

Categorical Abstract Algebraic Logic: Models of π -Institutions

George Voutsadakis

Abstract An important part of the theory of algebraizable sentential logics consists of studying the algebraic semantics of these logics. As developed by Czelakowski, Blok, and Pigozzi and Font and Jansana, among others, it includes studying the properties of logical matrices serving as models of deductive systems and the properties of abstract logics serving as models of sentential logics. The present paper contributes to the development of the categorical theory by abstracting some of these model theoretic aspects and results from the level of sentential logics to the level of π -institutions.

1 Introduction

In [3], Blok and Pigozzi introduced the Leibniz congruence corresponding to a theory of a given deductive system. The map sending a theory to its Leibniz congruence is now known as the Leibniz operator of the deductive system. The Leibniz operator may be applied, more generally, to any algebra and maps a filter on the algebra to the largest congruence that is compatible with the filter, that is, the largest congruence such that the filter is a union of some of its equivalence classes. If a deductive system is such that the Leibniz operator is bijective and commutes with substitutions in a precise technical sense, then it is algebraizable, that is, it has an equivalent algebraic semantics. In this case, various metalogical properties of the deductive system are reflected in corresponding algebraic properties of its equivalent algebraic semantics. Other weaker properties of the Leibniz operator give rise to weaker ties between a deductive system and classes of algebras over the same algebraic signature. Thus, the well-known hierarchy of deductive systems consisting mainly of the

Received April 14, 2004; accepted October 6, 2004; printed December 8, 2005
2000 Mathematics Subject Classification: Primary, 03Gxx; Secondary, 18Axx, 68N05
Keywords: abstract algebraic logic, deductive systems, institutions, equivalent deductive systems, algebraizable deductive systems, adjunctions, equivalent institutions, algebraizable institutions, Leibniz congruence, Tarski congruence, algebraizable sentential logics

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protoalgebraic (Blok and Pigozzi [2]), the equivalential (Czelakowski [6], [7]), and the algebraizable deductive systems (Blok and Pigozzi [3]) arises. In Czelakowski [8] an excellent account of the main results in this area and in Font et al. [10] an excellent brief overview are provided. The protoalgebraic logics are at the lowest end of the algebraic hierarchy and are quite well understood. However, many of the non-protoalgebraic logics have also naturally associated classes of algebras with them, which are not the algebras that the traditional theory of deductive systems predicts. This is, in part, due to the fact that the ties between the metalogical properties and the corresponding algebraic properties at this level are rather weak.

With the motivation to study the classes of algebras associated with nonprotoalgebraic logics, Font and Jansana [11] replaced logical matrices, which had been used as models of deductive systems in the classical theory, with abstract logics. Abstract logics consist of an algebra over the same signature as that of the deductive system and of a closure operator on its universe, rather than just an algebra and a filter on the algebra. The Leibniz operator, mapping filters to congruences, is now replaced by the Tarski operator, which maps closed set systems to congruences. Font and Jansana were able to develop the theory of abstract logics and, using a variety of results pertaining to the use of abstract logics as models for deductive systems, extended the theory of algebraizability to a scope beyond the levels of the classical hierarchy. Their work had a second significant byproduct. It contains some of the missing elements for extending the theory of algebraizability, that is based on the Leibniz operator, from the deductive system framework to the π -institution framework.

In Voutsadakis [17], [21], and [20], a categorical theory of algebraizability was developed with the goal of extending the results obtained for deductive systems to the framework of π -institutions. The notion of an algebraic and of an algebraizable π -institution were defined based on their corresponding counterparts for deductive systems. This theory has proven useful in providing more natural alternatives to the classical algebraizations of some specific important multisignature logics, namely, equational (Voutsadakis [22], see also Voutsadakis [23]) and first-order logic (Voutsadakis [25], see also Voutsadakis [24]). It has been lacking in terms of providing analogs of the most fundamental and powerful results of the classical theory to this more abstract level. An attempt to develop the theory toward this direction, starting from an even more abstract definition of the Tarski operator, suitable for the π -institution framework, was begun in Voutsadakis [19]. In Voutsadakis [18] the limits of the generality of that theory were tested. This paper is viewed as a sequel to [19] and quotes many of the definitions and results of [19]. However, a brief review of the basics of the theory needed will be given in the following section so as to make the present paper as self-contained as possible.

Some of the references to different areas related to this paper are summarized here for the reader's convenience. For the very general background needed from category theory, the reader is referred to any of Barr and Wells [1], Borceux [4], or MacLane [16]. A second level of references, also quite general, consists of background references pertaining to institutions and π -institutions and comes mainly from the world of theoretical computer science. Apart from the fundamental papers Goguen and Burstall [12], Goguen and Burstall [15], and Fiadeiro and Sernadas [9], the reader may also consult Cerioli and Meseguer [5], Goguen and Diaconescu [13],

and Goguen and Rosu [14] and the references therein for some more recent developments. Focusing now on the abstract algebraic logic area, the reader is encouraged to study the exposition [10] and the book [8]. For introductions to categorical abstract algebraic logic and its reason for being, see [17], [20], and [21]. Finally, narrowing down to the material that has been the main inspiration for and has significantly influenced the present work, the reader is encouraged to study the works of Blok and Pigozzi [3], Font and Jansana [11], and the preceding works [19] and [18] by the author. As mentioned before, [19] has been summarized in the next section to give a degree of self-sufficiency to the present writing.

2 π -Institutions, Translations, Equivalence Systems, and Quotients

This section summarizes background material that is needed for the theory developed in subsequent sections. Recall that a π -institution [9] $\mathcal{I} = \langle \mathbf{Sign}, \mathbf{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ is a triple consisting of

- (i) a category \mathbf{Sign} , whose objects are called *signatures* and whose morphisms are called *assignments*;
- (ii) a functor $\mathbf{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ from the category of signatures to the category of small sets giving, for each $\Sigma \in |\mathbf{Sign}|$, the set of Σ -sentences $\mathbf{SEN}(\Sigma)$ and mapping an assignment $f : \Sigma_1 \rightarrow \Sigma_2$ to a *substitution* $\mathbf{SEN}(f) : \mathbf{SEN}(\Sigma_1) \rightarrow \mathbf{SEN}(\Sigma_2)$;
- (iii) a mapping $C_\Sigma : \mathcal{P}(\mathbf{SEN}(\Sigma)) \rightarrow \mathcal{P}(\mathbf{SEN}(\Sigma))$, for each $\Sigma \in |\mathbf{Sign}|$, called Σ -closure, such that
 - (a) $A \subseteq C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|$, $A \subseteq \mathbf{SEN}(\Sigma)$,
 - (b) $C_\Sigma(C_\Sigma(A)) = C_\Sigma(A)$, for all $\Sigma \in |\mathbf{Sign}|$, $A \subseteq \mathbf{SEN}(\Sigma)$,
 - (c) $C_\Sigma(A) \subseteq C_\Sigma(B)$, for all $\Sigma \in |\mathbf{Sign}|$, $A \subseteq B \subseteq \mathbf{SEN}(\Sigma)$,
 - (d) $\mathbf{SEN}(f)(C_{\Sigma_1}(A)) \subseteq C_{\Sigma_2}(\mathbf{SEN}(f)(A))$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $A \subseteq \mathbf{SEN}(\Sigma_1)$.

A family $\{C_\Sigma : \mathcal{P}(\mathbf{SEN}(\Sigma)) \rightarrow \mathcal{P}(\mathbf{SEN}(\Sigma))\}_{\Sigma \in |\mathbf{Sign}|}$ will be referred to as a *closure system* on $\mathbf{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ if it satisfies (iii)(a)–(d) above. For two closure systems C, C' on $\mathbf{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$, we write $C \leq C'$ to denote that $C_\Sigma(\Phi) \subseteq C'_\Sigma(\Phi)$, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathbf{SEN}(\Sigma)$.

Given two π -institutions $\mathcal{I}, \mathcal{I}'$, a *translation* [21] $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ consists of a functor $F : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ together with a natural transformation $\alpha : \mathbf{SEN} \rightarrow \mathcal{P}\mathbf{SEN}'F$. A translation depends only on the categories of signatures and on the sentence functors and not on the closure systems. If a closure system is not present, the above translation may be denoted by $\langle F, \alpha \rangle : \mathbf{SEN} \rightarrow \mathbf{SEN}'$ or, if only one of the two closure systems is present, it will be written $\langle F, \alpha \rangle : \mathbf{SEN} \rightarrow \mathcal{I}'$ or $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathbf{SEN}'$, accordingly. A translation is said to be a *singleton translation* [19], written $\langle F, \alpha \rangle : \mathcal{I} \rightarrow^s \mathcal{I}'$, if, for every $\Sigma \in |\mathbf{Sign}|$, $\varphi \in \mathbf{SEN}(\Sigma)$, $|\alpha_\Sigma(\varphi)| = 1$. A singleton translation is said to be *surjective* if

1. $F : \mathbf{Sign} \rightarrow \mathbf{Sign}'$ is surjective and
2. $\alpha_\Sigma : \mathbf{SEN}(\Sigma) \rightarrow \mathbf{SEN}'(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}|$.

A translation $\langle F, \alpha \rangle : \mathcal{L} \rightarrow \mathcal{L}'$ is a *semi-interpretation*, written $\langle F, \alpha \rangle : \mathcal{L} \text{--} \mathcal{L}'$, if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$,

$$\varphi \in C_{\Sigma}(\Phi) \quad \text{implies} \quad \alpha_{\Sigma}(\varphi) \subseteq C'_{F(\Sigma)}(\alpha_{\Sigma}(\Phi)).$$

It is an *interpretation*, denoted $\langle F, \alpha \rangle : \mathcal{L} \vdash \mathcal{L}'$, if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$,

$$\varphi \in C_{\Sigma}(\Phi) \quad \text{iff} \quad \alpha_{\Sigma}(\varphi) \subseteq C'_{F(\Sigma)}(\alpha_{\Sigma}(\Phi)).$$

Let \mathbf{Sign} be a category and $\text{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ a functor. The *clone of all natural transformations on SEN* is the locally small category with collection of objects $\{\text{SEN}^{\alpha} : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \text{SEN}^{\alpha} \rightarrow \text{SEN}^{\beta}$ β -sequences of natural transformations $\tau : \text{SEN}^{\alpha} \rightarrow \text{SEN}$. Composition

$$\text{SEN}^{\alpha} \xrightarrow{\langle \tau_i : i < \beta \rangle} \text{SEN}^{\beta} \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \text{SEN}^{\gamma}$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory of this category containing *all* objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \text{SEN}^k \rightarrow \text{SEN}$, $i < k$, $k < \omega$, with $p_{\Sigma}^{k,i} : \text{SEN}(\Sigma)^k \rightarrow \text{SEN}(\Sigma)$ given by

$$p_{\Sigma}^{k,i}(\vec{\varphi}) = \varphi_i, \quad \text{for all} \quad \vec{\varphi} \in \text{SEN}(\Sigma)^k,$$

will be referred to as a *category of natural transformations on SEN*.

A few remarks are in order here concerning this definition. First, it is emphasized that a category of natural transformations has as its objects all finite powers SEN^k , $k < \omega$. This definition intends to capture, in the categorical framework, the algebraic structure underlying matrices, abstract logics, and sentential logics in the universal algebraic framework. It is inspired by a similar construction of the algebraic clone of operations associated with an algebraic theory in monoid form in the category of sets. See, for example, [16]. Unlike in the case of algebraic theories, the construction is applied here to the arbitrary functor SEN rather than to a free algebra functor of an algebraic theory in set.

Let \mathbf{Sign} be a category, $\text{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ be a functor, and N be a category of natural transformations on SEN . Given $\Sigma \in |\mathbf{Sign}|$, an equivalence relation θ_{Σ} on $\text{SEN}(\Sigma)$ is said to be an N -congruence if, for all $\sigma : \text{SEN}^k \rightarrow \text{SEN}$ in N and all $\vec{\varphi}, \vec{\psi} \in \text{SEN}(\Sigma)^k$,

$$\vec{\varphi} \theta_{\Sigma}^k \vec{\psi} \quad \text{imply} \quad \sigma_{\Sigma}(\vec{\varphi}) \theta_{\Sigma} \sigma_{\Sigma}(\vec{\psi}).$$

A collection $\theta = \{(\Sigma, \theta_{\Sigma}) : \Sigma \in |\mathbf{Sign}|\}$ is called a (*first-order*) *equivalence system* of SEN if

1. θ_{Σ} is an equivalence relation on $\text{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$,
2. $\text{SEN}(f)^2(\theta_{\Sigma_1}) \subseteq \theta_{\Sigma_2}$, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$.

If, in addition, N is a category of natural transformations on SEN and θ_{Σ} is an N -congruence, for all $\Sigma \in |\mathbf{Sign}|$, then θ is said to be a (*first-order*) N -congruence system of SEN .

Let now $\mathcal{I} = \langle \mathbf{Sign}, \text{SEN}, \{C_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ be a π -institution. An equivalence system θ of SEN is called a *logical equivalence system* of \mathcal{I} if, for all $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \text{SEN}(\Sigma)$,

$$\langle \varphi, \psi \rangle \in \theta_{\Sigma} \quad \text{implies} \quad C_{\Sigma}(\varphi) = C_{\Sigma}(\psi).$$

An N -congruence system of SEN is a *logical N -congruence system* of \mathcal{I} if it is logical as an equivalence system of \mathcal{I} .

It is proven in [19] that the collection of all logical N -congruence systems of a given π -institution \mathcal{I} forms a complete lattice under signature-wise inclusion and the largest element of the lattice is termed the *Tarski N -congruence system* of \mathcal{I} and denoted by $\tilde{\Omega}^N(\mathcal{I})$. Theorem 4 of [19] fully characterizes the Tarski N -congruence system of a π -institution.

Given two π -institutions $\mathcal{I} = \langle \mathbf{Sign}, \text{SEN}, C \rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \text{SEN}', C' \rangle$ and categories of natural transformations N, N' , respectively, on SEN, SEN' , a singleton translation (semi-interpretation or interpretation) $\langle F, \alpha \rangle$ from \mathcal{I} to \mathcal{I}' is said to be (N, N') -*homomorphic* if, for every natural transformation $\tau : \text{SEN}^k \rightarrow \text{SEN}$ in N , there exists a natural transformation $\sigma : \text{SEN}'^k \rightarrow \text{SEN}'$ in N' such that, for every $\Sigma \in |\mathbf{Sign}|$ and every $\vec{\varphi} \in \text{SEN}(\Sigma)^k$,

$$\begin{array}{ccc} \text{SEN}(\Sigma)^k & \xrightarrow{\alpha_\Sigma^k} & \text{SEN}'(F(\Sigma))^k \\ \tau_\Sigma \downarrow & & \downarrow \sigma_{F(\Sigma)} \\ \text{SEN}(\Sigma) & \xrightarrow{\alpha_\Sigma} & \text{SEN}'(F(\Sigma)) \end{array}$$

$$\alpha_\Sigma(\tau_\Sigma(\vec{\varphi})) = \sigma_{F(\Sigma)}(\alpha_\Sigma^k(\vec{\varphi})). \quad (1)$$

It is said to be (N, N') -*epimorphic* if it is (N, N') -homomorphic and, in addition, for every $\sigma : \text{SEN}'^k \rightarrow \text{SEN}'$ in N' , there exists $\tau : \text{SEN}^k \rightarrow \text{SEN}$ in N such that equation 1 holds, for all $\Sigma \in |\mathbf{Sign}|$, $\vec{\varphi} \in \text{SEN}(\Sigma)^k$. We denote homomorphic by the superscript h and epimorphic by the superscript e , respectively, assuming that the categories N and N' are clear from context.

An (N, N') -*logical morphism* $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ from \mathcal{I} to \mathcal{I}' is a singleton (N, N') -epimorphic semi-interpretation from \mathcal{I} to \mathcal{I}' , denoted accordingly by $\langle F, \alpha \rangle : \mathcal{I} \xrightarrow{\text{se}} \mathcal{I}'$. A logical morphism is *strong* if it is an interpretation, denoted $\langle F, \alpha \rangle : \mathcal{I} \vdash^{\text{se}} \mathcal{I}'$. Finally, it is called an (N, N') -*bilogical morphism* if it is a surjective strong (N, N') -logical morphism.

After having introduced the analogs of the logical and the bilogical morphisms of Font and Jansana [11] for π -institutions, a few clarifying remarks are in order. In previous work by the author on the categorical theory ([17], [21], and [20]), condition (d) in the definition of a closure system played the role of structurality, that is, invariance under substitutions. The signature morphisms were perceived as being the assignments with the induced sentence morphisms as being the corresponding substitutions. In the present framework, where π -institutions assume the role of abstract logics of the sentential logic framework, the role of endomorphisms or substitutions is played by the morphisms of the category N of natural transformations on SEN . So, in a certain sense, since the closure system and the category N are not necessarily related in any way, the assumption of the substitution role by the morphisms in N lifts the structurality condition from the framework and brings it closer to that of abstract logics, which are not required to be structural.

Given a π -institution $\mathcal{I} = \langle \mathbf{Sign}, \text{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ and a logical equivalence system θ of \mathcal{I} , the triple $\mathcal{I}/\theta = \langle \mathbf{Sign}, \text{SEN}^\theta, \{C_\Sigma^\theta\}_{\Sigma \in |\mathbf{Sign}|} \rangle$, where

1. $\text{SEN}^\theta : \mathbf{Sign} \rightarrow \mathbf{Set}$ is defined by $\text{SEN}^\theta(\Sigma) = \text{SEN}(\Sigma)/\theta_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$, and, given $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$ $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\varphi \in \text{SEN}(\Sigma_1)$

$$\text{SEN}^\theta(f)(\varphi/\theta_{\Sigma_1}) = \text{SEN}(f)(\varphi)/\theta_{\Sigma_2};$$

2. for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\psi\} \subseteq \text{SEN}(\Sigma)$,

$$\psi/\theta_\Sigma \in C_\Sigma^\theta(\Phi/\theta_\Sigma) \quad \text{iff} \quad \psi/\theta_\Sigma \subseteq C_\Sigma(\bigcup_{\varphi \in \Phi} \varphi/\theta_\Sigma),$$

is also a π -institution (see [19]), called the *quotient* of \mathcal{L} by θ and denoted by \mathcal{L}/θ . Furthermore, there exists a surjective singleton interpretation $\langle \mathbf{ISign}, \pi^\theta \rangle : \mathcal{L} \rightarrow \mathcal{L}/\theta$ called the *canonical projection* from \mathcal{L} onto \mathcal{L}/θ . If the logical equivalence system θ is a logical N -congruence system of \mathcal{L} , then the canonical projection is also an (N, N^θ) -biological morphism, where N^θ is the category of natural transformations on SEN^θ inherited by N , that is such that $N^\theta = \{\sigma^\theta : \sigma \in N\}$, where

$$\sigma_\Sigma^\theta(\vec{\varphi}/\theta_\Sigma) = \sigma_\Sigma(\vec{\varphi})/\theta_\Sigma, \quad \text{for all } \vec{\varphi} \in \text{SEN}(\Sigma)^k.$$

The quotient $\mathcal{L}/\tilde{\Omega}^N(\mathcal{L})$ is said to be the N -*reduct* of \mathcal{L} and is denoted by \mathcal{L}^N . The notation \mathcal{L}^N replaces \mathcal{L}^* , which would have been inherited by the notation for abstract logics (see [11]), so as to make the dependence on N in the current context transparent.

3 Closure System Generation

In this section it is shown how a collection of translations from a given sentence functor to various π -institutions may be used to endow the sentence functor with a closure system, that is, to give rise to a π -institution. This construction corresponds in this framework to the well-known way of defining a logic on the algebra of formulas by using the closure system of a given abstract logic (in that case over the same algebraic signature).

Let \mathbf{Sign} be a category, $\text{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ a functor, and $\mathcal{L}^i = \langle \mathbf{Sign}^i, \text{SEN}^i, \{C_\Sigma^i\}_{\Sigma \in |\mathbf{Sign}^i|} \rangle$, $i \in I$, a collection of π -institutions and $\mathcal{F} = \{\langle F^i, \alpha^i \rangle : i \in I\}$ a collection of translations $\langle F^i, \alpha^i \rangle : \text{SEN} \rightarrow \mathcal{L}^i$, $i \in I$. Define $C^\mathcal{F} = \{C_\Sigma^\mathcal{F} : \Sigma \in |\mathbf{Sign}|\}$, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$, by

$$\varphi \in C_\Sigma^\mathcal{F}(\Phi) \quad \text{iff,} \quad \text{for all } i \in I, \alpha_\Sigma^i(\varphi) \subseteq C_{F^i(\Sigma)}^i(\alpha_\Sigma^i(\Phi)).$$

Note that this condition is equivalent to the condition

$$\varphi \in C_\Sigma^\mathcal{F}(\Phi) \quad \text{iff,} \quad \text{for all } i \in I, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \\ \text{SEN}^i(F^i(f))(\alpha_\Sigma^i(\varphi)) \subseteq C_{F^i(\Sigma')}^i(\text{SEN}^i(F^i(f))(\alpha_\Sigma^i(\Phi))).$$

In the sequel, when the system $\mathcal{F} = \{\langle F, \alpha \rangle\}$ consists of a single translation $\langle F, \alpha \rangle : \text{SEN} \rightarrow \mathcal{L}$, $C^\mathcal{F}$ will be denoted simply by $C^{(F, \alpha)}$.

Proposition 3.1 *The system $C^\mathcal{F}$ is a closure system on SEN .*

Proof Since $C_\Sigma^\mathcal{F} = \bigcap_{i \in I} C_\Sigma^{(F^i, \alpha^i)}$, for all $\Sigma \in |\mathbf{Sign}|$, and the collection of closure systems is closed under intersections, it suffices to show that, for all $i \in I$, $C^{(F^i, \alpha^i)}$ is a closure system on SEN . Properties (a) and (c) of a closure system are straightforward, so only (b) and (d) will be presented in detail.

For (b), suppose that $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$ such that $\varphi \in C_{\Sigma}^{\langle F^i, \alpha^i \rangle}(C_{\Sigma}^{\langle F^i, \alpha^i \rangle}(\Phi))$. Then, by the definition of $C^{\langle F^i, \alpha^i \rangle}$, we get

$$\alpha_{\Sigma}^i(\varphi) \subseteq C_{F^i(\Sigma)}^i(\alpha_{\Sigma}^i(C_{\Sigma}^{\langle F^i, \alpha^i \rangle}(\Phi))) \quad (2)$$

and

$$\alpha_{\Sigma}^i(C_{\Sigma}^{\langle F^i, \alpha^i \rangle}(\Phi)) \subseteq C_{F^i(\Sigma)}^i(\alpha_{\Sigma}^i(\Phi)), \quad (3)$$

for all $\Sigma \in |\mathbf{Sign}|$. Combining (2) and (3) and exploiting property (b) of C^i yields $\alpha_{\Sigma}^i(\varphi) \subseteq C_{F^i(\Sigma)}^i(\alpha_{\Sigma}^i(\Phi))$, which proves property (b) for $C^{\langle F^i, \alpha^i \rangle}$.

For (d), suppose $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $\Phi \subseteq \text{SEN}(\Sigma)$. If $\varphi \in C_{\Sigma}^{\langle F^i, \alpha^i \rangle}(\Phi)$, then $\alpha_{\Sigma}^i(\varphi) \subseteq C_{F^i(\Sigma)}^i(\alpha_{\Sigma}^i(\Phi))$, whence

$$\text{SEN}^i(F^i(f))(\alpha_{\Sigma}^i(\varphi)) \subseteq \text{SEN}^i(F^i(f))(C_{F^i(\Sigma)}^i(\alpha_{\Sigma}^i(\Phi))).$$

Therefore, since α is a natural transformation and C^i is a closure system, we get

$$\begin{array}{ccc} \text{SEN}(\Sigma) & \xrightarrow{\alpha_{\Sigma}^i} & \text{SEN}^i(F^i(\Sigma)) \\ \text{SEN}(f) \downarrow & & \downarrow \text{SEN}^i(F^i(f)) \\ \text{SEN}(\Sigma') & \xrightarrow{\alpha_{\Sigma'}^i} & \text{SEN}^i(F^i(\Sigma')) \end{array}$$

$\alpha_{\Sigma'}^i(\text{SEN}(f)(\varphi)) \subseteq C_{F^i(\Sigma')}^i(\text{SEN}^i(F^i(f))(\alpha_{\Sigma}^i(\Phi)))$. Hence, again by the naturality of α , $\alpha_{\Sigma'}^i(\text{SEN}(f)(\varphi)) \subseteq C_{F^i(\Sigma')}^i(\alpha_{\Sigma'}^i(\text{SEN}(f)(\Phi)))$. But this is exactly equivalent to $\text{SEN}(f)(\varphi) \in C_{\Sigma'}^{\langle F^i, \alpha^i \rangle}(\text{SEN}(f)(\Phi))$, which proves property (d). \square

By Proposition 3.1, $\langle \mathbf{Sign}, \text{SEN}, \{C_{\Sigma}^{\mathcal{F}}\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ is a π -institution. It will be called the π -institution generated by \mathcal{F} and denoted by $\mathcal{I}^{\mathcal{F}}$. \mathcal{I} will be said to be generated by \mathcal{I}' or to be \mathcal{I}' -generated if $\mathcal{I} = \mathcal{I}^{\langle F, \alpha \rangle}$ for some $\langle F, \alpha \rangle : \text{SEN} \rightarrow \mathcal{I}'$.

Next, it is shown that all translations $\langle F^i, \alpha^i \rangle : \text{SEN} \rightarrow \mathcal{I}^i$ in \mathcal{F} that generate a π -institution $\mathcal{I}^{\mathcal{F}}$ become semi-interpretations $\langle F^i, \alpha^i \rangle : \mathcal{I}^{\mathcal{F}} \dashv \mathcal{I}^i$.

Proposition 3.2 *Suppose $\mathcal{F} = \{\langle F^i, \alpha^i \rangle : \text{SEN} \rightarrow \mathcal{I}^i, i \in I\}$ is a collection of translations. For all $i \in I$, $\langle F^i, \alpha^i \rangle : \mathcal{I}^{\mathcal{F}} \dashv \mathcal{I}^i$ is a semi-interpretation. Moreover, $\langle F^i, \alpha^i \rangle : \mathcal{I}^{\langle F^i, \alpha^i \rangle} \vdash \mathcal{I}^i$ is an interpretation.*

Proof Let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$. Then we have $\varphi \in C_{\Sigma}^{\mathcal{F}}(\Phi)$ if and only if $\alpha_{\Sigma}^i(\varphi) \subseteq C_{F^i(\Sigma)}^i(\alpha_{\Sigma}^i(\Phi))$, for all $i \in I$. Therefore $\langle F^i, \alpha^i \rangle : \mathcal{I}^{\mathcal{F}} \dashv \mathcal{I}^i$ is a semi-interpretation, for all $i \in I$.

Now fix $i \in I$. Let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$. Suppose that $\alpha_{\Sigma}^i(\varphi) \subseteq C_{F^i(\Sigma)}^i(\alpha_{\Sigma}^i(\Phi))$. Then, by the definition of $C^{\langle F^i, \alpha^i \rangle}$, we get that $\varphi \in C_{\Sigma}^{\langle F^i, \alpha^i \rangle}(\Phi)$, that is, $\langle F^i, \alpha^i \rangle : \mathcal{I}^{\langle F^i, \alpha^i \rangle} \vdash \mathcal{I}^i$ is an interpretation. \square

Next, the way a given generation is affected when composing with a semi-interpretation is studied. The first proposition compares the π -institution generated by a given translation, on the one hand, and by the composite of that translation with a given semi-interpretation, on the other.

Proposition 3.3 Suppose that **Sign** is a category and $\text{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$ a functor. Let $\mathcal{I}' = \langle \mathbf{Sign}', \text{SEN}', \{C'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|} \rangle$ and $\mathcal{I}'' = \langle \mathbf{Sign}'', \text{SEN}'', \{C''_\Sigma\}_{\Sigma \in |\mathbf{Sign}''|} \rangle$ be two π -institutions and $\langle K, \kappa \rangle : \text{SEN} \rightarrow \mathcal{I}'$ a translation.

1. If $\langle F, \alpha \rangle : \mathcal{I}' \dashv \mathcal{I}''$ is a semi-interpretation, then $C^{\langle K, \kappa \rangle} \leq C^{\langle FK, \alpha_K \kappa \rangle}$.
2. If $\langle F, \alpha \rangle : \mathcal{I}' \vdash \mathcal{I}''$ is an interpretation, then $C^{\langle K, \kappa \rangle} = C^{\langle FK, \alpha_K \kappa \rangle}$.

$$\text{SEN} \xrightarrow{\langle K, \kappa \rangle} \mathcal{I}' \xrightarrow{\langle F, \alpha \rangle} \mathcal{I}''$$

Proof Suppose that $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$. Then, if $\varphi \in C^{\langle K, \kappa \rangle}_\Sigma(\Phi)$, we get $\kappa_\Sigma(\varphi) \subseteq C'_{K(\Sigma)}(\kappa_\Sigma(\Phi))$. Therefore, since $\langle F, \alpha \rangle : \mathcal{I}' \dashv \mathcal{I}''$,

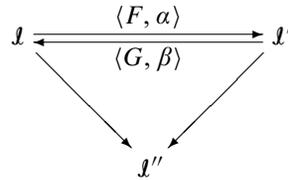
$$\alpha_{K(\Sigma')}(\kappa_\Sigma(\varphi)) \subseteq C''_{F(K(\Sigma))}(\alpha_{K(\Sigma)}(\kappa_\Sigma(\Phi))).$$

Hence, by the definition of $C^{\langle FK, \alpha_K \kappa \rangle}$, $\varphi \in C^{\langle FK, \alpha_K \kappa \rangle}_\Sigma(\Phi)$ and, therefore, $C^{\langle K, \kappa \rangle} \leq C^{\langle FK, \alpha_K \kappa \rangle}$. Note that, if $\langle F, \alpha \rangle : \mathcal{I}' \vdash \mathcal{I}''$ is an interpretation, then every step in the proof above is reversible. \square

The following two corollaries result by combining Propositions 3.2 and 3.3. They deal with the case of two π -institutions \mathcal{I} and \mathcal{I}' that are mutually interpretable in each other. The first says that, in the case there exist mutual translations between \mathcal{I} and \mathcal{I}' , \mathcal{I} is generated by a third π -institution \mathcal{I}'' if and only if \mathcal{I}' is. The second says that, in case the translations are interpretations, a third π -institution \mathcal{I}'' is \mathcal{I} -generated if and only if it is \mathcal{I}' -generated. Therefore, roughly speaking, mutually interpretable π -institutions have the same generating power.

Corollary 3.4 Suppose that there exist mutual translations $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ and $\langle G, \beta \rangle : \mathcal{I}' \rightarrow \mathcal{I}$ between two given π -institutions \mathcal{I} and \mathcal{I}' . Then \mathcal{I} is \mathcal{I}'' -generated if and only if \mathcal{I}' is \mathcal{I}'' -generated.

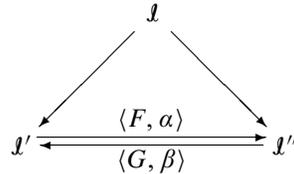
Proof One has to follow the interpretation paths in the diagram



taking account of Propositions 3.2 and 3.3. \square

Corollary 3.5 Suppose that $\mathcal{I}', \mathcal{I}''$ are mutually interpretable in each other via $\langle F, \alpha \rangle : \mathcal{I}' \vdash \mathcal{I}''$ and $\langle G, \beta \rangle : \mathcal{I}'' \vdash \mathcal{I}'$. Then a π -institution \mathcal{I} is \mathcal{I}' -generated if and only if it is \mathcal{I}'' -generated.

Proof One has to follow the interpretation paths in the diagram



again taking account of Propositions 3.2 and 3.3. \square

Proposition 3.3 also has the following corollary that expresses formally the fact that the closure generated by a given translation is the same as the closure generated by the translation followed by a canonical projection interpretation onto a logical quotient.

Corollary 3.6 *Let \mathcal{I}' be a π -institution and θ a logical equivalence system of \mathcal{I}' . If $\langle K, \kappa \rangle : \text{SEN} \rightarrow \mathcal{I}'$ is a translation, then $\mathcal{I}^{\langle K, \kappa \rangle} = \mathcal{I}^{\langle K, \pi_K^\theta \kappa \rangle}$.*

$$\text{SEN} \xrightarrow{\langle K, \kappa \rangle} \mathcal{I}' \xrightarrow{\langle \text{I}_{\text{Sign}'}, \pi^\theta \rangle} \mathcal{I}'/\theta$$

Proof By Proposition 23 of [19], the quotient interpretation $\langle \text{I}_{\text{Sign}'}, \pi^\theta \rangle : \mathcal{I}' \rightarrow \mathcal{I}'/\theta$ is a surjective singleton interpretation from \mathcal{I}' to \mathcal{I}'/θ . Combining this with Proposition 3.3 yields the result. \square

The following lemma is a reflection of the associativity of translation composition in this context. It expresses the fact that the closure generated by the composite of two translations is the same as that generated by the first translation applied to the closure generated by the second translation.

Lemma 3.7 *Suppose that $\mathbf{Sign}, \mathbf{Sign}'$ are categories and $\text{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$, $\text{SEN}' : \mathbf{Sign}' \rightarrow \mathbf{Set}$ are functors. Let $\langle K, \kappa \rangle : \text{SEN} \rightarrow \text{SEN}'$ and $\langle \Lambda, \lambda \rangle : \text{SEN}' \rightarrow \mathcal{I}''$ be translations. Then $\mathcal{I}^{\langle \Lambda K, \lambda_K \kappa \rangle} = \mathcal{I}^{\langle K, \kappa \rangle}$, where in $\mathcal{I}^{\langle K, \kappa \rangle}$, $\langle K, \kappa \rangle : \text{SEN} \rightarrow \mathcal{I}^{\langle \Lambda, \lambda \rangle}$.*

$$\text{SEN} \xrightarrow{\langle K, \kappa \rangle} \text{SEN}' \xrightarrow{\langle \Lambda, \lambda \rangle} \mathcal{I}''$$

Proof Consider the diagram

$$\begin{array}{ccc} & \text{SEN} & \\ \langle K, \kappa \rangle \swarrow & & \searrow \langle \Lambda K, \lambda_K \kappa \rangle \\ \mathcal{I}^{\langle \Lambda, \lambda \rangle} & \xrightarrow{\langle \Lambda, \lambda \rangle} & \mathcal{I}'' \end{array}$$

and, once more, combine Propositions 3.2 and 3.3. \square

4 Models of π -institutions

Next, the notion of a model of a given π -institution is introduced. First, the analog of the definition in [11] for an abstract logic being a model of a given sentential logic is used. Roughly speaking, a π -institution \mathcal{M} is called a model of a given π -institution \mathcal{I} via a translation $\langle M, \mu \rangle : \mathcal{I} \rightarrow \mathcal{M}$ in case the closure generated by $\langle M, \mu \rangle$ on SEN is bigger than the original closure of \mathcal{I} in the \leq -ordering. This condition, as reflected in the present context, is proved to be equivalent to semi-interpretability. Therefore, \mathcal{M} will be shown to be a model of \mathcal{I} via a translation $\langle M, \mu \rangle : \mathcal{I} \rightarrow \mathcal{M}$ if and only if $\langle M, \mu \rangle$ is a semi-interpretation.

Definition 4.1 $\mathcal{I}' = \langle \mathbf{Sign}', \text{SEN}', \{C'_\Sigma\}_{\Sigma \in |\mathbf{Sign}'|} \rangle$ is a *model* of $\mathcal{I} = \langle \mathbf{Sign}, \text{SEN}, \{C_\Sigma\}_{\Sigma \in |\mathbf{Sign}|} \rangle$ via $\langle F, \alpha \rangle$, if $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ is a translation such that $C \leq C^{\langle F, \alpha \rangle}$.

Lemma 4.2 \mathcal{I}' is a model of \mathcal{I} if and only if \mathcal{I} is semi-interpretable in \mathcal{I}' , in symbols $\mathcal{I} \dashv \mathcal{I}'$.

Proof Suppose, first, that $C \leq C^{(F, \alpha)}$ and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$ such that $\varphi \in C_\Sigma(\Phi)$. Then $\varphi \in C_\Sigma^{(F, \alpha)}(\Phi)$, whence $\alpha_\Sigma(\varphi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, and $\langle F, \alpha \rangle : \mathcal{I} \dashv \mathcal{I}'$ is a semi-interpretation.

Suppose, conversely, that $\langle F, \alpha \rangle : \mathcal{I} \dashv \mathcal{I}'$ is a semi-interpretation and let $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$ such that $\varphi \in C_\Sigma(\Phi)$. Therefore $\alpha_\Sigma(\varphi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$. Hence $\varphi \in C_\Sigma^{(F, \alpha)}(\Phi)$. \square

Notice the difference between the definition of model for π -institutions and the corresponding definition for sentential logics. In the π -institution framework a model of a π -institution \mathcal{I} is a π -institution that respects the closure of \mathcal{I} as the sentences of \mathcal{I} get translated into those of the model via a specific translation. In the sentential logic framework one considers all possible valuations of the sentential logic into the abstract logic serving as its model. Consideration of single translations is adopted for two reasons. The first, which is rather technical, is that the π -institution \mathcal{I} , whose models one wants to consider, is not, in general, assumed to have a sentence part with a freeness property with respect to some class of possible models. Therefore, it is impossible, in that generality, to carry out many of the crucial constructions on valuations that one has at hand when mapping a free algebra of a class into other algebras of the same class. Fixing the valuation and studying some analogs of the properties in this fixed setting overcomes some of the problems relating to this deficiency. The second reason has to do with the role that π -institutions have traditionally played in the literature. They are modeling entire logical systems, incorporating all their morphisms and system valuations. A translation from one logical system to another is usually performed via a specific “preferred” formal translation or class of translations that have some specific logical meaning. In this context, if that distinguished translation respects the closure systems, it is natural to say that one institution is a model of the other via the translation.

Returning to our formal treatment of models, a π -institution is said to be complete with respect to a class of its models if all of them collectively generate its closure system. More formally, the following definition applies.

Definition 4.3 \mathcal{I} is complete with respect to a class of models $\mathcal{I}^i, i \in I$, via the translations $\mathcal{F} = \{\langle F^i, \alpha^i \rangle : \mathcal{I} \rightarrow \mathcal{I}^i : i \in I\}$ if $C = C^{\mathcal{F}}$. If \mathcal{I} is complete with respect to $\{\mathcal{M}\}$, we say that \mathcal{I} is complete with respect to \mathcal{M} for simplicity.

Lemma 4.4 \mathcal{I} is complete with respect to \mathcal{I}' via the translation $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ if and only if $\langle F, \alpha \rangle : \mathcal{I} \vdash \mathcal{I}'$ is an interpretation.

Proof It was shown in Proposition 3.2 that, if $C = C^{(F, \alpha)}$, then $\langle F, \alpha \rangle : \mathcal{I} \vdash \mathcal{I}'$ becomes an interpretation. Conversely, if $\langle F, \alpha \rangle : \mathcal{I} \vdash \mathcal{I}'$ is an interpretation, then, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$, $\varphi \in C_\Sigma(\Phi)$ if and only if $\alpha_\Sigma(\varphi) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, whence $C = C^{(F, \alpha)}$ and \mathcal{I} is complete with respect to \mathcal{I}' via $\langle F, \alpha \rangle$. \square

The following proposition investigates possible ways in which a given model of a π -institution may give rise to other models by either “pulling back” along translations

or by “pushing forward” along translations, provided some additional commutativity conditions on the translations under discussion are present in the same context.

Proposition 4.5 *Suppose that \mathbf{Sign}' is a category, $\mathbf{SEN}' : \mathbf{Sign}' \rightarrow \mathbf{Set}$ is a functor, $\mathcal{I}, \mathcal{I}''$ two π -institutions, and $\langle K, \kappa \rangle : \mathbf{SEN}' \rightarrow \mathcal{I}''$ a translation.*

1. *If \mathcal{I}'' is a model of \mathcal{I} via $\langle F, \alpha \rangle : \mathcal{I} \vdash \mathcal{I}''$, then, if $\langle G, \beta \rangle : \mathcal{I} \rightarrow \mathcal{I}^{(K, \kappa)}$ is such that $\langle K, \kappa \rangle \langle G, \beta \rangle = \langle F, \alpha \rangle$, $\mathcal{I}^{(K, \kappa)}$ is a model of \mathcal{I} via $\langle G, \beta \rangle$.*

$$\begin{array}{ccc}
 & \mathcal{I} & \\
 \langle G, \beta \rangle \swarrow & & \searrow \langle F, \alpha \rangle \\
 \mathcal{I}^{(K, \kappa)} & \xrightarrow{\langle K, \kappa \rangle} & \mathcal{I}''
 \end{array}$$

2. *If $\mathcal{I}^{(K, \kappa)}$ is a model of \mathcal{I} via $\langle G, \beta \rangle : \mathcal{I} \vdash \mathcal{I}^{(K, \kappa)}$, then \mathcal{I}'' is a model of \mathcal{I} via $\langle K, \kappa \rangle \langle G, \beta \rangle$.*

Proof For the first part, suppose that \mathcal{I}'' is a model of \mathcal{I} via $\langle F, \alpha \rangle : \mathcal{I} \vdash \mathcal{I}''$ and that $\langle G, \beta \rangle : \mathcal{I} \rightarrow \mathcal{I}^{(K, \kappa)}$ is such that $\langle K, \kappa \rangle \langle G, \beta \rangle = \langle F, \alpha \rangle$. Then we have, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathbf{SEN}(\Sigma)$,

$$\begin{aligned}
 \varphi \in C_{\Sigma}(\Phi) \quad \text{implies} \quad & \alpha_{\Sigma}(\varphi) \subseteq C''_{F(\Sigma)}(\alpha_{\Sigma}(\Phi)) \\
 \text{iff} \quad & \kappa_{G(\Sigma)}(\beta_{\Sigma}(\varphi)) \subseteq C''_{K(G(\Sigma))}(\kappa_{G(\Sigma)}(\beta_{\Sigma}(\Phi))) \\
 \text{iff} \quad & \beta_{\Sigma}(\varphi) \subseteq C^{(K, \kappa)}_{G(\Sigma)}(\beta_{\Sigma}(\Phi)),
 \end{aligned}$$

whence $\mathcal{I}^{(K, \kappa)}$ is a model of \mathcal{I} via $\langle G, \beta \rangle$.

For the second part, suppose that $\mathcal{I}^{(K, \kappa)}$ is a model of \mathcal{I} via the semi-interpretation $\langle G, \beta \rangle : \mathcal{I} \vdash \mathcal{I}^{(K, \kappa)}$. Again, consider $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\varphi\} \subseteq \mathbf{SEN}(\Sigma)$. We have

$$\begin{aligned}
 \varphi \in C_{\Sigma}(\Phi) \quad \text{implies} \quad & \beta_{\Sigma}(\varphi) \subseteq C^{(K, \kappa)}_{G(\Sigma)}(\beta_{\Sigma}(\Phi)) \\
 \text{iff} \quad & \kappa_{G(\Sigma)}(\beta_{\Sigma}(\varphi)) \subseteq C''_{K(G(\Sigma))}(\kappa_{G(\Sigma)}(\beta_{\Sigma}(\Phi)))
 \end{aligned}$$

whence \mathcal{I}'' is a model of \mathcal{I} via $\langle K, \kappa \rangle \langle G, \beta \rangle$. □

Proposition 4.5 has the following consequence when one considers in place of the translation $\langle K, \kappa \rangle$ a canonical projection interpretation from a π -institution to a logical quotient by a logical equivalence system.

Proposition 4.6 *Suppose $\mathcal{I}, \mathcal{I}'$ are π -institutions, θ is a logical equivalence system of \mathcal{I}' . Then \mathcal{I}' is a model of \mathcal{I} via $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ if and only if \mathcal{I}'/θ is a model of \mathcal{I} via $\langle F, \pi_F^{\theta} \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'/\theta$.*

Proof If θ is a logical equivalence system, then, by Propositions 22 and 23 of [19], C'^{θ} is a closure system on \mathbf{SEN}'^{θ} and $\langle \mathbf{ISign}', \pi^{\theta} \rangle : \mathcal{I}' \vdash \mathcal{I}'/\theta$ is an interpretation, whence, if \mathcal{I}' is a model of \mathcal{I} via $\langle F, \alpha \rangle$, \mathcal{I}'/θ is a model of \mathcal{I} via $\langle \mathbf{ISign}', \pi^{\theta} \rangle \langle F, \alpha \rangle$.

$$\begin{array}{ccc}
 & \mathcal{I} & \\
 \langle F, \alpha \rangle \swarrow & & \\
 \mathcal{I}' & \xrightarrow{\langle \mathbf{ISign}', \pi^{\theta} \rangle} & \mathcal{I}'/\theta
 \end{array}$$

The converse is given by Proposition 4.5. □

Recall from [19] (see also Section 2) that, given a π -institution \mathcal{I} and a category N of natural transformations on SEN , the Tarski N -congruence system of \mathcal{I} is denoted by $\tilde{\Omega}^N(\mathcal{I})$. The logical quotient of \mathcal{I} by its Tarski N -congruence system is denoted by \mathcal{I}^N , the category of natural transformations $N^{\tilde{\Omega}^N(\mathcal{I})}$ on $\text{SEN}^N := \text{SEN}^{\tilde{\Omega}^N(\mathcal{I})}$ is denoted by \bar{N} , for simplicity, and the associated projection (N, \bar{N}) -bilogical morphism by $\langle \mathbf{ISign}, \pi^N \rangle : \mathcal{I} \vdash \mathcal{I}^N$. Using this notation, Proposition 4.6 yields the following corollary. It specializes the content of Proposition 4.6 to the case of the N' -reduct $\mathcal{I}'^{N'}$ of a given π -institution \mathcal{I}' by its Tarski N' -congruence system $\tilde{\Omega}^{N'}(\mathcal{I}')$.

Corollary 4.7 *Let \mathcal{I}' be a π -institution and N' a category of natural transformations on SEN' . \mathcal{I}' is a model of a π -institution \mathcal{I} via $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ if and only if $\mathcal{I}'^{N'}$ is a model of \mathcal{I} via $\langle F, \pi_F^{N'} \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'^{N'}$.*

The following proposition characterizes the property of an equivalence system of a π -institution \mathcal{I} being logical. It shows that this property is equivalent to the existence of a closure system on the quotient sentence structure that is induced by the given equivalence system such that it forms a model of \mathcal{I} with respect to which \mathcal{I} is complete via the canonical projection.

Proposition 4.8 *Suppose that \mathcal{I} is a π -institution. An equivalence system θ of \mathcal{I} is a logical equivalence system if and only if there exists a closure system C' on SEN^θ such that \mathcal{I} is complete with respect to $\mathcal{I}' = \langle \mathbf{Sign}, \text{SEN}^\theta, C' \rangle$ via $\langle \mathbf{ISign}, \pi^\theta \rangle$.*

Proof If θ is a logical equivalence system, the closure system C^θ on SEN^θ is such that \mathcal{I} is complete with respect to \mathcal{I}/θ , by Proposition 23 of [19] and Lemma 4.4.

Conversely, if for a closure system C' on SEN^θ , \mathcal{I} is complete with respect to $\mathcal{I}' = \langle \mathbf{Sign}, \text{SEN}^\theta, C' \rangle$ via $\langle \mathbf{ISign}, \pi^\theta \rangle$, then, if $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \text{SEN}(\Sigma)$, $\langle \varphi, \psi \rangle \in \theta_\Sigma$ implies that $\varphi/\theta_\Sigma = \psi/\theta_\Sigma$, whence $C'_\Sigma(\varphi/\theta_\Sigma) = C'_\Sigma(\psi/\theta_\Sigma)$, that is, $C'_\Sigma(\pi_\Sigma^\theta(\varphi)) = C'_\Sigma(\pi_\Sigma^\theta(\psi))$, whence $C_\Sigma(\varphi) = C_\Sigma(\psi)$ and θ is a logical equivalence system of \mathcal{I} . \square

Using Proposition 4.8, one immediately obtains the following completeness result of a given π -institution \mathcal{I} with respect to any class of models containing \mathcal{I} or any of its N -reducts. This proposition provides an analog of Proposition 2.6 of [11].

Proposition 4.9 *Let $\mathcal{I} = \langle \mathbf{Sign}, \text{SEN}, C \rangle$ be a π -institution, N a category of natural transformations on SEN , and θ a logical N -congruence system of \mathcal{I} . If \mathfrak{M} is a class of models of \mathcal{I} containing $\langle \mathbf{ISign}, \iota \rangle : \mathcal{I} \vdash \mathcal{I}$ or $\langle \mathbf{ISign}, \pi^N \rangle : \mathcal{I} \vdash \mathcal{I}^N$, then \mathcal{I} is complete with respect to \mathfrak{M} . In particular, \mathcal{I} is complete with respect to the class of all its models and also with respect to the class of all its N -reduced models.*

5 Min and Full Models

Recall from [19] the definitions of the functor $F^\# : \mathbf{Th}(\mathcal{I}) \rightarrow \mathbf{Th}(\mathcal{I}')$ and of the logical equivalence system $\theta^{(F, \alpha)}$. More precisely, given a semi-interpretation $\langle F, \alpha \rangle : \mathcal{I} \dashv \mathcal{I}'$, the functor $F^\# : \mathbf{Th}(\mathcal{I}) \rightarrow \mathbf{Th}(\mathcal{I}')$ from the category of theories of \mathcal{I} to that of \mathcal{I}' is defined at the object level by

$$F^\#(\langle \Sigma, T \rangle) = \langle F(\Sigma), C'_{F(\Sigma)}(\alpha_\Sigma(T)) \rangle, \quad \text{for all } \langle \Sigma, T \rangle \in |\mathbf{Th}(\mathcal{I})|,$$

and, at the morphism level, given $f : \langle \Sigma_1, T_1 \rangle \rightarrow \langle \Sigma_2, T_2 \rangle$, by

$$F^\#(f) = F(f) : \langle F(\Sigma_1), C'_{F(\Sigma_1)}(\alpha_{\Sigma_1}(T_1)) \rangle \rightarrow \langle F(\Sigma_2), C'_{F(\Sigma_2)}(\alpha_{\Sigma_2}(T_2)) \rangle.$$

Furthermore, given a singleton interpretation $\langle F, \alpha \rangle : \mathcal{I} \vdash^s \mathcal{I}'$, by $\theta^{(F, \alpha)} = \{ \langle \Sigma, \theta_\Sigma^{(F, \alpha)} \rangle : \Sigma \in |\mathbf{Sign}| \}$ is denoted the logical equivalence system defined by

$$\theta_\Sigma^{(F, \alpha)} = \{ \langle \varphi, \psi \rangle \in \text{SEN}(\Sigma)^2 : \alpha_\Sigma(\varphi) = \alpha_\Sigma(\psi) \}.$$

See Proposition 26 of [19] for more details.

Proposition 5.1 *Let $\mathcal{I}, \mathcal{I}'$ be two π -institutions and $\langle F, \alpha \rangle : \mathcal{I} \vdash^s \mathcal{I}'$ a surjective singleton semi-interpretation. Then the following are equivalent:*

1. $\langle F, \alpha \rangle : \mathcal{I} \vdash^s \mathcal{I}'$ is an interpretation;
2. $F^\# : \mathbf{Th}(\mathcal{I}) \rightarrow \mathbf{Th}(\mathcal{I}')$ is a functorial bijection;
3. for all $\Sigma \in |\mathbf{Sign}|$, $\langle \Sigma, T \rangle \in |\mathbf{Th}(\mathcal{I})|$ implies $\langle F(\Sigma), \alpha_\Sigma(T) \rangle \in |\mathbf{Th}(\mathcal{I}')|$ and $\theta^{(F, \alpha)}$ is a logical equivalence of \mathcal{I} .

Proof

(1 \rightarrow 2) This is the content of Lemma 17 of [19].

(2 \rightarrow 3) First, to show that $\langle F(\Sigma), \alpha_\Sigma(T) \rangle \in |\mathbf{Th}(\mathcal{I}')|$, let $\psi \in C'_{F(\Sigma)}(\alpha_\Sigma(T))$. Then, since $\langle F, \alpha \rangle$ is surjective, there exists $\varphi \in \text{SEN}(\Sigma)$ such that $\psi = \alpha_\Sigma(\varphi)$. Therefore $\alpha_\Sigma(\varphi) \in C'_{F(\Sigma)}(\alpha_\Sigma(T))$, whence, by hypothesis, $\varphi \in T$. Hence $\psi = \alpha_\Sigma(\varphi) \in \alpha_\Sigma(T)$ and $\alpha_\Sigma(T)$ is indeed an $F(\Sigma)$ -theory. For the last part, we have

$$\begin{aligned} \langle \varphi, \psi \rangle \in \theta_\Sigma^{(F, \alpha)} & \text{ implies } \alpha_\Sigma(\varphi) = \alpha_\Sigma(\psi) \\ & \text{ implies } C'_{F(\Sigma)}(\alpha_\Sigma(\varphi)) = C'_{F(\Sigma)}(\alpha_\Sigma(\psi)) \\ & \text{ implies } C'_{F(\Sigma)}(\alpha_\Sigma(C_\Sigma(\varphi))) = C'_{F(\Sigma)}(\alpha_\Sigma(C_\Sigma(\psi))) \\ & \text{ implies } C_\Sigma(\varphi) = C_\Sigma(\psi). \end{aligned}$$

(3 \rightarrow 1) It suffices, by Lemma 14 of [19], to show that

$$\alpha_\Sigma^{-1}(C'_{F(\Sigma)}(\alpha_\Sigma(\varphi))) = C_\Sigma(\varphi).$$

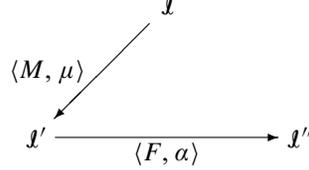
Since $\langle F, \alpha \rangle$ is a semi-interpretation, $C_\Sigma(\varphi) \subseteq \alpha_\Sigma^{-1}(C'_{F(\Sigma)}(\alpha_\Sigma(\varphi)))$. Thus, it suffices, in turn, to show that $\alpha_\Sigma^{-1}(C'_{F(\Sigma)}(\alpha_\Sigma(\varphi))) \subseteq C_\Sigma(\varphi)$. Since α_Σ is surjective and $\theta^{(F, \alpha)}$ is a logical equivalence, this is equivalent to $C'_{F(\Sigma)}(\alpha_\Sigma(\varphi)) \subseteq \alpha_\Sigma(C_\Sigma(\varphi))$, which follows immediately from the given condition in 3. \square

A crucial role in the theory of algebraizability of a sentential logic \mathcal{S} is played by those models $\mathbb{L} = \langle \mathbf{A}, \mathcal{C} \rangle$, which are abstract logics whose closure system \mathcal{C} consists of all \mathcal{S} -filters on the underlying universe A of the algebra \mathbf{A} . The notion of an $\langle F, \alpha \rangle$ -min-model \mathcal{M} of a π -institution \mathcal{I} for a given translation $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{M}$ in the present context is an attempt to capture the essence of an abstract logic model having a closure system consisting of all logical filters.

Definition 5.2 Let $\mathcal{I} = \langle \mathbf{Sign}, \text{SEN}, C \rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \text{SEN}', C' \rangle$ be two π -institutions and $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ a translation. \mathcal{I}' is said to be the $\langle F, \alpha \rangle$ -min model of \mathcal{I} if, for every model $\mathcal{I}'' = \langle \mathbf{Sign}'', \text{SEN}'', C'' \rangle$ of \mathcal{I} via $\langle F, \alpha \rangle$, $C' \leq C''$. \mathcal{I}' is said to be a min model of \mathcal{I} if it is $\langle F, \alpha \rangle$ -min for some translation $\langle F, \alpha \rangle : \text{SEN} \rightarrow \text{SEN}'$.

Min models are preserved by surjective singleton interpretations. This is the content of the following proposition.

Proposition 5.3 *Suppose that $\langle F, \alpha \rangle : \mathcal{I}' \vdash^s \mathcal{I}''$ is a surjective singleton interpretation from \mathcal{I}' to \mathcal{I}'' . Then, if \mathcal{I}' is an $\langle M, \mu \rangle$ -min model of \mathcal{I} , then \mathcal{I}'' is an $\langle FM, \alpha_M \mu \rangle$ -min model of \mathcal{I} .*



Proof Roughly speaking, the result follows from Proposition 5.1, that is, from the fact that the surjective singleton interpretation induces a bijection between the theories of the two models.

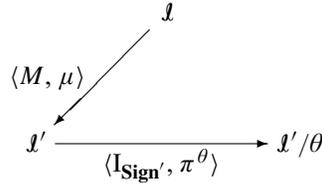
Suppose that \mathcal{I}'' is not the $\langle FM, \alpha_M \mu \rangle$ -min model of \mathcal{I} . Therefore, there exists a closure system C''' on SEN'' such that $C''' < C''$ and $\mathcal{I}''' = \langle \mathbf{Sign}'' , \text{SEN}'' , C''' \rangle$ is a model of \mathcal{I} via $\langle FM, \alpha_M \mu \rangle$. But then, by Proposition 5.1, the closure $C^{(F, \alpha)}$ generated by $\langle F, \alpha \rangle : \text{SEN}' \rightarrow \mathcal{I}'''$ is such that $C^{(F, \alpha)} < C'$ and $\langle M, \mu \rangle : \mathcal{I} \rightarrow \mathcal{I}^{(F, \alpha)}$ is still a semi-interpretation since

$$\begin{aligned}
 \varphi \in C_\Sigma(\Phi) \quad & \text{implies} \quad \alpha_{M(\Sigma)}(\mu_\Sigma(\varphi)) \subseteq C_{F(M(\Sigma))}'''(\alpha_{M(\Sigma)}(\mu_\Sigma(\Phi))) \\
 & \text{implies} \quad \mu_\Sigma(\varphi) \subseteq C_{M(\Sigma)}^{(F, \alpha)}(\mu_\Sigma(\Phi)).
 \end{aligned}$$

Thus \mathcal{I}' is not the $\langle M, \mu \rangle$ -min model on SEN' either, which contradicts our hypothesis. \square

Proposition 5.3 has the following two corollaries. The first describes the special case in which the surjective singleton interpretation is a canonical projection onto the quotient π -institution by a logical equivalence system. The second deals with the case of two π -institutions that are mutually interpretable in one another.

Corollary 5.4 *Let \mathcal{I}' be a π -institution and θ a logical equivalence system of \mathcal{I}' . If \mathcal{I}' is an $\langle M, \mu \rangle$ -min model of \mathcal{I} , then \mathcal{I}'/θ is an $\langle M, \pi_M^\theta \mu \rangle$ -min model of \mathcal{I} .*



The second corollary of Proposition 5.3 asserts that, if two π -institutions are mutually interpretable in one another, then one is a min model of a third π -institution if and only if the other is.

Corollary 5.5 *Let $\mathcal{I}' , \mathcal{I}''$ be two π -institutions that are mutually interpretable in one another via surjective singleton interpretations. Then \mathcal{I}' is a min model of a π -institution \mathcal{I} if and only if \mathcal{I}'' is a min model of \mathcal{I} .*

The following definition has a two-fold purpose. It first introduces the notion of an (N, N') -model, which is a model of a π -institution $\mathcal{I} = \langle \mathbf{Sign}, \text{SEN}, C \rangle$ via an (N, N') -epimorphic semi-interpretation, where N is a category of natural transformations on SEN . Roughly speaking, it is a model that preserves the natural transformation structure N as it is translated onto the structure N' . Secondly, it introduces the notion of a full model. This is an attempt to capture the essence of the notion

of a full model from the theory of sentential logics. Full models in that theory (see [11] for details) are abstract logic models whose reducts are models consisting of all logical filters on the universes of their algebras.

Definition 5.6 Let $\mathcal{I} = \langle \mathbf{Sign}, \mathbf{SEN}, C \rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathbf{SEN}', C' \rangle$ be π -institutions, N, N' categories of natural transformations on $\mathbf{SEN}, \mathbf{SEN}'$, respectively. \mathcal{I}' is said to be an (N, N') -model of \mathcal{I} via $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathcal{I}'$ if $\langle F, \alpha \rangle$ is an (N, N') -logical morphism. It is said to be an (N, N') -full model of \mathcal{I} via $\langle F, \alpha \rangle$ if and only if $\mathcal{I}'^{N'}$ is a min (N, N') -model of \mathcal{I} via $\langle F, \pi_F^{N'} \alpha \rangle$.

$$\mathcal{I} \xrightarrow{\langle F, \alpha \rangle} \mathcal{I}' \xrightarrow{\langle \mathbf{I}_{\mathbf{Sign}'}, \pi^{N'} \rangle} \mathcal{I}'^{N'}$$

Proposition 5.7 below justifies the use of the word model in the term ‘full model’.

Proposition 5.7 *If \mathcal{I}' is an (N, N') -full model of \mathcal{I} via $\langle F, \alpha \rangle$, then \mathcal{I}' is an (N, N') -model of \mathcal{I} .*

Proof Following the diagram

$$\mathcal{I} \xrightarrow{\langle F, \alpha \rangle} \mathcal{I}' \xrightarrow{\langle \mathbf{I}_{\mathbf{Sign}'}, \pi^{N'} \rangle} \mathcal{I}'^{N'}$$

we get, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathbf{SEN}(\Sigma)$,

$$\begin{aligned} \varphi \in C_{\Sigma}(\Phi) & \text{ implies } \alpha_{\Sigma}(\varphi) / \tilde{\Omega}_{F(\Sigma)}^{N'}(\mathcal{I}') \subseteq C'_{F(\Sigma)}(\alpha_{\Sigma}(\Phi) / \tilde{\Omega}_{F(\Sigma)}^{N'}(\mathcal{I}')) \\ & \text{ iff } \alpha_{\Sigma}(\varphi) \subseteq C'_{F(\Sigma)}(\alpha_{\Sigma}(\Phi)), \end{aligned}$$

where the implication holds by hypothesis and the equivalence by the definition of the quotient π -institution. \square

Min models of π -institutions form a subclass of full models.

Proposition 5.8 *Let $\mathcal{I}, \mathcal{I}'$ be π -institutions. If \mathcal{I}' is the $\langle F, \alpha \rangle$ -min (N, N') -model of \mathcal{I} then \mathcal{I}' is an (N, N') -full model of \mathcal{I} via $\langle F, \alpha \rangle$.*

Proof This result follows by combining the definitions of a min model and of a full model with Corollary 5.4. \square

Furthermore, full models are preserved by biological morphisms, whose first components (functor components) are isomorphisms. This is a weak analog of Proposition 2.11 of [11] in the context of π -institutions. Before proving this result, an auxiliary technical lemma is needed to the effect that closure systems may be transferred ‘‘forward’’ through surjective singleton translations with bijective components. These translations then become surjective singleton interpretations between the two π -institutions.

Lemma 5.9 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathbf{SEN}, C \rangle$ is a π -institution, \mathbf{Sign}' is a category, $\mathbf{SEN}' : \mathbf{Sign}' \rightarrow \mathbf{Set}$ is a functor, and $\langle F, \alpha \rangle : \mathcal{I} \rightarrow \mathbf{SEN}'$ a surjective singleton translation such that F and α_{Σ} , $\Sigma \in |\mathbf{Sign}|$, are bijections. Then there exists a closure system C' on \mathbf{SEN}' such that $\langle F, \alpha \rangle : \mathcal{I} \vdash^s \langle \mathbf{Sign}', \mathbf{SEN}', C' \rangle$ is a surjective singleton interpretation.*

Proof Define, for every $\Sigma' \in |\mathbf{Sign}'|$, $C'_{\Sigma'} : \mathcal{P}\text{SEN}'(\Sigma') \rightarrow \mathcal{P}\text{SEN}'(\Sigma')$ by setting, for all $\Psi \cup \{\psi\} \subseteq \text{SEN}'(\Sigma')$,

$$\psi \in C'_{\Sigma'}(\Psi) \quad \text{iff} \quad \alpha_{F^{-1}(\Sigma')}^{-1}(\psi) \in C_{F^{-1}(\Sigma')}(\alpha_{F^{-1}(\Sigma')}^{-1}(\Psi)).$$

It is not very difficult to show that $\{C'_{\Sigma'}\}_{\Sigma' \in |\mathbf{Sign}'|}$ is a closure system on SEN' .

Properties (a) and (c) of a closure system are very easy to establish for C' based on the corresponding properties for C . For instance, for property (a), if $\Sigma' \in |\mathbf{Sign}'|$, $\Psi \cup \{\psi\} \subseteq \text{SEN}'(\Sigma')$ such that $\psi \in \Psi$, we get that $\alpha_{F^{-1}(\Sigma')}^{-1}(\psi) \in \alpha_{F^{-1}(\Sigma')}^{-1}(\Psi)$, whence, by property (a) for C , $\alpha_{F^{-1}(\Sigma')}^{-1}(\psi) \in C_{F^{-1}(\Sigma')}(\alpha_{F^{-1}(\Sigma')}^{-1}(\Psi))$. Therefore, by the definition of C' , $\psi \in C'_{\Sigma'}(\Psi)$, which establishes (a) for C' .

For property (b), suppose that $\Sigma' \in |\mathbf{Sign}'|$, $\Psi \cup \{\psi\} \subseteq \text{SEN}'(\Sigma')$ such that $\psi \in C'_{\Sigma'}(C'_{\Sigma'}(\Psi))$. Then we get that

$$\alpha_{F^{-1}(\Sigma')}^{-1}(\psi) \in C_{F^{-1}(\Sigma')}(\alpha_{F^{-1}(\Sigma')}^{-1}(C'_{\Sigma'}(\Psi)))$$

and also that

$$C'_{\Sigma'}(\Psi) \subseteq \alpha_{F^{-1}(\Sigma')}^{-1}(C_{F^{-1}(\Sigma')}(\alpha_{F^{-1}(\Sigma')}^{-1}(\Psi))).$$

Therefore, we obtain that

$$\begin{aligned} \alpha_{F^{-1}(\Sigma')}^{-1}(\psi) &\in C_{F^{-1}(\Sigma')}(\alpha_{F^{-1}(\Sigma')}^{-1}(\alpha_{F^{-1}(\Sigma')}^{-1}(C_{F^{-1}(\Sigma')}(\alpha_{F^{-1}(\Sigma')}^{-1}(\Psi)))) \\ &= C_{F^{-1}(\Sigma')}(\alpha_{F^{-1}(\Sigma')}^{-1}(C_{F^{-1}(\Sigma')}(\alpha_{F^{-1}(\Sigma')}^{-1}(\Psi))) \\ &= C_{F^{-1}(\Sigma')}(\alpha_{F^{-1}(\Sigma')}^{-1}(\Psi)), \end{aligned}$$

that is, that $\psi \in C'_{\Sigma'}(\Psi)$, which proves (b).

For (d), suppose that $\Sigma'_1, \Sigma'_2 \in |\mathbf{Sign}'|$, $f' \in \mathbf{Sign}'(\Sigma'_1, \Sigma'_2)$, $\Phi \subseteq \text{SEN}'(\Sigma'_1)$. If $\varphi \in \text{SEN}'(f')(C'_{\Sigma'_1}(\Phi))$, there exists $\psi \in C'_{\Sigma'_1}(\Phi)$ such that $\varphi = \text{SEN}'(f')(\psi)$. Hence, we obtain $\alpha_{F^{-1}(\Sigma'_1)}^{-1}(\psi) \in C_{F^{-1}(\Sigma'_1)}(\alpha_{F^{-1}(\Sigma'_1)}^{-1}(\Phi))$ and also $\alpha_{F^{-1}(\Sigma'_2)}^{-1}(\varphi) = \text{SEN}(F^{-1}(f'))(\alpha_{F^{-1}(\Sigma'_1)}^{-1}(\psi))$. Hence

$$\begin{aligned} \alpha_{F^{-1}(\Sigma'_2)}^{-1}(\varphi) &\in \text{SEN}(F^{-1}(f'))(C_{F^{-1}(\Sigma'_1)}(\alpha_{F^{-1}(\Sigma'_1)}^{-1}(\Phi))) \\ &\subseteq C_{F^{-1}(\Sigma'_2)}(\text{SEN}(F^{-1}(f'))(\alpha_{F^{-1}(\Sigma'_1)}^{-1}(\Phi))) \\ &= C_{F^{-1}(\Sigma'_2)}(\alpha_{F^{-1}(\Sigma'_2)}^{-1}(\text{SEN}'(f')(\Phi))), \end{aligned}$$

that is, by the definition of C' , we get that $\varphi \in C'_{\Sigma'_2}(\text{SEN}'(f')(\Phi))$, which proves (d) for C' and completes the proof that C' is a closure system on SEN' .

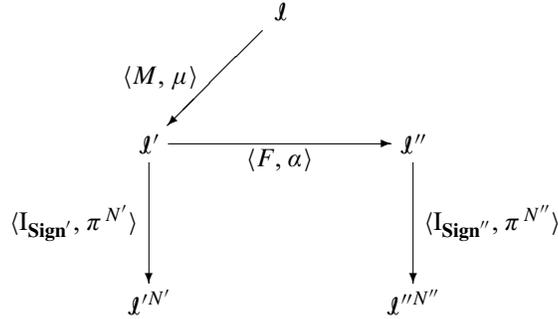
Finally, $\langle F, \alpha \rangle$ is a surjective singleton interpretation from \mathcal{I} to $\langle \mathbf{Sign}', \text{SEN}', C' \rangle$. In fact, if $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \text{SEN}(\Sigma)$, we get $\varphi \in C_{\Sigma}(\Phi)$ if and only if, since both F and α_{Σ} are bijections, $\alpha_{\Sigma}^{-1}(\alpha_{\Sigma}(\varphi)) \in C_{\Sigma}(\alpha_{\Sigma}^{-1}(\alpha_{\Sigma}(\Phi)))$, if and only if, by the definition of C' , $\alpha_{\Sigma}(\varphi) \in C'_{F(\Sigma)}(\alpha_{\Sigma}(\Phi))$ and $\langle F, \alpha \rangle : \mathcal{I} \vdash^s \langle \mathbf{Sign}', \text{SEN}', C' \rangle$ is a surjective singleton interpretation. \square

Proposition 5.10 *Let $\mathcal{I}, \mathcal{I}', \mathcal{I}''$ be π -institutions and $\langle F, \alpha \rangle : \mathcal{I}' \rightarrow \mathcal{I}''$ an (N', N'') -bilogical morphism.*

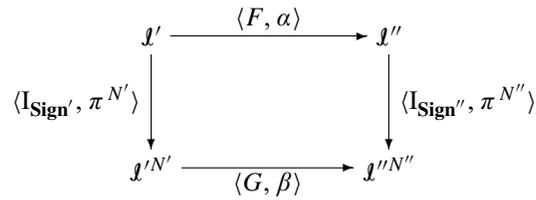
1. *If \mathcal{I}' is an (N, N') -full model of \mathcal{I} via $\langle M, \mu \rangle$, then \mathcal{I}'' is an (N, N'') -full model of \mathcal{I} via $\langle F, \alpha \rangle \langle M, \mu \rangle$.*
2. *If F is a bijection and \mathcal{I}'' is an (N, N'') -full model of \mathcal{I} via $\langle F, \alpha \rangle \langle M, \mu \rangle$, then \mathcal{I}' is an (N, N') -full model of \mathcal{I} via $\langle M, \mu \rangle$.*

Proof

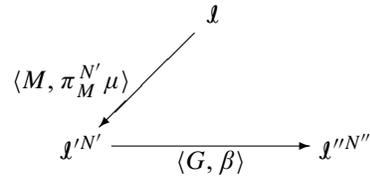
1 Suppose first that \mathcal{I}' is an (N, N') -full model of \mathcal{I} via $\langle M, \mu \rangle$. Consider the following diagram



Then, by Proposition 32 of [19], there exists an $(\overline{N'}, \overline{N''})$ -bilogical morphism $\langle G, \beta \rangle : \mathcal{I}'^{N'} \dashv^{se} \mathcal{I}''^{N''}$ making the square

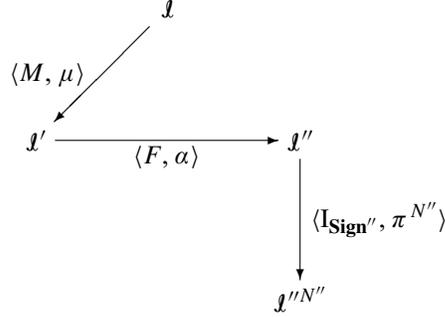


commutative. Now consider the following triangle and apply Proposition 5.3



to conclude that $\mathcal{I}''^{N''}$ is a $\langle \mathbf{ISign}'', \pi^{N''} \rangle \langle F, \alpha \rangle \langle M, \mu \rangle$ -min $(N, \overline{N''})$ -model of \mathcal{I} , whence \mathcal{I}'' is an (N, N'') -full model of \mathcal{I} via $\langle F, \alpha \rangle \langle M, \mu \rangle$.

2 Suppose next that \mathcal{I}'' is an (N, N'') -full model of \mathcal{I} via $\langle F, \alpha \rangle \langle M, \mu \rangle$. Then, by definition, $\mathcal{I}''^{N''}$ is a $\langle \mathbf{ISign}'', \pi^{N''} \rangle \langle F, \alpha \rangle \langle M, \mu \rangle$ -min $(N, \overline{N''})$ -model of \mathcal{I} .



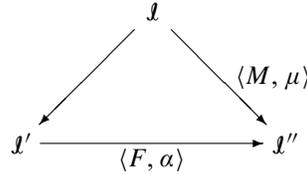
Suppose for the sake of obtaining a contradiction that \mathcal{I}' is not an (N, N') -full model of \mathcal{I} via $\langle M, \mu \rangle$. Then $\mathcal{I}'^{N'}$ is not a $\langle \mathbf{ISign}', \pi^{N'} \rangle \langle M, \mu \rangle$ -min $(N, \overline{N'})$ -model of \mathcal{I} . Thus, there exists a closure system C''' on $\text{SEN}^{N'}$ such that $C''' < C'^{N'}$. Since F is a bijection by hypothesis, and β_Σ is bijective, by Theorem 21 of [19], for all $\Sigma \in |\mathbf{Sign}|$, there exists, by Lemma 5.9, a closure system $C^{(4)}$ on $\text{SEN}''^{N''}$ such that $\langle G, \beta \rangle : \langle \mathbf{Sign}', \text{SEN}^{N'}, C''' \rangle \vdash \langle \mathbf{Sign}'', \text{SEN}'', C^{(4)} \rangle$ is an interpretation. But then, by Proposition 5.1, since $C''' < C'^{N'}$, we get that $C^{(4)} < C''^{N''}$ and, therefore, $\mathcal{I}''^{N''}$ is not the $\langle \mathbf{ISign}'', \pi^{N''} \rangle \langle F, \alpha \rangle \langle M, \mu \rangle$ -min $(N, \overline{N''})$ -model of \mathcal{I} , which contradicts our hypothesis. \square

Proposition 5.10 has the following corollary showing that the property of being a full model is preserved by logical quotients. Note that one of the two directions is obvious from the definition of an (N, N') -full model.

Corollary 5.11 *If $\mathcal{I}, \mathcal{I}'$ are π -institutions, then \mathcal{I}' is an (N, N') -full model of \mathcal{I} via $\langle M, \mu \rangle$ if and only if $\mathcal{I}'^{N'}$ is an $(N, \overline{N'})$ -full model of \mathcal{I} via $\langle \mathbf{ISign}', \pi^{N'} \rangle \langle M, \mu \rangle$.*

Proposition 5.12 may now be proved, which is the analog of Corollary 2.12 of [11] for π -institutions. It expresses the fact that full models may be viewed as inverse images of min models under suitable bilogical morphisms.

Proposition 5.12 *A π -institution \mathcal{I}' is an (N, N') -full model of a π -institution \mathcal{I} if and only if there is an (N', N'') -bilogical morphism $\langle F, \alpha \rangle$, with F an isomorphism, from \mathcal{I}' to an (N, N'') -min model $\langle M, \mu \rangle : \mathcal{I} \xrightarrow{\text{se}} \mathcal{I}''$ of \mathcal{I} through which $\langle M, \mu \rangle$ factors.*



Proof First, suppose that \mathcal{I}' is an (N, N') -full model of a π -institution \mathcal{I} via the (N, N') -logical morphism $\langle M, \mu \rangle : \mathcal{I} \rightarrow \mathcal{I}'$. But then

$$\mathcal{I} \xrightarrow{\langle M, \mu \rangle} \mathcal{I}' \xrightarrow{\langle \mathbf{ISign}', \pi^{N'} \rangle} \mathcal{I}'^{N'}.$$

$\mathcal{I}'^{N'}$ is, by definition, an $(N, \overline{N'})$ -min model of \mathcal{I} via $\langle M, \pi_M^{N'} \mu \rangle$ and the factorization property is clear.

Conversely, suppose that $\langle F, \alpha \rangle : \mathcal{I}' \vdash^{se} \mathcal{I}''$ is an (N', N'') -biological morphism from \mathcal{I}' to \mathcal{I}'' , with F an isomorphism, and that \mathcal{I}'' is an (N, N'') -min model of \mathcal{I} via the (N, N'') -logical morphism $\langle M, \mu \rangle : \mathcal{I} \dashv^{se} \mathcal{I}''$ such that $\langle M, \mu \rangle = \langle F, \alpha \rangle \langle G, \beta \rangle$, for some singleton (N, N') -epimorphic translation $\langle G, \beta \rangle : \mathcal{I} \rightarrow^s \mathcal{I}'$.

$$\begin{array}{ccc}
 & \mathcal{I} & \\
 \langle G, \beta \rangle \swarrow & & \searrow \langle M, \mu \rangle \\
 \mathcal{I}' & \xrightarrow{\langle F, \alpha \rangle} & \mathcal{I}''
 \end{array}$$

Then, by Proposition 5.8, \mathcal{I}'' is an (N, N'') -full model of \mathcal{I} via $\langle M, \mu \rangle$, whence, by Proposition 5.10, \mathcal{I}' is also an (N, N') -full model of \mathcal{I} via $\langle G, \beta \rangle$. \square

Proposition 5.12 immediately yields the following analog of Corollary 2.13 of [11] which characterizes the class $\text{FMod}^N(\mathcal{I})$ of (N, N') -full models \mathcal{I}' of a given π -institution \mathcal{I} .

Corollary 5.13 *The class $\text{FMod}^N(\mathcal{I})$ of (N, N') -full models of a π -institution \mathcal{I} is the smallest class of π -institutions that contains all (N, N') -min models \mathcal{I}' of \mathcal{I} and is closed under factors of min models through (N, N') -biological morphisms with isomorphic functor components, for some category N' of natural transformations on the sentence functor SEN' of \mathcal{I}' .*

Finally, Theorem 2.14 of [11], characterizing the closed sets of the closure system of a full model of a sentential logic \mathcal{B} , has the following analog in the π -institution framework.

Theorem 5.14 *A π -institution \mathcal{I}' is an (N, N') -full model of a π -institution \mathcal{I} via the (N, N') -logical morphism $\langle F, \alpha \rangle : \mathcal{I} \dashv^{se} \mathcal{I}'$ if and only if*

$$\tilde{\Omega}^{N'}(C') \leq \tilde{\Omega}^{N'}(C'') \quad \text{iff} \quad C' \leq C'',$$

for every closure system C'' on SEN' such that $\mathcal{I}'' = \langle \text{Sign}', \text{SEN}', C'' \rangle$ is an (N, N') -model of \mathcal{I} via $\langle F, \alpha \rangle$.

Proof Suppose that \mathcal{I}' is an (N, N') -full model of \mathcal{I} via $\langle F, \alpha \rangle : \mathcal{I} \dashv^{se} \mathcal{I}'$. If $C' \leq C''$, then, by Corollary 9 of [19], $\tilde{\Omega}^{N'}(C') \leq \tilde{\Omega}^{N'}(C'')$. If, on the other hand, $\tilde{\Omega}^{N'}(C') \leq \tilde{\Omega}^{N'}(C'')$, then $\tilde{\Omega}^{N'}(C')$ is a logical N' -congruence system of \mathcal{I}'' , whence, by Proposition 5.8, there exists a closure system C''' on $\text{SEN}'^{N'}$ such that \mathcal{I}'' is complete with respect to $\mathcal{I}''' = \langle \text{Sign}', \text{SEN}'^{N'}, C''' \rangle$ via $\langle \text{I}_{\text{Sign}'}, \pi^{N'} \rangle$.

$$\begin{array}{ccccc}
 \mathcal{I} & \xrightarrow{\langle F, \alpha \rangle} & \mathcal{I}' & \xrightarrow{\langle \text{I}_{\text{Sign}'}, \iota \rangle} & \mathcal{I}'' \\
 & & \downarrow \langle \text{I}_{\text{Sign}'}, \pi^{N'} \rangle & & \downarrow \langle \text{I}_{\text{Sign}'}, \pi^{N'} \rangle \\
 & & \mathcal{I}'^{N'} & & \mathcal{I}'''
 \end{array}$$

Since \mathcal{I}' is an (N, N') -full model of \mathcal{I} , it follows that $\mathcal{I}'^{N'}$ is an $(N, \overline{N'})$ -min model of \mathcal{I} , whence $C'^{N'} \leq C'''$ and, therefore, since both $\langle \text{I}_{\text{Sign}'}, \pi^{N'} \rangle$ are $(N', \overline{N'})$ -biological morphisms, we get $C' \leq C''$.

Suppose, conversely, that $\tilde{\Omega}^{N'}(C') \leq \tilde{\Omega}^{N'}(C'')$ if and only if $C' \leq C''$, for every closure system C'' of an (N, N') -model $\mathcal{I}'' = \langle \text{Sign}', \text{SEN}', C'' \rangle$ of \mathcal{I} via $\langle F, \alpha \rangle$.

Then, for every closure operator C''' on $\text{SEN}^{N'}$ such that $\langle F, \pi_F^{N'} \alpha \rangle : \mathcal{L} \text{---}^{se} \mathcal{L}'''$ is an $(N, \overline{N'})$ -model, since $\langle \text{I}_{\text{Sign}}, \pi^{N'} \rangle$ is an $(N', \overline{N'})$ -biological morphism, there exists C'' on $\text{SEN}^{N'}$ such that \mathcal{L}'' is complete with respect to \mathcal{L}''' . Hence, by Proposition 5.8, $\tilde{\Omega}^{N'}(C')$ is a logical N' -congruence system of \mathcal{L}'' , whence $\tilde{\Omega}^{N'}(C') \leq \tilde{\Omega}^{N'}(C'')$ and therefore $C' \leq C''$. Thus also $C'^{N'} \leq C'''$. Hence $\mathcal{L}'^{N'}$ is an $\langle F, \pi_F^{N'} \alpha \rangle$ -min model of \mathcal{L} , which shows that \mathcal{L}' is an (N, N') -full model of \mathcal{L} . \square

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Acknowledgments

The research reported in this paper continues work that started with the author’s doctoral dissertation, written under the supervision of Professor Don Pigozzi at Iowa State. I acknowledge Don’s support and guidance. Thanks also to the colleagues Josep Maria Font and Ramon Jansana from Barcelona, whose work provided the basis and the inspiration for the current developments. Last, but very significantly, I would like to thank the anonymous referee for substantial contributions in adding some explaining remarks

and in pointing out several inaccuracies and errors in a previous version of the paper. I appreciated a lot his/her very constructive criticism and, largely due to him/her, I enjoyed very much doing the revision. I would like to mention, in particular, that, despite my best intentions, I had given an incomplete and/or misleading definition of the notion of *category of natural transformations* in a previous version of the paper. The recommendation of the anonymous referee on this point is gratefully acknowledged.

School of Mathematics and Computer Science
Lake Superior State University
650 W Easterday Avenue
Sault Sainte Marie MI 49783
gvoutsad@lssu.edu
<http://pigozzi.lssu.edu/WWW/main.html>