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Book Review

Michael Potter. *Reason's Nearest Kin. Philosophies of Arithmetic from Kant to Carnap*. Oxford University Press, Oxford, 2000. x + 305 pages

1 Potter's Project

Potter's book *Reason's Nearest Kin* tells a very important and interesting story in a novel and insightful way. It is the story of how some of the greatest philosophers and mathematicians of the late 19th and early 20th century have attempted to give philosophical accounts of arithmetic. The figures whose work Potter discusses are Frege, Dedekind, Whitehead and Russell, Wittgenstein, Ramsey, Hilbert, Gödel, and Carnap. Of course, they all were directly or indirectly influenced by Kant, and so Kant's philosophy of arithmetic also receives extensive treatment. Potter frames his discussion by two questions, which he takes all of these writers to be seeking to answer:

> Can we give an account of arithmetic that does not make it depend for its truth on the way the world is? And if so, what constrains the world to conform to arithmetic? (p. 1)

Potter, and certainly also most of the figures he considers, takes it as given that arithmetic is necessary and that the main difficulties a philosophical account of arithmetic faces are those of (a) explaining why that is and (b) how this necessity can be reconciled with the applicability of arithmetic to the world.

Potter's choice of topics is certainly novel, both in what it includes and in what it excludes. Standard treatments of the philosophy of mathematics around 1900 commonly give significant attention to Brouwer and Weyl's intuitionism, and few pay as much attention to Wittgenstein, Ramsey, and Carnap as Potter does. But it is clear from the questions framing the book why that is: Brouwer's intuitionism is decidedly subjectivist; the emphasis in intuitionism on the mathematical constructing subject arises from a desire to answer not the question of why mathematics is necessary and objective, but why (and to what extent) it is certain. One might of course think of other positions in the philosophy of mathematics worthy of treatment, such as Mill's

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and Husserl's, but they also wouldn't fit in Potter's particular view of the development of the subject. So although it is perhaps regrettable that Brouwer (and others) are left out, it is justified (and even refreshing). And it is also justified and decidedly refreshing to include Ramsey, Wittgenstein, and Carnap.

Potter's book is divided into thirteen chapters. In the Introduction, Potter sets the stage by setting out the constraints philosophies of arithmetic have to satisfy and considers and rejects some of the approaches to the philosophy of arithmetic which immediately fail one or another of these constraints, such as empiricism, psychologism, and several versions of formalism. Chapter 1 is devoted to Kant's views on arithmetic. Here Potter gives a good survey of Kant's scattered remarks on the role of intuition and reason in accounting for the synthetic a priori nature of arithmetic. It is by no means an in-depth study of Kant's work, but it is one of the better introduction to Kant's thinking on arithmetic I've seen. The main purpose it serves is to set up Kant as a paradigm, and a foil, for the views of the later figures Potter studies. It serves this purpose very well. Chapter 2 is a discussion of Frege's *Grundlagen*. It is followed by a chapter on Dedekind's account of number in *Was sind und was sollen die Zahlen?* Chapter 4 is devoted to Frege's account of classes in *Grundgesetze*. The next four chapters concern the further development of logicism: Chapter 5 is devoted to the development of Russell's work, from "On denoting" through the work with Whitehead in *Principia* and ending with *Introduction to Mathematical Philosophy*. Chapter 6 is an informative and unconventional excursion into Wittgenstein's views on number in the *Tractatus*. It is followed by a short chapter on the second edition of *Principia* (Chapter 7) and one on Ramsey (Chapter 8). Chapter 9 deals with Hilbert's finitary standpoint and Hilbert's program for the foundation of arithmetic. Chapter 10 discusses Gödel's incompleteness theorems and their impact on Hilbert's program. Chapter 11 is devoted to Carnap's account of arithmetic, mainly in *Logical Syntax*. A final conclusion closes the book.

Reason's Nearest Kin is a fine book in many respects. The questions Potter uses to frame his discussion make for a nice arc in the story he tells—an arc that is often lacking in writings on the foundational projects of the period in question. This virtue is also a slight drawback, for the writers in question (in particular those who were not primarily concerned with these questions per se) put the emphasis in their own work elsewhere. This is in particular true of Dedekind and Hilbert, who were engaged in broader foundational projects, and also in more properly mathematical work. But introducing more focus into the discussion is not a bad thing. Even in light of this focus, restrictions of space no doubt prevented Potter from giving a comprehensive account of much of what he does cover, and, in particular, of the (extensive) secondary literature. Nevertheless, the absence of more detailed notes and references to the literature is sometimes frustrating. The specialist will be able to fill in for herself much of the context, historical and critical, but the book is hardly introductory. The large amounts of technical detail—a laudable trait, in my view, and very useful even to the specialist—will be a barrier for a wider readership. So *Reason's Nearest Kin* will best be read *after* one has already a good understanding of the issues and of the technical work covered.

Since this is a critical notice, I will permit myself to be a little critical of the details. I will focus on Potter's discussion of Hilbert's program and Gödel's incompleteness theorems (Chapters 9 and 10). Others more knowledgeable about the preceding chapters have had things to say about them: Demopoulos [3] about Potter's treatment of Frege, and Landini [15] about that of Russell; see also the review by MacFarlane [16]. Potter's discussion of Hilbert's views is divided into roughly four parts. Hilbert's early views are treated in §3.3. Chapter 9 is entirely devoted to Hilbert and deals with the finitary standpoint in §§9.2 and 9.3 and his consistency program in §§9.4–9.6. Chapter 10 contains a presentation of Gödel's second incompleteness theorem and discusses how it impacts Hilbert's program.

2 Potter on Hilbert's Finitism

Hilbert's philosophical views are notoriously difficult to pin down. This is in part due to the fact that Hilbert's published writings were primarily directed at a mathematical audience, and his writings on the philosophy of mathematics were almost all published versions of lectures he gave in various venues, usually meetings of mathematical societies. So it is not surprising that in them we do not find the deep engagement with philosophical issues we find in, say, Frege or Russell, and that references to established philosophical positions on which one might hang an interpretation of Hilbert's philosophical remarks are short and sparse. Interpretation of Hilbert's writings, more so than that of more philosophically oriented figures like Frege and Russell, must then not only take into account Hilbert's published writings, but also that of his collaborators in "Hilbert's program" (above all, Paul Bernays), the technical work carried out in pursuit of the program, and the copious unpublished material. In recent years, the availability (in English translation) of almost all the relevant published writings of Hilbert and Bernays of the 1920s (especially in Ewald [6] and Mancosu [17]) has made things a good deal easier. Hilbert's unpublished lectures, however, are only now being prepared for publication (Hilbert [12]). Quite a number of commentators have already made use of these materials in the last twenty years or so. (Unfortunately, Potter's account of Hilbert makes reference only to Hallett [9].) Nevertheless, even the published writings raise many important and difficult questions about Hilbert's views which Potter is tracing out in Chapter 9.

Let me begin with Potter's discussion of the finitary standpoint. Potter follows Parsons [18] in emphasizing the intuitive character of finitary evidence. For Parsons, as for Hilbert himself, the role of intuition in finitary mathematics was that of providing a bedrock of certainty on which Hilbert's metamathematics could be erected. In his discussions of finitism, Hilbert himself did not, to the best of my knowledge, concern himself to any substantial extent with Potter's questions, namely, why arithmetic is necessary and how it is that arithmetic applies to the world. Hilbert rejected the Kantian picture of space and time as a priori conditions of experience—for Hilbert "the structure of space and time," according to Potter, "[is] a question for physicists to settle, not for philosophers." Hilbert did not explain how, say, the conditions of experience account for the necessary character of finitary evidence, nor did he give a detailed account of the applicability of finitary arithmetic (see, however, the discussion of how finitary stroke symbols relate to cardinality in Hilbert [11] and especially in Hilbert and Bernays [13], pp. 28–9). Potter is certainly right in criticizing Hilbert on this point, but Hilbert saw his task not as this: it is a task for the philosopher, not the mathematician to provide an account of intuition that supplies the required knowledge of finitary mathematics. In this case, it was a task for Bernays and others in Göttingen, such as Leonard Nelson. Hilbert took the certainty of knowledge obtained by finitary evidence as given and tried to show how a foundation for the *rest* of mathematics can be obtained from that starting point. This also explains why in Hilbert's own writings there does not seem to be one account of finitary intuition which he had in mind. So the interpretation of Hilbert's views on finitary intuition faces the difficulty of sorting out which notion of finitary evidence he is appealing to at any given time, and the assessment of Hilbert's philosophical claims faces the difficulty of picking one of these notions against which to assess these claims.

Potter, like Parsons, focuses primarily on the early account of intuition of finite spatio-temporal combinations (of strokes) as underlying the finitary standpoint. This account faces two difficulties: The first is that it is difficult to see how it yields answers to Potter's two questions, noted above. The second problem is that appeal to intuition does not justify all primitive recursive functions as finitary, in particular, not exponentiation. Potter tries to help Hilbert get around the first problem by suggesting that a way out of the difficulties Hilbert faces on this account would be to appeal to Wittgenstein's account of arithmetic in the *Tractatus*. Hand [10] also suggested something like this, but the connection is more immediate when we look at later accounts of finitism, in particular, by Bernays [2], where sequences of strokes as the primitive objects of finitary mathematics are abandoned in favor of iteratively generated *Wiederholungsfiguren* (of which the sequences of strokes are just one canonical representation). Of course, this by itself does not provide an account of the necessity and applicability of finitary arithmetic, but it does go a long way to defusing another problem Potter (and Parsons) find with Hilbert's finitism. This is the question of how much arithmetic can be justified using the epistemological underpinnings of finitism. If one takes spatio-temporal manipulation of stroke symbols as the basis, it is indeed not at all clear how one can obtain exponentiation as a finitary function. But once finite iteration is accepted as the basis, it is not hard to see that each primitive recursive function can be finitarily justified (this is what Tait [20] does).

This highlights the point I made earlier: there are several notions of finitary evidence at play in Hilbert's writings at various stages, and an assessment of Hilbert's claims has to take this into account. Potter seems to agree that in Hilbert's writings there are at least two different conceptions of finitism at work, a "narrow" one throughout the 1920s, and a "broad" conception from 1930 onward. He claims (p. 237) that there is nothing in Hilbert's writings before 1930 which would support attribution of the "broad" conception of finitism to him, according to which every primitive recursive function counts as finitary. If this claim is understood as a claim about what functions can be justified as finitary on the basis of Hilbert's epistemological remarks before 1930, this is perhaps defensible. But on the straightforward reading, according to which it means, "nothing in Hilbert's writings before 1930 supports attribution to Hilbert of the view that all primitive recursive functions fall under the finitary standpoint," this is surely false. Hilbert gave plenty of examples of—putatively—finitary functions and operations which go beyond addition of stroke symbols. It was certainly not just in 1930 that exponentiation was considered finitary! Indeed, I would urge one to understand "the finitary standpoint" as a body of restricted mathematical means, not as a philosophical doctrine about what can be justified on the basis of some epistemological foundation. For otherwise it is hard to understand what Hilbert and his collaborators were doing throughout the 1920s. It is then still an interesting question which epistemological position of those available and hinted at by Hilbert underwrites the finitary standpoint. The outcome may very well be that a particular position does not go far enough; another one may very well, though!

Potter's argument (p. 236–7) that on the "narrow" conception of finitism primitive recursion is not finitarily justified is interesting, but a little problematic. He concedes that multiplication is so justified, because it can be defined by primitive recursion using an operation which can be "schematically represented," namely, addition. Presumably, by this he means that we can make finitary sense of multiplication of $\mathfrak{a}$ by $\mathfrak{b}$ because $\mathfrak{a}\mathfrak{b}$ can be obtained by $\mathfrak{a}$-fold repeated addition of $\mathfrak{b}$. The difference with exponentiation, so Potter, is that to obtain $\mathfrak{a}^{\mathfrak{b}}$ one would have to repeat the operation of multiplication by $\mathfrak{a}$ $\mathfrak{b}$-many times. But from the preceding discussion of addition it seems that Potter takes addition itself to be defined as repeated application of the process of adding a single stroke. This would make multiplication itself nonfinitary. It seems to me that one either finds finite iteration of *any* procedure problematic (because, perhaps, a "procedure" is not a finite object) or one should accept finite iteration of a procedure which has already been seen to be finitary acceptable. Since this yields a new procedure which is thus seen to be finitarily acceptable, this can be iterated, and hence we have all primitive recursive functions as finitarily justified.

The second issue Potter discusses with regard to the finitary standpoint is that of the character of finitary "schematic" reasoning. This is another hairy issue, and Potter is quite right to point out that if one takes the finitary standpoint as applying only to "objects concretely given" (in representation) then it is difficult to see how one can even state any general claims. In the literature, this difficulty has been labeled—although by no means satisfactorily explained—by drawing a distinction between the unproblematic finitary claims which relate only to specific numbers, and the problematic claims which are schematic and general. It is the latter that are of course of primary importance to the proof-theoretic program, since the claim of consistency itself is such a general claim: for every $\mathfrak{a}$, $\mathfrak{a}$ is not the (code of) a proof of a contradiction. Potter's discussion goes some way to making plausible that such claims can be proved by a finitary rule of induction. Incidentally, in Tait's discussion, schematic proof and finitary operations are dealt with uniformly: both are treated as constructions from an arbitrary number (iteration).

3 Hilbert's Program and Gödel's Incompleteness Theorems

Potter concludes his exposition of Hilbert's program in §9.6 by describing the aims of the consistency program. In the literature, the aim of the program is stated either in the form Hilbert himself gave it—give a finitary proof that, for all sequences of formulas $\mathfrak{a}$, $\mathfrak{a}$ is not a proof of $0 = 1$ (simple consistency)—or in the form of a claim of conservativity of the ideal theory I over its real, finitary subtheory R for (problematic) real statements (a tradition initiated by Kreisel [14])—for all sequences of formulas $\mathfrak{a}$ and Π^1_0 sentences $(x)\varphi(x)$, if $\mathfrak{a}$ is a proof with end formula $(x)\varphi(x)$, then $(x)\varphi(x)$ is true (Π^0_1-soundness). Potter sets up things slightly differently, namely, by defining the property of *outer consistency* as 'Whenever a sentence of the form $(x)f(x) = 0$ (with f primitive recursive) is provable in the ideal theory I, then each instance $f(\mathfrak{a}) = 0$ is true'. Note that this must be understood as a schematic statement, that is, $\mathfrak{a}$ is a schematic variable. It would not do, for example, to have infinitely many separate proofs of this, one for each particular numeral $\mathfrak{a}$. So Potter's outer consistency is equivalent to the second of the above formulations, that is,

Π^0_1-soundness. The aim of Hilbert's program, then, is to prove the outer consistency of ideal mathematical theories I by finitary means. Potter gives a brief summary of the attempts made in the 1920s to achieve this aim. The interested reader will find a more detailed discussion in Zach [23].

Chapter 10, then, is devoted to the demise of Hilbert's program at the hands of Gödel. §10.1 gives a very condensed presentation of the first incompleteness theorem. In §10.2, Potter discusses Hilbert's 1930 version of the ω-rule (a complementary discussion can be found in Tait [21]). §10.3 is the central section. Here, he sketches a proof of the unprovability of "outer consistency," that is, the version of the second incompleteness theorem according to which, for any theory I extending **PRA**, I does not prove the outer consistency of I. This version is attributed to Gödel in a footnote to *Gödel 1932b* dating from 1966, and an unpublished remark *Gödel 1972a* (see his *Collected Works*, vols. 1 and 3).

I should say a little bit about Potter's formalization of the claim of outer consistency, since unfortunately his formulas are somewhat difficult to read. Instead of the usual proof *predicate*, Potter uses its primitive recursive characteristic function $\mathrm{prf}_I(x, z)$ which is (provably) $= 1$ if x is the Gödel number of a proof of the formula with Gödel number z and $= 0$ otherwise. For our purposes we may imagine that z ranges only over Gödel numbers of formulas $(w) f(w) = 0$ with f primitive recursive (i.e., Π^0_1 sentences). Furthermore, he makes use of a function $\mathrm{inst}(z, y)$ which takes the Gödel number z of a formula $(w) f(w) = 0$ and an argument y and returns $f(y)$. His formalization of outer consistency then is

$$(x)(y)(z)(\mathrm{prf}_I(x, z) = 0 \vee \mathrm{inst}(z, y) = 0). \tag{1}$$

To see what this says, turn the disjunction into a conditional and move the quantifier (x) into the antecedent. Then the above formula becomes

$$(y)(z)((\exists x)\mathrm{prf}_I(x, z) = 1 \supset \mathrm{inst}(z, y) = 0).$$

or, even more readably but less precisely,

$$(\ulcorner f \urcorner)(y)(\mathrm{prov}_I(\ulcorner (w) f(w) = 0 \urcorner) \supset f(y) = 0).$$

(Here, I abbreviate $(\exists x)\mathrm{prf}_I(x, z) = 1$ as $\mathrm{prov}_I(z)$.) The claim then is that (1) is not provable in I provided that I contains **PRA** and is consistent. The theorem and the proof, however, are not entirely correct. The problem is that the function inst defined above does not exist and hence the Lemma on p. 248 is false. Suppose there is a primitive recursive function $\mathrm{inst}(x, y)$ so that for every primitive recursive function f, **PRA** $\vdash$ $(y)(\mathrm{inst}(\ulcorner (w) f(w) = 0 \urcorner, y) = f(y))$ (this is the claim of the Lemma). Now consider the 1-place primitive recursive function $d(x) = \mathrm{inst}(x, x) + 1$ and let $\overline{e_d} = \ulcorner (w)d(w) = 0 \urcorner$. We have, on the one hand,

$$\mathbf{PRA} \vdash d(\overline{e_d}) = \mathrm{inst}(\overline{e_d}, \overline{e_d}) + 1$$

from the defining equations of d, and, on the other hand,

$$\mathbf{PRA} \vdash \mathrm{inst}(\underbrace{\ulcorner (w)d(w) = 0 \urcorner}_{\overline{e_d}}, \overline{e_d}) = d(\overline{e_d})$$

from the claim of the Lemma, and so **PRA** $\vdash d(\overline{e_d}) = d(\overline{e_d}) + 1$ and **PRA** would be inconsistent.

The error has one main consequence. Without inst, outer consistency cannot be formalized the way Potter does, and the technical formulations of §§10.3 and 10.5

must be corrected. There are two ways to do this. First, one could use the standard formalization of outer consistency as a schema. Indeed, Feferman in his introduction to *Gödel 1972a* in Gödel [8], p. 285, construes Gödel's "best and most general version of the unprovability of consistency in the same system" as the claim that an instance of the Π^1_0 reflection schema

$$(y)(\text{prov}_I(\ulcorner(w) f(w) = 0\urcorner) \supset f(y) = 0) \tag{2}$$

is unprovable. In terms of the discussion of the impact of Gödel's result on Hilbert's program, this formalization has the drawback that we are then dealing with infinitely many sentences, one for each primitive recursive function f. This makes it unclear to what extent the "outer consistency" of I can fairly be characterized as what's required to be proved in Hilbert's program, since it cannot even be finitely stated.

Another possibility would be to consider a version of a single-sentence outer consistency statement involving a truth definition Tr for primitive recursive formulas. Such a definition can be given by a Π^0_1 formula, and outer consistency could then be written down as the single sentence

$$(\ulcorner f\urcorner)(y)(\text{prov}_I(x, \ulcorner(w) f(w) = 0\urcorner) = 1 \supset \text{Tr}(\ulcorner f(\dot{y}) = 0\urcorner)).$$

This would not suffer from the difficulty noted above, but it would have to be verified whether the truth definition required satisfies the necessary properties (e.g., that already $\mathbf{PRA} \vdash \text{Tr}(\ulcorner f(\dot{y}) = 0\urcorner) \equiv f(y) = 0$).

The reader familiar with the discussion about the impact of Gödel's theorem on Hilbert's program will have noticed that outer consistency is, at least prima facie, a stronger requirement than simple consistency. As I remarked above, it is common in the literature on Gödel's results and Hilbert's program to argue that Hilbert's program in fact requires a proof of outer consistency or "real soundness." Depending on one's leanings on this issue, one might praise Potter for getting to the core of the issue and formulating consistency in the form that Hilbert really needed for his program—or feel that Potter is cheating. For, truth be told, there is little in Hilbert's writings that supports an attribution of the provability of outer consistency by finitary means as the aim of the proof-theoretic program. Potter does address this worry in §10.5 where he compares the unprovability of outer consistency to the unprovability of other versions of consistency and shows that under the assumption that prf_I is "well presented," the outer consistency statement for I is provable from its consistency statement. Well-presentedness here is Potter's criterion for when the provability function is well behaved (see p. 255 for the precise definition). His particular formalization of consistency and the proof of the theorem suffers from the same problem about inst discussed above. It would have to be verified that the proof of a corrected version of the formalization of outer consistency from a corrected version of the formalization of simple consistency goes through under only the condition that prf_I is well presented. I suspect that it does, since well-presentedness is essentially condition D3 in Feferman's introduction to *Gödel 1972a* (see p. 283 of [8]), which is the crucial condition needed. It remains true in any case that as long as prf_I and I satisfy some further derivability conditions in addition to the conditions needed for the first incompleteness theorem, Π^0_1 reflection is provable from Con_I.

The nice feature of the version of Gödel's second incompleteness theorem using outer consistency is that its unprovability does not depend on any derivability conditions, but only on the outer consistency of I itself. In fact, while Gödel himself stated outer consistency of I as a condition of his unprovability theorem, in Potter's version of the theorem the only requirements are that I is consistent and that it contains **PRA**. Let me give here a simple proof (which I owe to Torkel Franzén) of the unprovability of an instance of the reflection principle (2): Since I contains **PRA**, we can arithmetize syntax and get a Gödel sentence for I, that is, a sentence G_I of the form $(y)f(y) = 0$ such that $I \vdash G_I \equiv \neg\text{prov}_I(\ulcorner G_I \urcorner)$. Suppose that $I \vdash (y)(\text{prov}_I(\ulcorner (w)f(w) = 0 \urcorner) \supset f(y) = 0)$; that is, $I \vdash \text{prov}_I(\ulcorner G_I \urcorner) \supset G_I$. Then $I \vdash \text{prov}_I(\ulcorner G_I \urcorner) \supset \neg\text{prov}_I(\ulcorner G_I \urcorner)$ and thus $I \vdash \neg\text{prov}_I(\ulcorner G_I \urcorner)$. But then $I \vdash G_I$, whence $I \vdash \text{prov}_I(\ulcorner G_I \urcorner)$ since I contains **PRA**, and hence also $I \vdash \neg G_I$. Thus I would be inconsistent.

This is interesting because the usual proofs of the unprovability of reflection principles go either via Löb's theorem, or via the equivalence of Π^0_1 reflection for I with Con_I—and in both the derivability conditions are needed. Potter's point that the derivability conditions are not required for the unprovability of outer consistency is definitely an important one to make. Those who would like to revive Hilbert's program by considering formal systems of ideal mathematics which prove their own consistency (see especially Detlefsen [4, 5]) will have to take this point into account. This does not settle the issue, however. There are primitive recursive definitions of the characteristic function of the proof relation other than the canonical prf_I—such as Rosser provability—for which the statement expressing simple consistency of I is provable in I. The proof above shows that reflection formalized with such functions is still unprovable. However, it is then no longer the case that Π^0_1 reflection of I is provable from Con_I.

4 Generalized and Relativized Hilbert Programs

Potter concludes his discussion of Hilbert's program by a brief discussion of the extension of the finitary standpoint by Gentzen in which consistency is proved by transfinite induction and by some general considerations on the impact of Gödel's theorem on versions of formalism. As far as the former is concerned, I think Potter ably makes clear the difficulties one faces when attempting to justify induction along ordinal notations up to ε_0 or even just ω^ω. Potter argues that it is unclear how one can come to accept the well-foundedness of ordinal notations $< \omega^\omega$ or $< \varepsilon_0$ based on an intuitive representation (an infinite array) of all notations less than a given one. This leaves the possibility open, however, that induction on such ordinal notations can be justified in a different way. For instance, on Tait's view, induction up to any $\alpha < \omega^\omega$ can be justified even though there are infinitely many $\beta < \alpha$ for $\alpha \geq \omega$. In line with my previous comment on the usefulness of paying attention to sources other than Hilbert's own programmatic papers from the 1920s, I would like to point out that already Ackermann's 1924 dissertation used such prima facie infinitary methods to prove the consistency (of an extension of **PRA**), and both Ackermann and Hilbert then considered this not to go beyond the finitary standpoint. This, of course, does not mean that they were justified in doing so, but it highlights how difficult it is to pin down Hilbert's views in the 1920s.

The conclusion Potter reaches about the importance of Gödel's theorem in settings other than Hilbert's own should also be relativized somewhat. In §10.6, he considers several interesting scenarios in which the real theory R in which the proof of consistency (or conservativity) of an ideal theory I is to be carried out are not Hilbert's pair

(R = finitary mathematics, I = first-order arithmetic), but some of the theories considered in earlier chapters. For instance, one might wonder if it is possible to prove in the system of *Principia* without the problematic axioms of infinity and reducibility (R) the consistency of the entire theory with these axioms included, or the case where R is second-order logic and I also contains Frege's numerical equivalence. I do not know the answers to these questions (although probably at least the latter has been investigated). Potter states (on p. 260) that Gödel's theorems show that the provability function cannot be "well presented" if the conservativity of I over R can be proved in R itself. Potter's claim is true insofar as Gödel's theorems show that (under suitable conditions) R doesn't prove the conservativity of I over R in the sense that $R \nvdash \mathrm{prov}_I(\varphi) \supset \varphi$ (even for simple φ, e.g., $0 = 1$). But R may very well prove the following version of conservativity of I over R: $R \vdash \mathrm{prov}_I(\varphi) \supset \mathrm{prov}_R(\varphi)$ (for suitable choices of classes of φs). Conservativity in this sense is often all that is needed to obtain philosophical payoff, if it can be argued that $\mathrm{prov}_R(A)$ is in fact a good presentation of provability inside R (in a sense in which it is justified to consider R to be "able to talk about provability in R"). In fact, these sorts of results are the entire point of so-called relativized Hilbert programs, which have been an important and lively area of research in proof theory. In relativized Hilbert programs, one is also interested in pairs of theories R, I where I is some ways stronger than R. For instance, R might be a first-order theory of arithmetic, and I an extension by second-order principles and set existence assumptions. Or R might be the weak subtheory of **PRA** known as **EA** which contains only addition, multiplication, and exponentiation, but no essentially more complex primitive recursive functions. Philosophically acute surveys of work in this regard can be found in Feferman [7] and Avigad [1].

5 Conclusion

Let me conclude on a positive note: *Reason's Nearest Kin* is a valuable contribution to the literature on the philosophy and foundations of arithmetic. It is well written and informative, and I enjoyed reading it enormously. Where Potter's discussion is unconventional in his choice of topics or exposition, it is refreshingly unconventional. The reader will find much that is new, insightful, and stimulating. I don't think it is the definitive treatment of the topics it covers, but it is significant, and future writers will have to take up the issues Potter has raised.

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