \ldots, \psi_n \vdash_{\text{Int}} \varphi) \text{ iff } \psi_1, \ldots, \psi_n \vdash_{\text{Int}} \varphi,$$

and

$$Meta(\text{Int}) \vdash (\psi_1, \ldots, \psi_n \vdash_{\text{Cl}} \varphi) \text{ iff } \psi_1, \ldots, \psi_n \vdash_{\text{Cl}} \varphi.$$

Proof Given Theorem 2.12, I simply must show that

$$Meta(\text{Int}) \vdash ((\psi_1, \ldots, \psi_n \vdash_{\text{Int}} \varphi) \leftrightarrow (\vdash_{\text{Int}} \psi_1 \wedge \ldots \wedge \psi_n \rightarrow \varphi)),$$

and the analogous statement for $\vdash_{\text{Cl}}$. The forward direction of this equivalence is given by Theorem 2.13, and the reverse direction is given by Definition 2.9 together with the fact that Int is deductively closed (Definition 1.8). □

Lastly I note that *Meta*(Int) is able to show that every $L \in \mathcal{L}_{\text{Prop}}$ classifies the same sets of formulas consistent.

Theorem 2.16 *Meta*(Int) *can prove that for any $L_1, L_2 \in \mathcal{L}_{\text{Prop}}$, for any set Γ of formulas of* Prop*, $\Gamma \vdash_{L_1} \bot$ if and only if $\Gamma \vdash_{L_2} \bot$.*

Even though I am reasoning exclusively in *Meta*(Int), I will do this proof without rigor. One could use the definitions in this section to rigorize it, more like the previous theorems, but it would become considerably longer.

Proof I show instead that for any set Γ of formulas of Prop, if $\Gamma \vdash_{\text{Cl}} \bot$ then $\Gamma \vdash_{\text{Int}} \bot$. Then given $L_1, L_2 \in \mathcal{L}_{\text{Prop}}$, if $\Gamma \vdash_{L_1} \bot$, then $\Gamma \vdash_{\text{Cl}} \bot$ because $L_1 \subseteq \text{Cl}$, and so $\Gamma \vdash_{\text{Int}} \bot$, giving $\Gamma \vdash_{L_2} \bot$ because $\text{Int} \subseteq L_2$. Note that *Meta*(Int) can do this because of Definition 1.8.

Assuming $\Gamma \vdash_{\text{Cl}} \bot$, Definition 2.9 provides us with a finite list of formulas $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\gamma_1, \ldots, \gamma_n \vdash_{\text{Cl}} \bot$. Because Cl is the deductive closure of $\text{Int} \cup \text{TND}$, there are a finite number of instances $\tau_1, \ldots, \tau_k \in \text{TND}$ such that $\gamma_1, \ldots, \gamma_n, \tau_1, \ldots, \tau_k \vdash_{\text{Int}} \bot$. By the Deduction Theorem immediately above we can write $\gamma_1, \ldots, \gamma_n \vdash_{\text{Int}} \tau_1 \wedge \cdots \wedge \tau_k \rightarrow \bot$.

Intuitionistically, $p \rightarrow q$ implies $(\neg q) \rightarrow (\neg p)$, and applying this principle twice gives us that

$$\gamma_1, \ldots, \gamma_n \vdash_{\text{Int}} \neg\neg(\tau_1 \wedge \cdots \wedge \tau_k) \rightarrow \neg\neg\bot.$$

Because $\vdash_{\text{Int}} \bot \leftrightarrow \neg\neg\bot$ and $\vdash_{\text{Int}} (\neg\neg(p \wedge q)) \leftrightarrow (\neg\neg p) \wedge (\neg\neg q)$, we also have that

$$\gamma_1, \ldots, \gamma_n \vdash_{\text{Int}} (\neg\neg\tau_1) \wedge \cdots \wedge (\neg\neg\tau_k) \rightarrow \bot.$$

Yet each τ_i is an instance of tertium non datur, and thus its double negation is a theorem of Int. All these intuitionistic derivations are possible in *Meta*(Int) by Corollary 2.15. So by the closure of Int (condition 1 of Definition 1.8), we have $\gamma_1 \ldots, \gamma_n \vdash_{\text{Int}} \bot$, and thus $\Gamma \vdash_{\text{Int}} \bot$, as desired. □

One final remark regarding the capabilities of the metatheories *Meta*(L) is necessary. I plan to investigate which metatheories have the power to prove completeness theorems, and it is important to note that *any* metatheory *Meta*(L_1) can prove the soundness theorem for any other $L_2 \in \mathcal{L}_{\text{Prop}}$, including itself. This is because the soundness theorem for Cl with respect to truth functions is a straightforward induction proof that simply uses Definition 1.3 to verify that each Cl-axiom is valid (e.g., Enderton [2], pp. 124–25). Even *Meta*(Int) can do such a proof. It follows that any stronger metatheory $Meta(L_1)$ can prove the soundness theorem for any weaker logic $L_2 \subseteq \text{Cl}$.

2.3 Injectivity of *Meta* Recall that the main question this paper seeks to answer is "Which logics can prove their own completeness?" That is, for which logics $L \in \mathscr{L}_{\text{Prop}}$ can $Meta(L)$ prove the completeness theorem for L? (I will formalize this in Definition 3.1.) Since this is my eventual use for *Meta*, it is important to ensure that *Meta* is injective. That is, the very question just mentioned seems to assume that each logic has its own metatheory, and the question would lose some of its meaning and significance if *Meta* were to amalgamate two different propositional logics into one metatheory. The injectivity of *Meta* keeps the question sensible as it was originally phrased; it tells us that adding an arithmetic metatheory to a propositional logic does not in any way alter or mar the logic but preserves its unique behavior. In addition to this motivation, one of the main classification results of this paper (Corollary 3.8) relies on the injectivity of *Meta*.

I prove that *Meta* is injective using the second-order part of each $Meta(L)$ to distinguish it from the others. This indeed demonstrates the result I need in order to continue with my work, but it does have the disadvantage of leaving unanswered the interesting question of how the first-order portions of the metatheories compare. Such a question is interesting because it is known that some first-order logics are identified by the addition of axioms for arithmetic, and thus one may ask if the *only* way to distinguish the metatheories I created in Definition 2.7 is by their second-order portions. This is a natural question, but this paper does not solve it.

The following basic result begins the process of showing the injectivity of *Meta*.

Theorem 2.17 *For any two logics $L_1, L_2 \in \mathscr{L}_{\text{Prop}}$,*

$$L_1 \subseteq L_2 \text{ implies } Meta(L_1) \subseteq Meta(L_2).$$

Proof If $L_1 \subseteq L_2$, then clearly $s(L_1) \subseteq s(L_2)$, and so $s(L_1) \cup \text{HAS} \subseteq s(L_2) \cup \text{HAS}$. Therefore any theorem derivable from $s(L_1) \cup \text{HAS}$ is derivable from $s(L_2) \cup \text{HAS}$ by weakening. Hence $Meta(L_1) = \overline{s(L_1) \cup \text{HAS}} \subseteq \overline{s(L_2) \cup \text{HAS}} = Meta(L_2)$. □

Given the previous theorem, it remains to show that when we have $L_1 \subsetneqq L_2$, we also have $Meta(L_1) \subsetneqq Meta(L_2)$. In this section I show that there are second-order formulas that behave analogously to the propositional letters. Thus the pattern of behavior of the original propositional logic is preserved within a portion of the second-order theory, and so the operation of creating the second-order theory is injective.

The following three lemmas provide us with some facts about Kripke models and frames. I need these two facts to prove the one theorem in this section, which establishes the injectivity of *Meta* by a semantic argument.

Lemma 2.18 *In any first-order Kripke model K whose nodes are all decorated with the same first-order structure M and whose transition functions are all the identity function on the universe of M, we have $K \Vdash \varphi \iff M \vDash \varphi$, for any first-order sentence φ.*

Proof I write A for the universe of M, and thus also for the universe at each node in K. I show by induction on φ that for any node α in K, and for any elements

$a_1, \ldots, a_n \in A$, $\alpha \Vdash \varphi[\overline{a}]$ if and only if $M \vDash \varphi[\overline{a}]$. In the case where φ is atomic, the definition of forcing specifies that for any node α, $\alpha \Vdash \varphi[\overline{a}]$ if and only if $M \vDash \varphi[\overline{a}]$.

When φ is a conjunction or a disjunction, the argument is quite straightforward. For example, here is the case for disjunction, $\varphi = \psi_1 \vee \psi_2$, assuming arbitrary α.

$$\begin{aligned} \alpha \Vdash (\psi_1 \vee \psi_2)[\overline{a}] &\iff \alpha \Vdash \psi_1[\overline{a}] \text{ or } \alpha \Vdash \psi_2[\overline{a}] \\ &\iff M \vDash \psi_1[\overline{a}] \text{ or } M \vDash \psi_2[\overline{a}] \\ &\qquad\qquad \text{by the Induction Hypothesis} \\ &\iff M \vDash (\psi_1 \vee \psi_2)[\overline{a}]. \end{aligned}$$

When $\varphi = \psi_1 \to \psi_2$, again let α be arbitrary.

$$\begin{aligned} \alpha \Vdash (\psi_1 \to \psi_2)[\overline{a}] &\iff \forall \beta \geq \alpha \text{ (if } \beta \Vdash \psi_1[\overline{a}] \text{ then } \beta \Vdash \psi_2)[\overline{a}] \\ &\qquad\qquad \text{all transition functions are the} \\ &\qquad\qquad \text{identity functions on } A \\ &\iff \forall \beta \geq \alpha \text{ (if } M \vDash \psi_1[\overline{a}] \text{ then } M \vDash \psi_2)[\overline{a}] \\ &\qquad\qquad \text{by the Induction Hypothesis} \\ &\iff \text{if } M \vDash \psi_1[\overline{a}] \text{ then } M \vDash \psi_2[\overline{a}] \\ &\qquad\qquad \text{no } \beta \text{ in scope of quantifier} \\ &\iff M \vDash (\psi_1 \to \psi_2)[\overline{a}]. \end{aligned}$$

When $\varphi = \exists x\, \psi$, again let α be arbitrary.

$$\begin{aligned} \alpha \Vdash (\exists x\, \psi)[\overline{a}] &\iff \exists a \in A\ (\alpha \Vdash \psi[\overline{a}, a]) \\ &\iff \exists a \in A\ (M \vDash \psi[\overline{a}, a]) \\ &\qquad\qquad \text{by the Induction Hypothesis} \\ &\iff M \vDash (\exists x\, \psi)[\overline{a}]. \end{aligned}$$

When $\varphi = \forall x\, \psi$, again let α be arbitrary.

$$\begin{aligned} \alpha \Vdash (\forall x\, \psi)[\overline{a}] &\iff \forall \beta \geq \alpha\ \forall a \in A\ (\beta \Vdash \psi[\overline{a}, a]) \\ &\iff \forall \beta \geq \alpha\ \forall a \in A\ (M \vDash \psi[\overline{a}, a]) \\ &\qquad\qquad \text{by the Induction Hypothesis} \\ &\iff \forall a \in A\ (M \vDash \psi[\overline{a}, a]) \\ &\qquad\qquad \text{no } \beta \text{ in scope of quantifier} \\ &\iff M \vDash (\forall x\, \psi)[\overline{a}]. \end{aligned}$$

This completes the proof, having exhausted the induction cases for φ. □

Lemma 2.19 *For any logic $L \in \mathcal{I}_{\text{Prop}}$, for any function $g : \mathbb{N} \to \mathbb{N}$, if $\varphi \in L$, then $\varphi[p_i := p_{g(i)}] \in L$. That is, L is closed under the operation of replacing the propositional letters in a formula, even with g not injective.*

Proof Let $g : \mathbb{N} \to \mathbb{N}$ and let P be a poset over whose frames L is sound and complete, as per part 3 of Definition 1.8. Then any model built on P will force φ. I show that any model built on P will also force $\varphi[p_i := p_{g(i)}]$.

Let K be a model built on P. Create a new model K' on the frame P as follows. For any $\alpha \in P$, for any $i \in \mathbb{N}$,

$$\alpha \Vdash_{K'} p_i \iff \alpha \Vdash_K p_{g(i)}.$$

One can show inductively that for any formula ψ of Prop and any node α in P,

$$\alpha \Vdash_{K'} \psi \iff \alpha \Vdash_K \psi[p_i := p_{g(i)}].$$

The base case (when ψ is atomic) is part of the construction of K'. The induction steps are all straightforward, relying only on the fact that substitution commutes with all connectives. Here is the case for the conditional, as an example.

$$\begin{aligned}
\alpha \Vdash_{K'} \psi_1 \to \psi_2 &\iff \forall \beta \geq \alpha \text{ (if } \beta \Vdash_{K'} \psi_1 \text{ then } \beta \Vdash_{K'} \psi_2)\\
&\iff \forall \beta \geq \alpha \text{ (if } \beta \Vdash_{K} \psi_1[p_i := p_{g(i)}]\\
&\qquad\qquad \text{then } \beta \Vdash_{K} \psi_2[p_i := p_{g(i)}])\\
&\qquad\qquad \text{by the Induction Hypothesis}\\
&\iff \alpha \Vdash_K \psi_1[p_i := p_{g(i)}] \to \psi_2[p_i := p_{g(i)}]\\
&\iff \alpha \Vdash (\psi_1 \to \psi_2)[p_i := p_{g(i)}].
\end{aligned}$$

Because K' is built on P, we have $K' \Vdash \varphi$, and so $K \Vdash \varphi[p_i := p_{g(i)}]$. Thus an arbitrary model built on P forces $\varphi[p_i := p_{g(i)}]$, and so $\varphi[p_i := p_{g(i)}] \in L$. □

Lemma 2.20 *Let L be a propositional logic and S be a set of posets such that L is sound over the set of all Kripke models one can build on the frames in S. Then for any second-order Kripke model K whose frame is in S, we have $K \Vdash s(L)$.*

Proof Enumerate all formulas of SO in a list $\chi = \langle \chi_i \mid i \in \mathbb{N} \rangle$ without repetition. Create a propositional Kripke model K' over the same frame as K as follows. Let node α in K' force propositional letter p_i if and only if formula χ_i is forced at node α in K. Clearly then, when a propositional letter is forced at one node, it is forced at all higher nodes, and so I have defined a valid propositional model K'.

Claim 1 For any node α in the frame of K and for any propositional formula φ, we have $\alpha \Vdash_{K'} \varphi$ if and only if $\alpha \Vdash_K \varphi_\chi$. That is, node α in the model K' forces φ if and only if the corresponding node in K forces φ with each p_i replaced by χ_i (as per Definition 2.4).

Proof of Claim 1: By structural induction on the propositional formula φ, with α arbitrary. The base case, when $\varphi = p_i$ for some $i \in \mathbb{N}$, follows immediately from my construction of K' based on K. The induction steps are nearly as straightforward; here is the case for the conditional, as an example. Assuming $\varphi = \psi_1 \to \psi_2$,

$$\begin{aligned}
\alpha \Vdash_{K'} (\psi_1 \to \psi_2) &\iff \forall \beta \geq \alpha \text{ (if } \beta \Vdash_{K'} \psi_1 \text{ then } \beta \Vdash_{K'} \psi_2)\\
&\iff \forall \beta \geq \alpha \text{ (if } \beta \Vdash_{K} (\psi_1)_\chi \text{ then } \beta \Vdash_{K} (\psi_2)_\chi)\\
&\qquad\qquad \text{by the Induction Hypothesis}\\
&\iff \alpha \Vdash_K (\psi_1)_\chi \to (\psi_2)_\chi\\
&\iff \alpha \Vdash_K (\psi_1 \to \psi_2)_\chi.
\end{aligned}$$

Therefore now that the claim has been established, assume toward a contradiction that there is an element $\varphi \in s(L)$ such that $K \nVdash \varphi$. Now because φ is a member of $s(L)$, it is a substitution instance of some propositional formula $\psi \in L$, as per Definition 2.5. That is, some map f from propositional letters $\{ p_i \mid i \in \mathbb{N} \}$ to the set $\{ \chi_i \mid i \in \mathbb{N} \}$ of all formulas of SO satisfies $\varphi = \psi[p_i := f(p_i)]$. Let g be the unique map from $\mathbb{N}$ to $\mathbb{N}$ such that $f(p_i) = \chi_{g(i)}$, which exists because the enumeration χ is without repetition.

Claim 2 The formula $\psi[p_i := p_{g(i)}]$ is in L, and yet K' does not force it.

The first half of Claim 2 is given to us by Lemma 2.19. As to the second half, note that

$$\varphi = \psi[p_i := f(p_i)] = \psi[p_i := \chi_{g(i)}] = (\psi[p_i := p_{g(i)}])[p_i := \chi_i],$$

and therefore by Claim 1,

$$\alpha \Vdash_{K'} \psi[p_i := p_{g(i)}] \iff \alpha \Vdash_K (\psi[p_i := p_{g(i)}])[p_i := \chi_i] \iff \alpha \Vdash_K \varphi.$$

Since $K \nVdash \varphi$, we have that not all nodes of K force φ, and thus not all nodes of K' force $\psi[p_i := p_{g(i)}]$. Thus $K' \nVdash \psi[p_i := p_{g(i)}]$, and so we have an element of L not satisfied by the model K', whose frame is a member of S. This contradicts the fact that L is sound over the set of models whose frames are in S. Thus we have reached a contradiction as desired and can conclude that every member of $s(L)$ is forced by K. □

Now we are ready to prove this section's main result, which leads directly to the desired injectivity of *Meta*.

Theorem 2.21 *For any $L_1, L_2 \in \mathcal{I}_{\text{Prop}}$,*

$$L_1 \nsubseteq L_2 \textit{ implies } Meta(L_1) \nsubseteq Meta(L_2).$$

Proof Assume $L_1 \nsubseteq L_2$. Then there is some $\varphi \in L_2 \backslash L_1$. Now because $L_1 \in \mathcal{I}_{\text{Prop}}$, there is a set S of posets such that L_1 is sound and complete over the set of Kripke models whose frames are in S. Thus there is a model K built on one of the posets in S such that $K \nVdash \varphi$ (and yet $K \Vdash L_1$). I build a second-order model K' on the same frame as K by the following procedure.

1. For each node α in K, decorate α in K' with an SO-structure M_α, whose first-order carrier set, which I will call A^1_α, is the set $\mathbb{N}$ of natural numbers, and whose second-order carrier set, which I will call A^2_α, is the powerset $\mathcal{P}(\mathbb{N})$.
2. Let each transition function $\langle \alpha, \beta \rangle$ in K' be the identity function.
3. Let the interpretations of 0, S, and f^n_i in M_α be just as in the standard model $\mathbb{N}$. Let the interpretation of each unary second-order constant C^1_i in M_α be the set $\{\, 0 \mid \alpha \Vdash p_i \,\}$. (This notation is from Section 1.3.) The interpretations of C^n_i for $n > 1$ are not relevant to this proof; let them be empty.

Now for any φ without second-order constants, Lemma 2.18 shows that $K' \Vdash \varphi$ if and only if $\mathbb{N} \vDash \varphi$, because the forcing of first-order formulas in no way references the second-order parts of K', and so we can appeal to Lemma 2.18. But for φ with second-order constants, this is not always true. The unary second-order constants in K' behave (on the number 0) like the propositional letters from K, behavior which is, in general, nonclassical.

$$\alpha \Vdash_K p_i \iff 0 \in [\![C^1_i]\!]^{M_\alpha} \iff \alpha \Vdash_{K'} (0 \in C^1_i)\,.$$

Thus one can show $K \Vdash \varphi \iff K' \Vdash \varphi[p_i := (0 \in C^1_i)]$ by an easy structural induction similar to the one in the proof of Lemma 2.19.

Therefore because $K \nVdash \varphi$, there is a $\psi \in s(L_2)$ satisfying $K' \nVdash \psi$, namely, $\psi = \varphi[p_i := (0 \in C^1_i)]$. Furthermore, $K' \Vdash s(L_1)$, by Lemma 2.20, and K' supports first-order Heyting Arithmetic because $\mathbb{N}$ does. We also have that K' supports the comprehension axiom of second-order Heyting Arithmetic because *every* subset of $\mathbb{N}$ is available in the second-order carrier set at each node, and so certainly those that are definable via the comprehension axiom are available.

Thus K' is a countermodel to $s(L_1) \cup \text{HAS} \vdash \psi$, and yet $\psi \in s(L_2) \cup \text{HAS}$. So we have the desired result:

$$Meta(L_1) = \overline{s(L_1) \cup \text{HAS}} \not\subseteq \overline{s(L_2) \cup \text{HAS}} = Meta(L_2) \qquad \square$$

Corollary 2.22 *For any two logics $L_1, L_2 \in \mathcal{L}_{\text{Prop}}$,*

1. $L_1 \subseteq L_2$ *iff* $Meta(L_1) \subseteq Meta(L_2)$,
2. $L_1 = L_2$ *iff* $Meta(L_1) = Meta(L_2)$, *and*
3. $L_1 \subsetneq L_2$ *iff* $Meta(L_1) \subsetneq Meta(L_2)$.

Proof The first equivalence is simply the combination of Theorems 2.17 and 2.21. The second follows from the first, and the third from the conjunction of the first two. $\square$

I have therefore achieved my goal of demonstrating the injectivity of *Meta*. This was not only necessary because an important theorem in the next section relies upon it, but also because it helps me ensure that an analysis of reflexivity for a logic refers only to that logic's unique behavior.

3 Propositional Reflexivity

3.1 Reasoning about members of $\mathcal{L}_{\text{Prop}}$ We are now in a position to use the metatheories I have constructed to reason about propositional logics. This accomplishes the main goal of the paper, analyzing reflexivity. I formalized this goal a bit in Definition 1.4, which introduced the term "reflexive." Yet Definition 1.4 remained largely informal. I can now give a formal definition of reflexivity.

Definition 3.1 (Reflexive) A logic $L \in \mathcal{L}_{\text{Prop}}$ is reflexive if and only if

$$Meta(L) \vdash Cpl(L),$$

where *Cpl* was defined in Definition 1.2.

In Section 1.1, I introduced the important difference between Int and Cl which motivates the work of this paper: A classical metatheory can prove the completeness of classical propositional logic, but a strictly intuitionistic metatheory cannot prove the completeness of intuitionistic propositional logic. Using the notation from Definition 3.1, I can write these two facts more formally.

$$Meta(\text{Cl}) \vdash Cpl(\text{Cl}).$$

$$Meta(\text{Int}) \nvdash Cpl(\text{Int}).$$

Thus one can say that Cl is reflexive and Int is not.

Definition 3.2 ($\mathcal{R}_{\text{Prop}}$) For the set of reflexive propositional logics, I write $\mathcal{R}_{\text{Prop}}$. That is, $L \in \mathcal{R}_{\text{Prop}}$ if and only if $L \in \mathcal{L}_{\text{Prop}}$ and $Meta(L) \vdash Cpl(L)$.

So I can write $\text{Cl} \in \mathcal{R}_{\text{Prop}}$ and $\text{Int} \notin \mathcal{R}_{\text{Prop}}$. It is natural to ask whether the elements of $\mathcal{R}_{\text{Prop}}$ are not simply scattered about $\mathcal{L}_{\text{Prop}}$, but perhaps a boundary line exists between Int and Cl delimiting $\mathcal{R}_{\text{Prop}}$. Indeed it does, and the following theorems find that boundary line explicitly.

I do the work of this section using ordinary mathematical notation, but all of it could be formalized in any metatheory whose language is SO. For instance, when I write $f : A \to B$, I could instead have written

$$(f \subseteq A \times B) \ \wedge \ \forall x \in A \, \exists y \in B \, (\langle x, y\rangle \in f) \\ \wedge \forall x \forall y \forall z (\langle x, y\rangle \in f \ \wedge \ \langle x, z\rangle \in f \to y = z),$$

provided that I had defined $A \times B$ formally. As another example, a sequence $\{ B_i \mid i \in \mathbb{N} \}$ of second-order objects can be thought of as a single second-order object B with higher arity such that $B_i(n)$ if and only if $B(i, n)$. Workarounds such as these are common, and there are several textbooks that use such tactics extensively for advanced work (e.g., Simpson [16], pp. 69, 76, etc.). I shall feel free to use ordinary notation below without converting to a more cumbersome method simply to demonstrate more clearly that the work is doable in SO.

3.2 Testability logic and maximal consistent sets Testability logic (Test, Definition 1.10) now takes center stage, because I show below that $\mathcal{R}_{\text{Prop}}$ is the collection of logics in $\mathcal{I}_{\text{Prop}}$ that are at least as strong as Test. The principle of testability has noteworthy qualities besides its being central to the work of this paper. It is intuitionistically equivalent to the De Morgan law rejected by intuitionists, and two papers by Johnstone demonstrate that this holds true in arbitrary topoi as well ([4], [5]). He further shows more than a dozen other interesting properties that are equivalent, in topos logic, to the principle of testability.

I proceed toward showing Test $\in \mathcal{R}_{\text{Prop}}$ by first performing a construction that is useful in a completeness proof and then proving some essential facts about it. The construction is performed and the related results proven using only testability logic as the metatheory, so they are available later when trying to establish that *Meta*(Test) $\vdash$ *Cpl*(Test).

Central to the arguments below is the fact that the consistency of a set of formulas is itself expressed through a negative formula. That is, the statement "Γ is consistent in testability logic," written $\Gamma \nvdash_{\text{Test}} \bot$, is formulated

$$\neg\exists n \, \exists\gamma_1, \ldots, \exists\gamma_n \in \Gamma \, ((\gamma_1 \wedge \cdots \wedge \gamma_n \to \bot) \in \text{Test})$$

by Definition 2.9. Because this formula begins with a negation, the Law of the Excluded Middle holds for it in Testability Logic. That is, *Meta*(Test) derives $(\Gamma \nvdash_{\text{Test}} \bot) \vee \neg(\Gamma \nvdash_{\text{Test}} \bot)$, because it is an instance of the principle of testability, $(\neg\varphi) \vee (\neg\neg\varphi)$. We see below that this is essential to the construction of maximal consistent sets.

Definition 3.3 (Construction of Σ_n, Σ) Given any set of formulas Δ that is consistent in Test, I construct a sequence of sets Σ_n as follows.

1. Let $\langle \varphi_i \mid i \in \mathbb{N} \rangle$ be an enumeration of all formulas in Prop.
2. Define the sequence Σ_n recursively by

$$\Sigma_0 = \Delta, \text{ and}$$

$$\Sigma_{n+1} = \Sigma_n \cup \left\{ \varphi_n \;\middle|\; \Sigma_n \cup \{ \varphi_n \} \nvdash_{\text{Test}} \bot \right\}.$$

3. Lastly, let

$$\Sigma = \bigcup_{n \in \mathbb{N}} \Sigma_n.$$

The notation used in the recursive case is from Section 1.3. I'll be a bit more rigorous than usual in the following proofs because I'm working in *only* *Meta*(Test), and I want to make it clear that I'm obeying that restriction.

Lemma 3.4 *Given a set of formulas* Δ *that is consistent in testability logic and an extension* Σ *constructed as in Definition 3.3,* *Meta*(*Test*) *is sufficient to show each of the following.*

1. $\Delta \subseteq \Sigma$.
2. Σ *is consistent:* $\Sigma \nvdash_{\text{Test}} \bot$.
3. $\varphi_m \in \Sigma$ *iff* $\Sigma_m \cup \{ \varphi_m \} \nvdash_{\text{Test}} \bot$.
4. Σ *is closed under derivability:* $\Sigma \vdash_{\text{Test}} \varphi \ \Rightarrow \ \varphi \in \Sigma$.
5. *For any formula* φ*, either* $\varphi \in \Sigma$ *or* $(\neg\varphi) \in \Sigma$.

Proof

1 The first claim is trivial:

$$\Delta = \Sigma_0 \subseteq \bigcup_{n \in \mathbb{N}} \Sigma_n = \Sigma$$

because 0 is a natural number.

2 To establish the second claim, I first show that every Σ_m is consistent, by induction on m. The base case is done, because the consistency of $\Delta = \Sigma_0$ is a premise.

The induction step requires care to use only reasoning allowed in testability logic. We have two cases,

$$(\Sigma_n \cup \{ \varphi_n \} \nvdash_{\text{Test}} \bot) \quad \text{and} \quad \neg\, (\Sigma_n \cup \{ \varphi_n \} \nvdash_{\text{Test}} \bot)\,,$$

recalling that *Meta*(Test) derives their disjunction because consistency is a negative formula. In the case where $\Sigma_n \cup \{ \varphi_n \} \nvdash_{\text{Test}} \bot$, clearly $\Sigma_{n+1} \nvdash_{\text{Test}} \bot$ because $\Sigma_{n+1} = \Sigma_n \cup \{ \varphi_n \}$. In the other case, $\Sigma_{n+1} = \Sigma_n$ and so consistency comes from the induction hypothesis. This completes the induction step, and thus all Σ_m are consistent.

If we assume toward a contradiction that $\Sigma \vdash_{\text{Test}} \bot$, then by Definition 1.7 we have

$$\exists n\, \exists \gamma_1, \ldots, \exists \gamma_n \in \Sigma\ [(\gamma_1 \wedge \cdots \wedge \gamma_n \to \bot) \in \text{Test}].$$

Now each γ_i was introduced to Σ at some phase Σ_j; that is, $\gamma_i \in \Sigma = \cup_{j \in \mathbb{N}} \Sigma_j$ implies $\exists j (\gamma_i \in \Sigma_j)$. Let m be the largest such j, so that Σ_m contains all the γ_i. Therefore $\Sigma_m \vdash_{\text{Test}} \bot$, contradicting my induction proof above. Thus $\Sigma \nvdash_{\text{Test}} \bot$.

3 ($\Rightarrow$) If $\varphi_m \in \Sigma$ then $\Sigma_m \cup \{ \varphi_m \} \subseteq \Sigma$, and so it is consistent because I have shown Σ to be consistent.

($\Leftarrow$) If $\Sigma_m \cup \{ \varphi_m \} \nvdash_{\text{Test}} \bot$ then by the construction in Definition 3.3, we have that $\Sigma_{m+1} = \Sigma_m \cup \{ \varphi_m \}$, and so $\varphi_m \in \Sigma_{m+1} \subseteq \Sigma$.

4 To show that Σ is closed under derivability, first assume that $\Sigma \vdash_{\text{Test}} \varphi$. Let $m \in \mathbb{N}$ such that $\varphi = \varphi_m$, and assume toward a contradiction that $\Sigma_m \cup \{ \varphi_m \} \vdash_{\text{Test}} \bot$. From this assumption, I can use the Deduction Theorem (as per Theorem 2.13) to achieve that $\Sigma_m \vdash_{\text{Test}} (\varphi_m \to \bot)$ and weakening to show that $\Sigma \vdash_{\text{Test}} (\varphi_m \to \bot)$. But because we already have $\Sigma \vdash_{\text{Test}} \varphi$, modus ponens gives us $\Sigma \vdash_{\text{Test}} \bot$, contradicting my earlier proof of the consistency of Σ.

Thus we have $\Sigma_m \cup \{ \varphi_m \} \nvdash_{\text{Test}} \bot$, which gives $\varphi = \varphi_m \in \Sigma$ by part 3.

5 That $(\neg\varphi) \in \Sigma$ implies $\varphi \notin \Sigma$ is immediate by the consistency of Σ, proven above. The other direction requires more work. Assume $\varphi = \varphi_m$.

I have just shown that $\varphi_m \in \Sigma$ holds if and only if the negative statement $\Sigma_m \cup \{ \varphi_m \} \nvdash_{\text{Test}} \bot$ holds, which means that membership in Σ is a negative statement. Thus for any formula φ, the statements $\varphi \in \Sigma$ and $\neg\neg(\varphi \in \Sigma)$ are equivalent. This enables us to show the other direction, that $\varphi \notin \Sigma$ implies $(\neg\varphi) \in \Sigma$.

$$\begin{array}{rll}
\varphi \notin \Sigma & \iff \varphi_m \notin \Sigma & \varphi = \varphi_m \\
& \iff \neg(\Sigma_m \cup \{ \varphi_m \} \nvdash_{\text{Test}} \bot) & \text{part 3 of this lemma} \\
& \iff \neg\neg(\Sigma_m \vdash_{\text{Test}} \neg\varphi_m) & \text{Theorem 2.13} \\
& \Rightarrow \neg\neg(\Sigma \vdash_{\text{Test}} \neg\varphi) & \Sigma_m \subseteq \Sigma, \varphi = \varphi_m \\
& \iff \neg\neg((\neg\varphi) \in \Sigma) & \text{part 4 of this lemma} \\
& \iff (\neg\varphi) \in \Sigma & \text{previous paragraph}
\end{array}$$

□

Lemma 3.5 *Given Σ_n, Σ constructed from Δ as in Definition 3.3, testability logic is sufficient to show that Σ is truth-functional. That is, defining $M(\varphi) = \{ 0 \mid \varphi \in \Sigma \}$ makes M a truth function as per Definition 1.3. This notation $\{ 0 \mid \varphi \in \Sigma \}$ is from Section 1.3.*

Proof

1 To show $M \vDash \neg\varphi \iff M \nvDash \varphi$, one needs only to show $(\neg\varphi) \in \Sigma \iff \varphi \notin \Sigma$, which has been done in Lemma 3.4.

2 To show that $M \vDash \varphi \wedge \psi \iff M \vDash \varphi$ and $M \vDash \psi$, note that both directions of the equivalence $(\varphi \wedge \psi) \in \Sigma \iff \varphi \in \Sigma$ and $\psi \in \Sigma$ are immediate from the closure of Σ under derivability.

3 To show that $M \vDash \varphi \vee \psi \iff M \vDash \varphi$ or $M \vDash \psi$, I show $(\varphi \vee \psi) \in \Sigma \iff \varphi \in \Sigma$ or $\psi \in \Sigma$ as follows.

- $(\Leftarrow)$ This direction is immediate from the closure of Σ under derivability.
- $(\Rightarrow)$ Assume $(\varphi \vee \psi) \in \Sigma$. Further assume that $\varphi \notin \Sigma$. Recall from above that $\varphi \notin \Sigma$ implies $\neg\varphi \in \Sigma$.

The deduction $\varphi \vee \psi, \neg\varphi \vdash \psi$ is intuitionistically valid, as the derivation below demonstrates. *Meta*(L) is aware of it by virtue of *Meta*$(L) \supseteq$ *Meta*(Int), together with Corollary 2.15.

$$\dfrac{\varphi \vee \psi \qquad \dfrac{\dfrac{[\varphi]^{①} \quad \neg\varphi}{\bot}\,\text{MP}}{\psi}\,\text{EFQ} \qquad [\psi]^{①}}{\psi}\ \vee\text{E}\ ①$$

Since $(\varphi \vee \psi) \in \Sigma$ and $(\neg\varphi) \in \Sigma$, we therefore have $\Sigma \vdash_{\text{Test}} \psi$, and thus $\psi \in \Sigma$ by the closure of Σ under derivability. Thus when $(\varphi \vee \psi) \in \Sigma$, we have that $\varphi \notin \Sigma$ implies $\psi \in \Sigma$.

Now membership in Σ is a negative formula, as per part 3 of Lemma 3.4. Thus we have the disjunction $(\varphi \in \Sigma) \vee (\varphi \notin \Sigma)$ to work with in *Meta*(Test), as an instance of the principle of testability. From the implication shown immediately above, from $(\varphi \in \Sigma) \vee (\varphi \notin \Sigma)$ we can conclude $(\varphi \in \Sigma) \vee (\psi \in \Sigma)$, as desired.

4 To show that $M \vDash (\varphi \to \psi) \iff$ if $M \vDash \varphi$ then $M \vDash \psi$, I show

$$(\varphi \to \psi) \in \Sigma \iff \text{if } \varphi \in \Sigma \text{ then } \psi \in \Sigma$$

as follows.

($\Rightarrow$) This direction is immediate by the closure of Σ under derivability.
($\Leftarrow$) Assume that $\varphi \in \Sigma$ implies $\psi \in \Sigma$.

If we have $(\varphi \to \psi) \notin \Sigma$, then by part 1, we must have $(\neg(\varphi \to \psi)) \in \Sigma$. Then we have $\neg\neg\varphi \in \Sigma$ by the following derivation and the fact that Σ is deductively closed.

$$\dfrac{\dfrac{\neg(\varphi \to \psi) \qquad \dfrac{\dfrac{\dfrac{[\varphi]^{①} \qquad [\neg\varphi]^{②}}{\bot}\ \text{MP}}{\psi}\ \text{EFQ}}{\varphi \to \psi}\ \to \text{I}\ ①}{\bot}\ \text{MP}}{\neg\neg\varphi}\ \neg\text{I}\ ②$$

Applying part 1 of this Lemma twice to $(\neg\neg\varphi) \in \Sigma$ gives us $\neg\neg(\varphi \in \Sigma)$, which is equivalent to $\varphi \in \Sigma$ because membership in Σ is negative, as established earlier.

My original assumption then gives us $\psi \in \Sigma$, and the closure of Σ under deduction gives $(\varphi \to \psi) \in \Sigma$. This contradicts my original assumption, yielding $\neg((\varphi \to \psi) \notin \Sigma)$. Again because membership in Σ is negative, this is equivalent to $(\varphi \to \psi) \in \Sigma$.

5 $M \nvDash \bot$ because $\bot \notin \Sigma$, as per the consistency of Σ established in Lemma 3.4.

The above parts together demonstrate that Σ is truth-functional as per Definition 1.3. □

Note that because the construction in Definition 3.3 and all the results that followed it can be proven in *Meta*(Test), they can also be proven in *Meta*(L), for any $L \in \mathcal{L}_{\text{Prop}}$ such that $L \supseteq$ Test.

3.3 Classifying reflexive propositional logics We now have the tools to characterize $\mathcal{R}_{\text{Prop}}$ explicitly. I first show that, for any L extending Test, $L \in \mathcal{R}_{\text{Prop}}$. I will then use a simplified version of a proof from [11] to show that *Meta*(L) $\supseteq$ *Meta*(Test) is a necessary condition for membership in $\mathcal{R}_{\text{Prop}}$. I then conclude from the injectivity of *Meta* that this condition can be simplified to $L \supseteq$ Test. The combination of these results establishes that $L \in \mathcal{R}_{\text{Prop}}$ if and only if $L \supseteq$ Test.

Theorem 3.6 *For any* $L \in \mathcal{L}_{\text{Prop}}$*, if* $L \supseteq$ Test *then* $L \in \mathcal{R}_{\text{Prop}}$.

Proof Let Δ be a set of formulas of Prop that is consistent in L, and I show (using only *Meta*(L)) that it has a model. From Δ construct an extension Σ as in Definition 3.3. By Lemma 3.5, I can create a truth function M by $M(\varphi) = \{ 0 \mid \varphi \in \Sigma \}$. So we have a structure $M \vDash \Sigma$, and therefore $M \vDash \Delta$, and each step was performed using only *Meta*(L) to do its work.

So I have shown in *Meta*(L) that, for any set Δ of formulas from Prop, if $\Delta \nvdash_L \bot$ then $\exists M(M \vDash \Delta)$. This places $L \in \mathcal{R}_{\text{Prop}}$ by Definition 3.2. □

As I have mentioned, the converse of this result comes easily by tailoring a theorem of [11] to suit our needs. I show that proof here, and then a straightforward corollary establishes the desired converse.

Theorem 3.7 (McCarty [11]) *If T is a metatheory that is at least as strong as $Meta(\mathrm{Int})$, and T is sufficient to prove completeness for an intermediate propositional logic L, then T derives every instance of the principle of testability.*

The original theorem differs from this in two noteworthy ways. It is stronger in that it assumes only that completeness with respect to subfinite sets of formulas is provable in T, and it guarantees that every negative instance of every classical tautology is provable. This proof's argument is a bit simpler, because it has the freedom to deal with a specific classical tautology, the principle of testability. The original theorem is only weaker in that I have replaced Int with L throughout, noting that there is no problem in doing so.

Less substantive differences include that McCarty did not use the notation $Meta(\mathrm{Int})$ to refer to a strictly intuitionistic metatheory, and there are some small alterations which make evident the uses of the results from Section 2.2.

Proof Assume that $T \vdash Cpl(L)$. Given an instance $\neg\varphi \vee \neg\neg\varphi$ of the principle of testability, I show that $T \vdash \neg\varphi \vee \neg\neg\varphi$. Define two sets of formulas Φ and Ψ as follows.

$$\Psi = \{\, p \mid \neg\varphi \,\} \cup \{\, \neg p \mid \neg\neg\varphi \,\}.$$

$$\Phi = \Psi \cup \{\, p \vee \neg p \,\}.$$

The notation used in defining Ψ is from Section 1.3.

I show that $T \vdash (\Phi \nvdash_L \bot)$. Working in the metatheory T, assume toward a contradiction that $\Phi \vdash_L \bot$. Then we would have $\Phi \vdash_{\mathrm{Cl}} \bot$, because $L \subseteq \mathrm{Cl}$. The deduction theorem for L in T (Corollary 2.14) together with the definition of Φ gives us that $\Psi \vdash_{\mathrm{Cl}} p \vee \neg p \to \bot$. The metatheory T is aware of the classical deduction $\vdash_{\mathrm{Cl}} p \vee \neg p$ because $Meta(\mathrm{Int})$ is, by Corollary 2.15. Therefore T can use the fact that L is deductively closed (Definition 1.8) to obtain $\Psi \vdash_{\mathrm{Cl}} \bot$.

Now if $\neg\varphi$ were to hold, then we would have $\Psi = \{\, p \,\}$, which is certainly a consistent set. This contradicts the fact that $\Psi \vdash_{\mathrm{Cl}} \bot$, and so I conclude $\neg\neg\varphi$. However, this gives us that $\Psi = \{\, \neg p \,\}$, which is also a consistent set. This, too, contradicts $\Psi \vdash_{\mathrm{Cl}} \bot$, allowing us to conclude that our assumption from the previous paragraph is false, giving $\Phi \nvdash_L \bot$. Because T has just shown $\Phi \nvdash_L \bot$ and we have assumed that $T \vdash Cpl(L)$, we can conclude in T that $\exists M (M \vDash \Phi)$.

Now working again in T, assume that $M \vDash p$, and assume further that $\neg\neg\varphi$ holds. We then have that $(\neg p) \in \Phi$ by the definition of Φ and Ψ, giving us $M \vDash \neg p$. This contradiction demonstrates that $M \vDash p$ implies $\neg\varphi$. By an analogous argument, T is also sufficient to show that $M \vDash \neg p$ implies $\neg\neg\varphi$.

From these implications, T can show that $M \vDash p \vee \neg p$ implies $\neg\varphi \vee \neg\neg\varphi$. Since the antecedent is true by $M \vDash \Phi$, so must the consequent be, and therefore T derives the desired instance of testability. □

Corollary 3.8 *For any $L \in \mathcal{I}_{\mathrm{Prop}}$, if $L \in \mathcal{R}_{\mathrm{Prop}}$, then $L \supseteq \mathrm{Test}$.*

Proof Given $Meta(L) \vdash Cpl(L)$, I can use Theorem 3.7 with $T = Meta(L)$ to obtain $Meta(L) \supseteq Meta(\mathrm{Test})$. From Corollary 2.22 one can conclude that $L \supseteq \mathrm{Test}$. □

Corollary 3.9 *For any $L \in \mathcal{I}_{\mathrm{Prop}}$, $L \in \mathcal{R}_{\mathrm{Prop}}$ if and only if $L \supseteq \mathrm{Test}$.*

Proof Theorem 3.6 and Corollary 3.8. □

The equivalence established in Corollary 3.9 is satisfying because it demonstrates that in $\mathcal{I}_{\text{Prop}}$, the predicate "is a reflexive propositional logic," whose complexity comes from its reliance on the complex definitions of *Meta* and *Cpl*, is equivalent to the simple characterization "is at least as big as Test." Therefore the natural and interesting concept of propositional reflexivity is also seen to have a pleasantly elegant representation.

4 Related Notes

4.1 Using Kripke semantics One may ask why I used a Tarskian semantics (one lone truth function) for all intermediate logics. Is it not the case that no logic besides Cl is complete for such a semantics? I began to address this question in Section 1.2, saying that truth functions do not always behave classically in a nonclassical metatheory. For example,

$$\mathit{Meta}(\text{Int}) \nvdash \forall M \forall \varphi (M \vDash \varphi \vee \neg \varphi).$$

One can show this using an argument like the one in Theorem 3.7.

Although one might use, for example, Kripke semantics in an attempt to ensure that all logics have ample opportunity to find needed countermodels, one would then be putting Kripke semantics in a nonclassical context. Kripke semantics (and Beth semantics and realizability models) behave constructively when the reasoning about them is done classically. Since I am not reasoning classically about these structures, I need no such assistance from within the structures themselves. The logic that is reasoning about the structure provides constructivity, or classicality, or anything in between. This tactic has been used in [11] and [10].

To ensure that this reliance on the metatheoretic context indeed accomplishes what I claim it does, I now show that the classification of $\mathcal{R}_{\text{Prop}}$ in this chapter does not change if one uses Kripke semantics rather than Tarskian semantics.

It is clear that countermodel completeneg (what I've been using so far) implies Kripke countermodel completeness, because every classical model is also a Kripke model. That is, because we know that

$$\exists M (M \vDash \Gamma) \text{ implies } \exists K (K \Vdash \Gamma),$$

then $\mathit{Cpl}(L)$, or

$$(\Gamma \nvdash_L \bot) \rightarrow \exists M (M \vDash \Gamma),$$

implies $\mathit{Cpl}_{\text{Kr}}(L)$, or

$$(\Gamma \nvdash_L \bot) \rightarrow \exists K (K \Vdash \Gamma).$$

Thus if $\mathit{Meta}(L) \vdash \mathit{Cpl}(L)$ then $\mathit{Meta}(L) \vdash \mathit{Cpl}_{\text{Kr}}(L)$, so using Kripke semantics gives at least as large a set of reflexive logics.

Conversely, consider the proof of Theorem 3.7. Note that nowhere therein were the properties of a classical structure exploited. That is, the one situation in which the structure M behaved classically (the case of the particular propositional letter p) was guaranteed not by the structure M itself, but by the set Φ. One could therefore repeat the proof of Theorem 3.7 using Kripke completeness and Kripke semantics, and the model K created would behave classically in the one instance required by Φ, just as M does. Thus $\mathit{Meta}(L) \vdash \mathit{Cpl}_{\text{Kr}}(L)$ implies $L \supseteq$ Test, and thus it also implies $\mathit{Meta}(L) \vdash \mathit{Cpl}(L)$. Thus $\mathit{Meta}(L) \vdash \mathit{Cpl}(L)$ is equivalent to $\mathit{Meta}(L) \vdash \mathit{Cpl}_{\text{Kr}}(L)$, and so using Kripke semantics would give the same set of reflexive logics as Tarskian semantics.

One might consider instead appealing to a technique like lawless sequences to prove this converse. That is, if $Meta(L)$ plus axioms for lawless sequences could prove that $Cpl_{\mathrm{Kr}}(L) \to Cpl(L)$, then perhaps a theorem on eliminability of lawless sequences (e.g., p. 33 of Troelstra [17]) could show that $Meta(L)$ proves the same result. However, axioms for lawless sequences are incompatible with classical logic, so such a technique cannot work for all $L \in \mathcal{I}_{\mathrm{Prop}}$. Thus I am content with the argument in the previous paragraph.

4.2 Zorn's Lemma In the work of Section 3.3, I only needed the principle of testability for the proof that the set Σ I constructed was maximal consistent. The existence of maximal consistent sets would be guaranteed by Zorn's Lemma, so one might naturally inquire as to the relationship between Zorn's Lemma and Testability. (Note that intuitionistically, Zorn's Lemma is not equivalent to the Axiom of Choice.)

If $Meta(L)$ contained Zorn's Lemma, then Theorem 3.6 would be sufficient to show $Meta(L) \vdash Cpl(L)$, simply replacing Lemma 3.5 with Zorn's Lemma to create Σ. Therefore Zorn's Lemma is at least as strong as Test, because if a logic supports Zorn's Lemma then it is in $\mathcal{R}_{\mathrm{Prop}}$, and therefore supports Test. But the converse is not true, because no metatheory $Meta(L)$ for $L \in \mathcal{I}_{\mathrm{Prop}}$ can contain Zorn's Lemma, because $Meta(\mathrm{Cl})$ does not contain it.

4.3 Broader implications Although the work of this paper has focused on investigating the power of metatheories reasoning about their underlying propositional logics, I have built up enough machinery to also settle the question, "When can $Meta(L_1) \vdash Cpl(L_2)$, for $L_1, L_2 \in \mathcal{I}_{\mathrm{Prop}}$?" The change here is that now the metatheory $Meta(L_1)$ need not be reasoning about its own underlying propositional logic L_1, but rather any other logic $L_2 \in \mathcal{I}_{\mathrm{Prop}}$.

The answer to the above question turns out to involve testability logic in much the same way as my original question about reflexivity. The following theorem answers this question, building on Theorem 2.16.

Theorem 4.1 *The following are equivalent for a logic $L_1 \in \mathcal{I}_{\mathrm{Prop}}$.*

1. *$L_1 \in \mathcal{R}_{\mathrm{Prop}}$ (i.e., $Meta(L_1) \vdash Cpl(L_1)$).*
2. *For some $L_2 \in \mathcal{I}_{\mathrm{Prop}}$, $Meta(L_1) \vdash Cpl(L_2)$.*
3. *For every $L_2 \in \mathcal{I}_{\mathrm{Prop}}$, $Meta(L_1) \vdash Cpl(L_2)$.*

Proof It is clear that (1) implies (2) by taking $L_2 = L_1$. It is also clear that (3) implies (1) by taking $L_2 = L_1$ again.

To see the remaining direction, that (2) implies (3), first assume that for some $L_2 \in \mathcal{I}_{\mathrm{Prop}}$, $Meta(L_1) \vdash Cpl(L_2)$. Now let L_3 be an arbitrary member of $\mathcal{I}_{\mathrm{Prop}}$ and note that Theorem 2.16 gives us that

$$Meta(\mathrm{Int}) \vdash (\Gamma \vdash_{L_2} \bot) \leftrightarrow (\Gamma \vdash_{L_3} \bot).$$

Thus the statement $Meta(L_1) \vdash Cpl(L_2)$, which is defined to be

$$Meta(L_1) \vdash \forall\Gamma\ (\Gamma \nvdash_{L_2} \bot \to \exists M(M \vDash \Gamma)),$$

is equivalent to

$$Meta(L_1) \vdash \forall\Gamma\ (\Gamma \nvdash_{L_3} \bot \to \exists M(M \vDash \Gamma)),$$

or $Meta(L_1) \vdash Cpl(L_3)$. Because L_3 was chosen arbitrarily, this gives us that for any $L_3 \in \mathcal{L}_{\mathrm{Prop}}$, $Meta(L_1) \vdash Cpl(L_3)$. This differs only from part 3 in the choice of the dummy variable. □

This gives a nice broadening of the earlier results regarding testability logic as the boundary of $\mathcal{R}_{\mathrm{Prop}}$. Not only is it the case that $L \supseteq$ Test if and only if L can prove its own completeness, but $L \supseteq$ Test if and only if it can prove anybody's completeness, if and only if it can prove everybody's completeness. In particular, *Meta*(Cl) can prove the completeness of Int with respect to Tarskian semantics but *Meta*(Int) cannot reciprocate, that is, prove *Cpl*(Cl).

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