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Finite and Physical Modalities

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Abstract The logic K_f of the modalities of finite, devised to capture the notion of 'there exists a finite number of accessible worlds such that . . . is true', was introduced and axiomatized by Fattorosi. In this paper we enrich the logical framework of K_f: we give consistency properties and a tableau system (which yields the decidability) explicitly designed for K_f, and we introduce a shorter and more natural axiomatization. Moreover, we show the strong and suggestive relationship between K_f and the much older logic of the physical modalities of Burks.

1 Introduction

The logic K_f of the modalities of finite is an extension of K by the operator $\Diamond_f$ (and dual $\Box_f$) whose truth condition is 'there exists a finite number of accessible worlds such that . . . is true'. This logic was introduced in Fattorosi Barnaba [5], where an extension of K with three axioms was proved to be complete. K_f was devised by Fattorosi Barnaba to get a finitary syntactical treatment of the finite (with respect to the set of worlds which are accessible from a fixed one) in modal logic, in the strong sense of a system with formulas of finite length and a finite set of axioms.

In this paper we enrich the logical framework of K_f. First, in Section 4, we introduce the notion of *consistency property* for K_f, which extends the one of K (Fitting [6]) with a single clause. The main result we prove is the *satisfiability theorem*: if $\mathcal{C}$ is a consistency property for K_f and $\Delta \in \mathcal{C}$ is a finite set of formulas, then Δ is satisfiable (in K_f). Second, in Section 5, we introduce a tableau system for K_f, denoted by TK_f, which extends the one of K [6] with a single rule. We show that $\mathcal{C}_{\mathrm{TK}_f} = \{\Delta \text{ ; no tableau for } \Delta \text{ is closed}\}$ is a consistency property for K_f and we get, via the satisfiability theorem, the completeness of TK_f. This yields a decision procedure for K_f. Third, in Section 6, we give a shorter axiomatization of K_f, denoted by HK_f ('H' simply stands for 'Hilbert'), obtained by replacing two axioms

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of the original axiom system [5] with a single (and more natural) one. We show that $\mathcal{C}_{\mathrm{HK}_f} = \{\Delta \;;\; \Delta \nvdash_{\mathrm{HK}_f} \bot\}$ is a consistency property for K_f and we get, via the satisfiability theorem, the completeness of HK_f.

In Section 7 we introduce the logic of physical modalities and we show that this logic is equivalent to K_f. The logic of physical modalities aims at formalizing two distinct notions of necessity: the logical necessity, symbolized by $\Box$, and the physical necessity, symbolized by $\Box^f$. The basic relation between these two notions is that what is logically necessary is physically necessary too. This is formalized by the axiom link $\Box A \to \Box^f A$. Unfortunately there is no general agreement on the other principles which these two notions fulfill. Perhaps the most controversial axiom is $\mathrm{T}^f = \Box^f A \to A$. In his calculus ([1], [2]), Burks included this axiom in order to formalize the logic of physical modalities correctly, but other authors disagreed. The problem is that it is not clear how to understand the notion of 'physical necessity'. For example, if it was 'deducibility from scientific laws', then it could be argued against T^f (see Montague [7] and Pizzi [9]) and in favor of the system of Montague [7] where T^f is rejected.

Anyway, these are philosophical questions and, at least in the present paper, we can ignore them. Indeed, K_f is proved to be equivalent to the "minimal" logic of physical modalities, which only admits the axiom link. This logic, which we denote by K^f, is a bimodal version of K with the axiom link. That is, on the syntactic side it contains a copy of K for $\Box$ and another for $\Box^f$, plus the axiom link. On the semantic side, we have birelational models $\langle W, R, R^f, V\rangle$ such that $R^f \subseteq R$, and we state the truth condition of $\Diamond^f$ as 'there exists a physically accessible world such that . . . is true', where 'physically accessible' means accessible via R^f.

The proof of equivalence between K_f and K^f will be given in a few lines. It turns out that our axiomatization of K_f is nearly identical to the one of K^f. This will lead to an obvious correspondence between the formulas of the two systems which preserves validity: let A be a formula of K_f, define A^2 by replacing each occurrence of $\Diamond_f$ and $\Box_f$ in A with $\neg\Diamond^f$ and $\neg\Box^f$, respectively. It will be easy to show that A is valid in K_f if and only if A^2 is valid in K^f.

The equivalence between K_f and K^f can improve the understanding of these modalities. The notion of 'physically necessary' has inspired lots of mathematicians and philosophers; therefore our equivalence provides the modalities of finite with a richer mathematical and philosophical background. For instance, in Section 7, we suggest that one can provide $\Diamond_f$ with the intuitive meaning of 'it is not reproducible'. On the other hand, since the right understanding of the notion of 'physically necessary' has been controversial, our results provide a further source of inspiration to go deeper into this notion.

In Appendix A we give a practical application of the equivalence. We show that a question on one system can have an illuminating translation (via the map $A \mapsto A^2$) into the other. That is, we show that the axioms given in [5] correspond to well-known theorems of normal systems. This will yield a syntactic proof of the equivalence between our axiomatization of K_f and the original one introduced in [5].

2 The Logic K_f of the Modalities of Finite

The *language of* K_f, denoted by $\mathcal{L}(\mathrm{K}_f)$, contains a denumerable set of propositional variables, denoted by $\mathcal{V}(\mathrm{K}_f)$, the propositional constants $\top$, $\bot$, the truth functional connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, and the modal operators $\Diamond$, $\Box$, $\Diamond_f$, $\Box_f$. The set of *formulas of* K_f, $\mathcal{F}(\mathrm{K}_f)$, is defined inductively as usual. We use $p, q, \ldots$ to range over $\mathcal{V}(\mathrm{K}_f)$, $A, B, \ldots$ to range over $\mathcal{F}(\mathrm{K}_f)$, and $\Gamma, \Delta, \ldots, \Gamma_0, \Delta_0, \ldots$ to range over subsets of $\mathcal{F}(\mathrm{K}_f)$. With $A \leftrightarrow B$ we abbreviate $(A \rightarrow B) \wedge (B \rightarrow A)$.

A *model of* K_f is a triple (W, R, V), where W is a nonempty set, R is a binary relation on W, and V is a *valuation* of $\mathcal{V}(\mathrm{K}_f)$ on W. We use $\mathcal{M}$ to range over models of K_f. Fixed $\mathcal{M}$, we assume $\mathcal{M} = (W, R, V)$ and let $x, y, \ldots$ range over W.

The *truth relation* $\models^{\mathcal{M}}$ is defined as usual, plus the following clauses:

$$x \models^{\mathcal{M}} \Diamond_f A \quad \text{iff} \quad \left|\left\{y\ ;\ xRy \textit{ and } y \models^{\mathcal{M}} A\right\}\right| < \omega;$$
$$x \models^{\mathcal{M}} \Box_f A \quad \text{iff} \quad \left|\left\{y\ ;\ xRy \textit{ and } y \not\models^{\mathcal{M}} A\right\}\right| \geq \omega.$$

The *truth set of* A *in* $\mathcal{M}$ is $\|A\|^{\mathcal{M}} = \{x\ ;\ x \models^{\mathcal{M}} A\}$ and *with respect to* x is $\|A\|^{\mathcal{M}}_x = \{y\ ;\ xRy \textit{ and } y \models^{\mathcal{M}} A\}$. Let $\|\Delta\|^{\mathcal{M}} = \bigcap_{A\in\Delta} \|A\|^{\mathcal{M}}$ and $\|\Delta\|^{\mathcal{M}}_x = \bigcap_{A\in\Delta} \|A\|^{\mathcal{M}}_x$.

A (Δ) is *true in* $\mathcal{M}$ if $\|A\|^{\mathcal{M}} = W$ $(\|\Delta\|^{\mathcal{M}} = W)$ and *satisfiable in* $\mathcal{M}$ if $\|A\|^{\mathcal{M}} \neq \varnothing$ $(\|\Delta\|^{\mathcal{M}} \neq \varnothing)$. A (Δ) is *valid in* K_f if it is true in every model of K_f and *satisfiable in* K_f if it is satisfiable in some model of K_f. With $\mathbf{K_f}$ we denote the set of valid formulas.

3 Unifying Notation

We extend the unifying notation given in Smullyan [10] and Fitting [6] to include $\Diamond_f$ and $\Box_f$. α, β, π, and ν-formulas and their components $\alpha_1, \alpha_2, \beta_1, \beta_2, \pi_0$, and ν_0 are defined as in [6]. Moreover, f and i-formulas and their components are defined as follows.

f	f_0
$\Diamond_f A$	A
$\neg\Box_f A$	$\neg A$

i	i_0
$\neg\Diamond_f A$	A
$\Box_f A$	$\neg A$

We use $\alpha, \beta, \pi, \nu, f, i$ to range over formulas of the corresponding type. For each type, a corresponding truth condition holds. In particular, the clauses for f and i-formulas are the following:

$$x \models^{\mathcal{M}} f \quad \text{iff} \quad \left|\left\{y\ ;\ xRy \text{ and } y \models^{\mathcal{M}} f_0\right\}\right| < \omega;$$
$$x \models^{\mathcal{M}} i \quad \text{iff} \quad \left|\left\{y\ ;\ xRy \text{ and } y \models^{\mathcal{M}} i_0\right\}\right| \geq \omega.$$

The *length* of A, $l(A)$, is the number of occurrences of symbols in A, and the *modal length*, $lm(A)$, is the number of occurrences of $\Diamond$, $\Box$, $\Diamond_f$, and $\Box_f$. The satisfiability theorem will be proved by induction on $C(A) = (lm(A), l(A))$, the *complexity* of A, lexicographically ordered. It is easy to see that the complexity of the component(s) is less than the complexity of the formula. Notice also that $C(\neg f_0) < C(f)$.

Next, we introduce the notion of *T-closure* of a set of formulas. Let $\mathrm{Sub}(A)$ be the set of subformulas of A, define $\mathrm{Sub}(\Delta) = \bigcup\{\mathrm{Sub}(A)\ ;\ A \in \Delta\}$. The *T-closure of* Δ, denoted by $[\Delta]$, is defined as follows:

$$[\Delta] = \mathrm{Sub}(\Delta) \cup \{\neg B\ ;\ B \in \mathrm{Sub}(\Delta)\} \cup \{\neg\neg B\ ;\ B \in \mathrm{Sub}(\Delta)\}.$$

This notion is designed to fulfil the following property.

Proposition 3.1 *Letting λ be an α or β-formula, and letting μ be a π, ν, f, or i-formula, the following clauses are satisfied:*

(i) *if* $\lambda \in [\Delta]$ *then* $\lambda_1 \in [\Delta]$ *and* $\lambda_2 \in [\Delta]$*;*
(ii) *if* $\mu \in [\Delta]$ *then* $\mu_0 \in [\Delta]$ *and* $\neg\mu_0 \in [\Delta]$.

Proof We show only $\mu = f$; the other cases are similar. If $f \in [\Delta]$, then for some $B \in \mathrm{Sub}(\Delta)$, $f = B$ or $f = \neg B$ or $f = \neg\neg B$. If $f = \Diamond_f A$, the only possibility is $f = B$; hence $\Diamond_f A \in \mathrm{Sub}(\Delta)$, $f_0 = A \in \mathrm{Sub}(\Delta)$, and $\neg f_0 = \neg A \in [\Delta]$. If $f = \neg\Box_f A$ there are two possibilities: if $f = B$ then $\neg\Box_f A \in \mathrm{Sub}(\Delta)$, $A \in \mathrm{Sub}(\Delta)$, $f_0 = \neg A \in [\Delta]$, and $\neg f_0 = \neg\neg A \in [\Delta]$; if $f = \neg B$, then $\Box_f A = B \in \mathrm{Sub}(\Delta)$, $A \in \mathrm{Sub}(\Delta)$, $f_0 = \neg A \in [\Delta]$, and $\neg f_0 = \neg\neg A \in [\Delta]$. □

4 Consistency Properties for K_f

We introduce the following notation: if λ denotes α or β, and μ denotes π, ν, f, or i, then $\Delta^{\lambda} = \{\lambda\ ;\ \lambda \in \Delta\}$, $\Delta^{\mu} = \{\mu\ ;\ \mu \in \Delta\}$, $\Delta^{\mu_0} = \{\mu_0\ ;\ \mu \in \Delta\}$, and $\Delta^{\neg\mu_0} = \{\neg\mu_0\ ;\ \mu \in \Delta\}$. Moreover, with the string $X_0, X_1, \ldots, X_n$, where X_i is either a formula or a set of formulas, we denote the union $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_n$, where $\Delta_i = \{X_i\}$ if X_i is a formula and $\Delta_i = X_i$ otherwise.

Definition 4.1 A consistency property for K_f is a family $\mathcal{C}$ of sets of formulas that satisfies the following clauses: for every $\Delta \in \mathcal{C}$,

(c0) Δ is not closed (that is, $\bot \notin \Delta$, $\neg\top \notin \Delta$, and for every A, $A \notin \Delta$ or $\neg A \notin \Delta$);
(cα) if $\alpha \in \Delta$ then $\Delta, \alpha_1, \alpha_2 \in \mathcal{C}$;
(cβ) if $\beta \in \Delta$ then $\Delta, \beta_1 \in \mathcal{C}$ or $\Delta, \beta_2 \in \mathcal{C}$;
(cπ) if $\pi \in \Delta$ then $\Delta^{\nu_0}, \pi_0 \in \mathcal{C}$;
(ci) if $i \in \Delta$ and Δ' is a finite subset of Δ then $\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0} \in \mathcal{C}$.

Therefore, the notion of consistency property for K_f extends the one of K [6] with clause (ci).

Define $\mathcal{C}_{\mathcal{M}} = \{\Delta\ ;\ \Delta \text{ is satisfiable in } \mathcal{M}\}$ and $\mathcal{C}_{\mathrm{K}_f} = \{\Delta\ ;\ \Delta \text{ is satisfiable in } \mathrm{K}_f\}$. These families are consistency properties for K_f. It can be easily proved by virtue of the following lemma.

Lemma 4.2 *If Δ, i is satisfiable and Δ' is a finite subset of Δ then $\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0}$ is satisfiable.*

Proof Suppose that Δ, i is satisfiable. Then there exists $\mathcal{M}$ such that $\|\Delta, i\|^{\mathcal{M}} \neq \varnothing$. Let $x \in \|\Delta, i\|^{\mathcal{M}}$. Then $x \in \|\Delta\|^{\mathcal{M}}$ and $x \in \|i\|^{\mathcal{M}}$. Since $x \in \|\Delta\|^{\mathcal{M}}$, if $f \in \Delta$ then $|\|f_0\|_x^{\mathcal{M}}| < \omega$. Since $x \in \|i\|^{\mathcal{M}}$, we get $|\|i_0\|_x^{\mathcal{M}}| \geq \omega$. Let Δ' be a finite subset of Δ. If $f \in \Delta'$ then $|\|f_0\|_x^{\mathcal{M}}| < \omega$, and because Δ' is finite we get $|\bigcup_{f \in \Delta'} \|f_0\|_x^{\mathcal{M}}| < \omega$. Therefore $|\|i_0\|_x^{\mathcal{M}}| \geq \omega$ and $|\bigcup_{f \in \Delta'} \|f_0\|_x^{\mathcal{M}}| < \omega$. We conclude $\varnothing \neq \|i_0\|_x^{\mathcal{M}} - \bigcup_{f \in \Delta'} \|f_0\|_x^{\mathcal{M}} \subseteq \|\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0}\|^{\mathcal{M}}$, that is, $\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0}$ is satisfiable. □

We are going to prove the *satisfiability theorem*; that is, if $\mathcal{C}$ is a consistency property for K_f and Δ is a finite set of $\mathcal{C}$, then Δ is satisfiable in K_f. We give the proof in three parts.

4.1 The extension of a consistency property

Lemma 4.3 *Let $\mathcal{C}$ be a consistency property for K_f and let $\mathcal{C}'$ be the family of all subsets of elements of $\mathcal{C}$. Then $\mathcal{C}'$ is a consistency property for K_f; moreover, $\mathcal{C}'$ extends $\mathcal{C}$ and is closed under subsets.*

Proof That $\mathcal{C}'$ satisfies clauses (c0) – (cπ) is proved in [6]. Moreover, that $\mathcal{C} \subseteq \mathcal{C}'$ and $\mathcal{C}'$ is closed under subsets is clear. It remains to show clause (ci). Let $\Delta \in \mathcal{C}'$. Suppose that $i \in \Delta$ and let Δ' be a finite subset of Δ. By definition of $\mathcal{C}'$ there exists $\Gamma \in \mathcal{C}$ such that $\Delta \subseteq \Gamma$. Thus $i \in \Gamma$ and Δ' is a finite subset of Γ. By (ci) $\Gamma^{\nu_0}, i_0, \Delta'^{\neg f_0} \in \mathcal{C}$. Since $\Delta^{\nu_0} \subseteq \Gamma^{\nu_0}$, we have $\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0} \subseteq \Gamma^{\nu_0}, i_0, \Delta'^{\neg f_0} \in \mathcal{C}$; by definition of $\mathcal{C}'$ we get $\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0} \in \mathcal{C}'$. □

A family $\mathcal{C}$ of sets is said to be of *finite character* provided for every Δ, $\Delta \in \mathcal{C}$ if and only if each finite subset of Δ is in $\mathcal{C}$. If $\mathcal{C}$ is of finite character then each element of $\mathcal{C}$ has a maximal extension in $\mathcal{C}$ [6]. Maximal elements of a consistency property are important because they enjoy the following closure property: let Φ be such an element; if $\alpha \in \Phi$ then $\alpha_1 \in \Phi$ and $\alpha_2 \in \Phi$, and if $\beta \in \Phi$ then $\beta_1 \in \Phi$ or $\beta_2 \in \Phi$.

Lemma 4.4 *Let $\mathcal{C}'$ be a consistency property for K_f closed under subsets and let $\mathcal{C}''$ be the family of all sets Δ such that all finite subsets of Δ are in $\mathcal{C}'$. Then $\mathcal{C}''$ is a consistency property for K_f; moreover, $\mathcal{C}''$ extends $\mathcal{C}'$ and is of finite character.*

Proof That $\mathcal{C}''$ satisfies clauses (c0) – (cπ), extends $\mathcal{C}'$, and is of finite character is proved in [6]. It remains to show clause (ci). Suppose that $i \in \Delta \in \mathcal{C}''$. We show that $\Delta^{\nu_0}, i_0, \Delta^{\neg f_0} \in \mathcal{C}''$. We have to prove that every finite subset of $\Delta^{\nu_0}, i_0, \Delta^{\neg f_0}$ is in $\mathcal{C}'$. Let Γ be a finite subset of $\Delta^{\nu_0}, i_0, \Delta^{\neg f_0}$. Then there exists a finite subset $\hat{\Delta}$ of Δ such that $\Gamma \subseteq \hat{\Delta}^{\nu_0}, i_0, \hat{\Delta}^{\neg f_0}$; moreover, we can assume $i \in \hat{\Delta}$ (otherwise take $\hat{\Delta}, i$). By definition of $\mathcal{C}''$ $\hat{\Delta} \in \mathcal{C}'$, by (c$i$) $\hat{\Delta}^{\nu_0}, i_0, \hat{\Delta}^{\neg f_0} \in \mathcal{C}'$, and by closure under subsets $\Gamma \in \mathcal{C}'$. Now, let Δ' be a finite subset of Δ and let Γ be a finite subset of $\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0}$. Then Γ is a finite subset of $\Delta^{\nu_0}, i_0, \Delta^{\neg f_0} \in \mathcal{C}''$; hence $\Gamma \in \mathcal{C}'$. □

Theorem 4.5 *Any consistency property for K_f may be extended to a consistency property for K_f of finite character.*

Proof By Lemmas 4.3 and 4.4. □

Consistency properties $\mathcal{C}_{\mathrm{TK}_f}$, $\mathcal{C}_{\mathrm{K}_f}$, $\mathcal{C}_{\mathcal{M}}$, and $\mathcal{C}_{\mathrm{HK}_f}$ are all closed under subsets. Moreover, $\mathcal{C}_{\mathrm{TK}_f}$ is of finite character because a tableau for K_f is a finite tree. $\mathcal{C}_{\mathrm{HK}_f}$ is of finite character too, by compactness of the deducibility relation. In contrast to this, $\mathcal{C}_{\mathrm{K}_f}$ and $\mathcal{C}_{\mathcal{M}}$ are not of finite character because K_f is not *compact*: there exist sets of formulas that are unsatisfiable but all of whose finite subsets are satisfiable [5].

4.2 The restriction of a consistency property

Theorem 4.6 *Let $\mathcal{C}$ be a consistency property for K_f closed under subsets. The restriction $\mathcal{C}|_{[\Delta]} = \{\Gamma \cap [\Delta]\ ;\ \Gamma \in \mathcal{C}\}$ is a consistency property for K_f; moreover, if $\mathcal{C}$ is of finite character then $\mathcal{C}|_{[\Delta]}$ is of finite character too.*

Proof Since $\mathcal{C}$ is closed under subsets we have $(*)$ $\mathcal{C}|_{[\Delta]} \subseteq \mathcal{C}$. Moreover, we have that $(**)$ if $\Gamma \in \mathcal{C}$ and $\Gamma \subseteq [\Delta]$ then $\Gamma \in \mathcal{C}|_{[\Delta]}$. We show clause (c$i$). The other clauses are proved similarly. Let $\Gamma \in \mathcal{C}|_{[\Delta]}$. Suppose that $i \in \Gamma$ and let Γ' be a finite subset of Γ. By $(*)$ $\Gamma \in \mathcal{C}$, by (ci) $\Gamma^{\nu_0}, i_0, \Gamma'^{\neg f_0} \in \mathcal{C}$. In order to apply $(**)$ we have to prove that $\Gamma^{\nu_0}, i_0, \Gamma'^{\neg f_0} \subseteq [\Delta]$. That's easy: since $\Gamma \in \mathcal{C}|_{[\Delta]}$ we have $\Gamma \subseteq [\Delta]$ and by Lemma 3.1 we get that if $\nu \in \Gamma$ then $\nu \in [\Delta]$ and $\nu_0 \in [\Delta]$; since $i \in \Gamma$ we have $i \in [\Delta]$ and $i_0 \in [\Delta]$; since $\Gamma' \subseteq \Gamma$ we have that if $f \in \Gamma'$ then $f \in [\Delta]$ and

$\neg f_0 \in [\Delta]$. Now suppose that $\mathcal{C}$ is of finite character and assume that each finite subset of Γ is in $\mathcal{C}|_{[\Delta]}$. By $(*)$ each finite subset of Γ is in $\mathcal{C}$, by the finite character of $\mathcal{C}$, $\Gamma \in \mathcal{C}$. Moreover, $\Gamma \subseteq [\Delta]$; in fact, if $A \in \Gamma$ then $\{A\}$ is a finite subset of Γ and so $\{A\} \in \mathcal{C}|_{[\Delta]}$. By $(**)$ the thesis follows. □

4.3 The satisfiability theorem

Lemma 4.7 *Let $\mathcal{C}$ be a consistency property for K_f of finite character and let Δ be a finite set of $\mathcal{C}$, then there exists a countable set $W(\Delta)$ of occurrences of maximal elements of $\mathcal{C}$ that satisfies the following clauses:*

(i) *if $\pi \in \Delta$ then there exists $\Phi \in W(\Delta)$ such that $\pi_0 \in \Phi$;*
(ii) *if $\nu \in \Delta$ then for every $\Phi \in W(\Delta)$, $\nu_0 \in \Phi$;*
(iii) *if $f \in \Delta$ then $|\{\Phi \in W(\Delta)\ ;\ \neg f_0 \notin \Phi\}| < \omega$;*
(iv) *if $i \in \Delta$ then $|\{\Phi \in W(\Delta)\ ;\ i_0 \in \Phi\}| = \omega$.*

Proof $W(\Delta)$ is defined in three steps.

1. If $\pi \in \Delta$ then $\Delta^{\nu_0}, \pi_0 \in \mathcal{C}$ and there exists a maximal extension $\Delta^{\nu_0}, \pi_0 \subseteq \Phi_\pi \in \mathcal{C}$. Let $W_1(\Delta)$ be the set of all Φ_π with $\pi \in \Delta$.
2. If $i \in \Delta$ then (Δ is finite) $\Delta^{\nu_0}, i_0, \Delta^{\neg f_0} \in \mathcal{C}$ and there exists a maximal extension $\Delta^{\nu_0}, i_0, \Delta^{\neg f_0} \subseteq \Phi_i \in \mathcal{C}$. Let $W_2(\Delta)$ be the set consisting, for every $i \in \Delta$, of denumerably many occurrences of Φ_i.
3. Let $W(\Delta)$ be the set of all occurrences in $W_1(\Delta)$ and $W_2(\Delta)$.

Since Δ is finite, we have that $W_1(\Delta)$ is finite and $W_2(\Delta)$ is countable, so we get that $W(\Delta)$ is countable. The clauses of the theorem are easily proved:

(i) if $\pi \in \Delta$ then $\pi_0 \in \Phi_\pi \in W_1(\Delta)$;
(ii) if $\Phi \in W(\Delta)$ then $\Delta^{\nu_0} \subseteq \Phi$;
(iii) let $f \in \Delta$; if $\Phi \in W_2(\Delta)$ then $\neg f_0 \in \Phi$ so that $\{\Phi \in W(\Delta)\ ;\ \neg f_0 \notin \Phi\} = \{\Phi \in W_1(\Delta)\ ;\ \neg f_0 \notin \Phi\} \subseteq W_1(\Delta)$ which is a finite set;
(iv) if $i \in \Delta$ then $i_0 \in \Phi_i$ which occurs denumerably many times in $W_2(\Delta)$.

□

Theorem 4.8 (Satisfiability Theorem) *Let $\mathcal{C}$ be a consistency property for K_f. If Δ is a finite set of $\mathcal{C}$ then Δ is satisfiable in a denumerable model.*

Proof By Theorem 4.5 there exists a consistency property $\mathcal{C}'$ of finite character that extends $\mathcal{C}$. By Theorem 4.6 $\mathcal{C}'|_{[\Delta]}$ is a consistency property for K_f of finite character. Moreover, since Δ is finite so also is $[\Delta]$ and the same is true for every element of $\mathcal{C}'|_{[\Delta]}$. Therefore, for every $\Gamma \in \mathcal{C}'|_{[\Delta]}$ there exists a countable set $W(\Gamma)$ of occurrences of maximal elements of $\mathcal{C}'|_{[\Delta]}$ that satisfies clauses (i)–(iv) of Lemma 4.7.

We construct a model of K_f. Let Φ_0 be a maximal extension of $\Delta \in \mathcal{C}'|_{[\Delta]}$. Let us define $W_0, W_1, \ldots$ inductively by the clauses $W_0 = \{\Phi_0\}$ and $W_{n+1} = \dot{\bigcup}\{W(\Phi)\ ;\ \Phi \in W_n\}$. Let $W = \dot{\bigcup}\{W_n\ ;\ n < \omega\}$. W is a countable set of maximal elements of $\mathcal{C}'|_{[\Delta]}$. Let us define $\Phi R \Psi$ if and only if $\Psi \in W(\Phi)$ and $V(\Phi, p) = t$ if and only if $p \in \Phi$. $\mathcal{M} = (W, R, V)$ is a model of K_f.

We prove that if $\Phi \in W$ and $A \in \Phi$ then $\Phi \models^{\mathcal{M}} A$. The proof is by induction on $(\mathrm{lm}(A), l(A))$ lexicographically ordered.

Base We prove the statement for literal formulas:

(p) if $p \in \Phi$ then $V(\Phi, p) = t$ and so $\Phi \models^{\mathcal{M}} p$;

($\neg p$) if $\neg p \in \Phi$ then by (c0) $p \notin \Phi$ so that $V(\Phi, p) = f$ and $\Phi \models^{\mathcal{M}} \neg p$.

We skip the easy cases of the propositional constants ($\top, \neg\top, \bot, \neg\bot$).

Inductive Step

(α) If $\alpha \in \Phi$ then by maximality $\alpha_1 \in \Phi$ and $\alpha_2 \in \Phi$; by inductive hypothesis ($l(\alpha_i) < l(\alpha)$ and $\mathrm{lm}(\alpha_i) \leq \mathrm{lm}(\alpha)$) $\Phi \models^{\mathcal{M}} \alpha_1$ and $\Phi \models^{\mathcal{M}} \alpha_2$; hence $\Phi \models^{\mathcal{M}} \alpha$.

(β) Similar to the previous case.

(π) If $\pi \in \Phi$ then by clause (i) of Lemma 4.7 there exists $\Psi \in W(\Phi)$ such that $\pi_0 \in \Psi$; hence there exists $\Psi \in W$ such that $\Phi R \Psi$ and $\pi_0 \in \Psi$; by inductive hypothesis ($\mathrm{lm}(\pi_0) < \mathrm{lm}(\pi)$) there exists $\Psi \in W$ such that $\Phi R \Psi$ and $\Psi \models^{\mathcal{M}} \pi_0$; therefore $\Phi \models^{\mathcal{M}} \pi$.

(ν) If $\nu \in \Phi$ then by clause (ii) of Lemma 4.7, for every $\Psi \in W(\Phi)$, $\nu_0 \in \Psi$; hence for every $\Psi \in W$ such that $\Phi R \Psi$ $\nu_0 \in \Psi$; by inductive hypothesis ($\mathrm{lm}(\nu_0) < \mathrm{lm}(\nu)$) for every $\Psi \in W$ such that $\Phi R \Psi$ $\Psi \models^{\mathcal{M}} \nu_0$; therefore $\Phi \models^{\mathcal{M}} \nu$.

(f) If $f \in \Phi$ then by clause (iii) of Lemma 4.7 $|\{\Psi \in W(\Phi)\ ;\ \neg f_0 \notin \Psi\}| < \omega$; hence $|\{\Psi \in W\ ;\ \Phi R \Psi \text{ and } \neg f_0 \notin \Psi\}| < \omega$; we note that if $\neg f_0 \in \Psi$ then by inductive hypothesis ($\mathrm{lm}(\neg f_0) < \mathrm{lm}(f)$) $\Psi \models^{\mathcal{M}} \neg f_0$; that is, $\Psi \not\models^{\mathcal{M}} f_0$; therefore $\{\Psi \in W\ ;\ \Phi R \Psi \text{ and } \Psi \models^{\mathcal{M}} f_0\} \subseteq \{\Psi \in W\ ;\ \Phi R \Psi \text{ and } \neg f_0 \notin \Psi\}$ and this is a finite set; therefore $\Phi \models^{\mathcal{M}} f$.

(i) If $i \in \Phi$ then by clause (iv) of Lemma 4.7 $|\{\Psi \in W(\Phi)\ ;\ i_0 \in \Psi\}| = \omega$; hence $|\{\Psi \in W\ ;\ \Phi R \Psi \text{ and } i_0 \in \Psi\}| = \omega$; by inductive hypothesis ($\mathrm{lm}(i_0) < \mathrm{lm}(i)$) $|\{\Psi \in W\ ;\ \Phi R \Psi \text{ and } \Psi \models^{\mathcal{M}} i_0\}| = \omega$; therefore $\Phi \models^{\mathcal{M}} i$.

Therefore, each formula of Δ is true in Φ_0. □

Corollary 4.9 *If a finite set of formulas is satisfiable, then it is satisfiable in a denumerable model.*

Proof Let Δ be finite and satisfiable in K_f. Then Δ is a finite set of $\mathcal{C}_{\mathrm{K}_f}$. By Theorem 4.8 Δ is satisfiable in a denumerable model. □

5 Tableaux for $\mathbf{K}_f$

An *extension rule* is presented in the form

$$\frac{\Delta}{\Delta_0 \mid \Delta_1 \mid \ldots \mid \Delta_n}(r).$$

It is *trivial* if $\Delta = \Delta_0 = \cdots = \Delta_n$.

Let $\mathcal{T}, \mathcal{T}'$ be trees of sets of formulas. We say that $\mathcal{T}'$ is an *r-extension* of $\mathcal{T}$ if Δ occurs in $\mathcal{T}$ as a leaf and $\mathcal{T}'$ is obtained from $\mathcal{T}$ by extending such an occurrence with the $n+1$ children $\Delta_0, \Delta_1, \ldots, \Delta_n$.

Definition 5.1 A tableau for K_f is a tree of sets of formulas defined inductively by the following clauses:

(i) the tree with the only node Γ is a tableau for K_f;

(ii) if $\mathcal{T}$ is a tableau for K_f and $\mathcal{T}'$ is an α, β, π, or i-extension of $\mathcal{T}$ then $\mathcal{T}'$ is a tableau for K_f, where the extension rules are the following:

$$\frac{\Delta, \alpha}{\Delta, \alpha, \alpha_1, \alpha_2}(\alpha) \qquad \frac{\Delta, \beta}{\Delta, \beta, \beta_1 \mid \Delta, \beta, \beta_2}(\beta)$$

$$\frac{\Delta, \pi}{\Delta^{\nu_0}, \pi_0}(\pi) \qquad \frac{\Delta, i}{\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0}}(i), \qquad \text{where } \Delta' \text{ is a finite subset of } \Delta;$$

(iii) nothing else is a tableau for K_f.

Therefore, the tableaux of K_f extend those of K [6] with rule i.

We use $\mathcal{T}, \mathcal{T}', \ldots$ to range over tableaux for K_f. We say that $\mathcal{T}$ *is satisfiable* if some leaf of it is satisfiable and $\mathcal{T}$ *is closed* if each leaf of it is closed (where Δ is *closed* if $\bot \in \Delta$ or $\neg\top \in \Delta$ or there exists A such that $A \in \Delta$ and $\neg A \in \Delta$). Clearly, a closed tableau cannot be satisfiable. A *proof in* TK_f *of* A is a closed tableau for $\neg A$ (that is, with root $\{\neg A\}$). A formula is a *theorem of* TK_f if there exists a proof of it. With $\mathbf{TK_f}$ we denote the set of theorems of TK_f.

Let us show the *correctness* of TK_f, that is, $\mathbf{TK_f} \subseteq \mathbf{K_f}$.

Lemma 5.2 *If $\mathcal{T}$ is satisfiable and $\mathcal{T}'$ is an α, β, π, or i-extension of $\mathcal{T}$ then $\mathcal{T}'$ is satisfiable.*

Proof Cases α, β, and π are proved in [6]; Case i follows by Lemma 4.2. □

Theorem 5.3 *If Γ is satisfiable and $\mathcal{T}$ is a tableau for Γ then $\mathcal{T}$ is satisfiable.*

Proof By Lemma 5.2 and by induction on the complexity of a tableau. □

Theorem 5.4 (Correctness of $\mathbf{TK_f}$) *A theorem of* TK_f *is valid in* K_f.

Proof If A is not valid then $\neg A$ is satisfiable; by Theorem 5.3, a tableau for $\neg A$ is satisfiable and cannot be closed. □

The *completeness* of TK_f, that is, $\mathbf{K_f} \subseteq \mathbf{TK_f}$, follows from the Satisfiability Theorem 4.8. Define $\mathcal{C}_{TK_f} = \{\Delta\ ;\ \text{no tableau for } \Delta \text{ is closed}\}$; it is easy to prove that $\mathcal{C}_{TK_f}$ is a consistency property for K_f.

Theorem 5.5 (Completeness of $\mathbf{TK_f}$) *A valid formula in* K_f *is a theorem of* TK_f.

Proof If $A \notin \mathbf{TK_f}$ then $\{\neg A\} \in \mathcal{C}_{TK_f}$. By Theorem 4.8, $\neg A$ is satisfiable; hence $A \notin \mathbf{K_f}$. □

Thus, we have a complete tableau system for K_f, which yields a decision procedure.

Theorem 5.6 (Decidability of $\mathbf{K_f}$) K_f *is decidable.*

Proof If Γ is finite, define its complexity as $c(\Gamma) = \max\{\mathrm{lm}(A)\ ;\ A \in \Gamma\}$. It turns out that $c(\Gamma, \alpha) = c(\Gamma, \alpha, \alpha_1, \alpha_2)$, $c(\Gamma, \beta) = c(\Gamma, \beta, \beta_1) = c(\Gamma, \beta, \beta_1)$, $c(\Gamma, \pi) > c(\Gamma^{\nu_0}, \pi_0)$, and $c(\Gamma, i) > c(\Gamma^{\nu_0}, i_0, \Gamma^{\neg f_0})$. Let Δ be finite and let $\mathcal{T}$ be a tableau for Δ free of trivial extensions. Let $X = \Gamma_0, \Gamma_1, \ldots, \Gamma_n$ be a branch of $\mathcal{T}$. Consider the sequence $c(\Gamma_0), c(\Gamma_1), \ldots, c(\Gamma_n)$. The maximum number of π and i extensions that we can meet along X is $c([\Delta]) = c(\Delta)$ and, by absence of trivial extensions, the maximum number of α and β consecutive extensions that we can meet along X is $|[\Delta]^{\alpha}|+2|[\Delta]^{\beta}|$. Therefore, $n \leq c(\Delta)\cdot(|[\Delta]^{\alpha}|+2|[\Delta]^{\beta}|)$. Thus, the depth of a tableau for $\neg A$ free of trivial extensions is at most $\mathrm{lm}(A)\cdot(|[\neg A]^{\alpha}|+2|[\neg A]^{\beta}|)$. This provides a limit for the number of different tableaux for $\neg A$ free of trivial extensions. □

6 Axiomatization of $\mathbf{K}_f$

In this section we introduce our axiomatization of K_f (the original one given in [5] is reported in Appendix A), denoted by HK_f, and we prove its completeness.

HK_f is defined by the following axioms and rules:

PL	tautologies of $\mathcal{L}(\mathrm{K}_f)$		
K$\Box$	$\Box(A \to B) \to (\Box A \to \Box B)$		
K$\Box_f$	$\neg\Box_f(A \to B) \to (\neg\Box_f A \to \neg\Box_f B)$	MP	$\dfrac{A \quad A \to B}{B}$
AL	$\Box A \to \neg\Box_f A$	RN$\Box$	$\dfrac{A}{\Box A}$
D$\Diamond$	$\Diamond A \leftrightarrow \neg\Box\neg A$		
D$\Diamond_f$	$\Diamond_f A \leftrightarrow \neg\Box_f\neg A$		

A is a theorem of HK_f, in symbols $\vdash_{\mathrm{HK}_f} A$, if there exists a proof of A in HK_f. We denote by $\mathbf{HK}_f$ the set of theorems of HK_f. We adopt the notions of *deducibility in* HK_f *of A from* Δ, in symbols $\Delta \vdash_{\mathrm{HK}_f} A$, and *consistency in* HK_f *of* Δ, in symbols $\mathrm{Con}_{\mathrm{HK}_f}\Delta$, as defined in Chellas [4]. Notice that these notions are designed to allow the deduction theorem.

The *correctness* of HK_f is easily proved by induction on the length of a proof in HK_f.

Theorem 6.1 (Correctness of $\mathbf{HK}_f$) *A theorem of* HK_f *is valid in* K_f.

Let us prove the *completeness* of HK_f. We first state (without proof) some derived rules and theorems of HK_f.

Proposition 6.2 $\mathbf{HK}_f$ *is closed under*

RN$\Box_f$	$\dfrac{A}{\neg\Box_f A}$	RK$\Box_f$	$\dfrac{A_1 \wedge \cdots \wedge A_n \to A}{\neg\Box_f A_1 \wedge \cdots \wedge \neg\Box_f A_n \to \neg\Box_f A}$
REP	$\dfrac{B \leftrightarrow B'}{A \leftrightarrow A[B/B']}$	EXC	$\varphi A \leftrightarrow \neg\varphi^*\neg A,$

(where φ is any finite—possibly empty—sequence of occurrences of $\neg$, $\Box$, $\Diamond$, $\Box_f$, and $\Diamond_f$, and φ^* denotes the result of interchanging $\Box$ and $\Diamond$, $\Box_f$, and $\Diamond_f$, throughout φ.)

Lemma 6.3 $\mathcal{C}_{\mathrm{HK}_f} = \{\Delta\ ;\ \mathrm{Con}_{\mathrm{HK}_f}\Delta\}$ *is a consistency property for* K_f.

Proof Cases (c0)–(cπ) are standard; we prove (ci). Let $\Delta \in \mathcal{C}_{\mathrm{HK}_f}$. Suppose that $i \in \Delta$, Δ' is a finite subset of Δ, but $\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0} \notin \mathcal{C}_{\mathrm{HK}_f}$. Then $\Delta^{\nu_0}, i_0, \Delta'^{\neg f_0} \vdash_{\mathrm{HK}_f} \bot$. By compactness of the deducibility relation there exists a finite subset Δ'' of Δ such that $\Delta''^{\nu_0}, i_0, \Delta'^{\neg f_0} \vdash_{\mathrm{HK}_f} \bot$. Consider the following proof:

1.	$\Delta''^{v_0}, i_0, \Delta'^{\neg f_0} \vdash \bot$	Hypothesis
2.	$\vdash \bigwedge \Delta''^{v_0} \bigwedge \Delta'^{\neg f_0} \to \neg i_0$	1, Deduction Theorem, PL
3.	$\vdash \bigwedge \neg\Box_f \Delta''^{v_0} \bigwedge \neg\Box_f \Delta'^{\neg f_0} \to \neg\Box_f \neg i_0$	2, RK$\Box_f$
4.	$\vdash \Box v_0 \to \neg\Box_f v_0$	AL
5.	$\vdash \bigwedge \Box\Delta''^{v_0} \bigwedge \neg\Box_f \Delta'^{\neg f_0} \to \neg\Box_f \neg i_0$	3, 4, PL
6.	$\vdash \Box v_0 \leftrightarrow v$	Easy
7.	$\vdash \neg\Box_f \neg f_0 \leftrightarrow f$	Easy
8.	$\vdash \neg\Box_f \neg i_0 \leftrightarrow \neg i$	Easy
9.	$\vdash \bigwedge \Delta''^{v} \bigwedge \Delta'^{f} \to \neg i$	5, 6, 7, 8, 9, REP
10.	$\Delta''^{v}, \Delta'^{f} \vdash \neg i$	9, Deduction Theorem

By weakening $\Delta \vdash \neg i$. But $i \in \Delta$ implies $\Delta \vdash i$. Therefore $\mathrm{Con}_{\mathrm{HK}_f}\,\Delta$ and we get the contradiction $\Delta \notin \mathcal{C}_{\mathrm{HK}_f}$. □

Theorem 6.4 (Completeness of $\mathbf{HK}_f$) *A valid formula in* K_f *is a theorem of* HK_f.

Proof If $A \notin \mathbf{HK}_f$ then $\{\neg A\} \in \mathcal{C}_{\mathrm{HK}_f}$. By Lemma 6.3 and Theorem 4.8, $\neg A$ is satisfiable; hence $A \notin \mathbf{K}_f$. □

7 The Logic $\mathbf{K}^f$ of Physical Modalities

In this section we introduce the logic K^f of physical modalities, and we show the equivalence between K_f and K^f.

The *language of* K^f is obtained from $\mathcal{L}(\mathrm{K}_f)$ by replacing $\Diamond_f$ and $\Box_f$ with $\Diamond^f$ and $\Box^f$. A *model of* K^f is a 4-tuple $\mathcal{M} = (W, R, R^f, V)$ where (W, R, V) is a model of K_f and $R^f \subseteq R$. The *truth relation* $\models^{\mathcal{M}}$ is defined as usual, plus the following clauses:

$$
\begin{array}{lll}
x \models^{\mathcal{M}} \Diamond^f A & \text{iff} & \text{there exists } y \text{ such that } x R^f y \text{ and } y \models^{\mathcal{M} A}; \\
x \models^{\mathcal{M}} \Box^f A & \text{iff} & \text{for every } y,\ \text{if } x R^f y \text{ then } y \models^{\mathcal{M}} A.
\end{array}
$$

The notions of *truth* and *satisfiability in* $\mathcal{M}$ and those of *satisfiability* and *validity in* K^f are defined as before.

Let us introduce the axiom system of K^f, which we denote by HK^f. It is defined by PL (the tautologies of $\mathcal{L}(\mathrm{K}^f)$), D$\Diamond$, D$\Diamond^f$, K$\Box$, plus the following two axioms,

$$
\begin{array}{ll}
\mathrm{K}\Box^f & \Box^f(A \to B) \to (\Box^f A \to \Box^f B), \\
\mathrm{AL} & \Box A \to \Box^f A,
\end{array}
$$

and the rules MP and RN$\Box$.

The completeness of HK^f is a standard result; see, for instance, Carnielli and Pizzi [3].

Theorem 7.1 (Correctness and Completeness of $\mathbf{HK}^f$) *A formula is a theorem of* HK^f *if and only if it is valid in* K^f.

Now look at HK_f and HK^f. We can indeed say they are almost identical. The next step should be obvious and, as we claimed in the introduction, the proof of equivalence will follow easily. Let A be a formula of K_f. Define A^2 by replacing each occurrence of $\Diamond_f$ and $\Box_f$ in A with $\neg\Diamond^f$ and $\neg\Box^f$, respectively. This map is an invariant for theorems of our systems.

Theorem 7.2 *A is a theorem of* HK_f *if and only if A^2 is a theorem of* HK^f.

On the semantic side, by Theorems 7.1, 6.1, and 6.4, we get that a formula is valid (satisfiable) in K_f if and only if A^2 is valid (satisfiable) in K^f.

Thus, we have a simple truth-preserving translation between the formulas of the two systems.[1] This proves the equivalence between K_f and K^f. Notice that this is easy by virtue of our axiomatization of K_f, whereas the original axiomatization of [5] (reported in Appendix A) does not give us any clue of the map.

Our translation $A \mapsto A^2$ and the proved equivalence can give us a better understanding of the modalities we are dealing with. For instance, the intuitive meaning of $\Diamond_f A$ can be 'A is not reproducible', and Theorem 7.2 establishes that 'A is physically possible' if and only if 'A^{-2} is reproducible' (formally, $\Diamond^f A$ is satisfiable in K^f if and only if $\neg\Diamond_f A^{-2}$ is satisfiable in K_f, where A^{-2} is any formula B of K_f such that $B^2 \leftrightarrow A$ is valid in K^f). This idea can be supported as follows. If $\mathcal{M}$ is a model of K^f and $x \models^{\mathcal{M}} B$, then we can build a model $\mathcal{M}'$ of K_f such that $x \models^{\mathcal{M}'} B^{-2}$. The construction proceeds by induction on the complexity of A. Assume that $B = \Diamond^f A$. Then there is y such that $xR^f y$ and $y \models^{\mathcal{M}} A$; in $\mathcal{M}'$ we make ω-copies of y which are accessible from x. Then the construction proceeds; for instance, if $A = \Diamond C$, then there is z such that yRz and $z \models^{\mathcal{M}} C$; in $\mathcal{M}'$ we make a single copy of z accessible from y. We omit the long formal treatment. Intuitively, we may think of A as describing a phenomenon of nature, which is physically possible at world x, and we may think of y as that "portion" of x which contains the causes that determine A. Now think of a scientist who observes the phenomenon A and tries to distinguish its causes. He tries to isolate the factors that influence the course of A. If he succeeds he may build a copy of y and reproduce the phenomenon. The experiment can then be repeated. He builds another copy of y and reproduces A, and so on. A is reproducible. Formally, he builds up a model of $\neg\Diamond_f A^{-2}$.

Finally, consider again $T^f = \Box^f A \to A$, which intuitively states 'if A is physically necessary then A is true'. We said in the introduction that the system of Burks [1] and [2] extends K^f with T^f, whereas the system of Montague [7] (which also extends K^f) rejects this axiom. We do not want to discuss the legitimacy of this axiom; we only want to suggest that this axiom (and any other additional axiom) may be analyzed in the setting of the modalities of finite. That is, the counterpart of T^f in K_f is $T_f = \neg\Box_f A \to A$, which intuitively states 'if $\neg A$ is not reproducible then A'. Clearly, Theorem 7.2 holds for the extended systems $HK_f + T_f$ and $HK^f + T^f$.

Appendix A Equivalence between H and $\mathbf{HK_f}$: Syntactic Proof

Let us denote by H the original axiomatization of K_f given in [5]. H differs from HK_f for $K\Box_f$ and AL, in place of which there are

A1. $\Box(A \to B) \to (\Diamond_f B \to \Diamond_f A)$,
A2. $\Diamond_f A \wedge \Diamond_f B \to \Diamond_f (A \vee B)$,
A3. $\neg\Diamond A \to \Diamond_f A$.

We can say our axiomatization improves the original one: it is shorter and clearer. Of course, by Theorems 6.1, 6.4, and the completeness of H (proved in [5]), there are proofs of A1, A2, and A3 in HK_f, and there are proofs of $K\Box_f$ and AL in H. We have these proofs. We can therefore give a syntactic proof of the equivalence between our axiomatization and the original one.

Theorem A.1 $\mathbf{H} = \mathbf{HK_f}$. *Syntactic proof.*

Proof First, we show that $\mathbf{H} \subseteq \mathbf{HK}_{\boldsymbol{f}}$. A3 is easily proved by $\Box\neg A \to \neg\Box_f\neg A$ (AL) and EXC. The proof of A1 is the following.

1.	$(A \to B) \land \neg B \to \neg A$	PL
2.	$\neg\Box_f(A \to B) \land \neg\Box_f\neg B \to \neg\Box_f\neg A$	1, RK$\Box_f$
3.	$\neg\Box_f(A \to B) \to (\neg\Box_f\neg B \to \neg\Box_f\neg A)$	2, PL
4.	$\neg\Box_f(A \to B) \to (\Diamond_f B \to \Diamond_f A)$	3, EXC, REP
5.	$\Box(A \to B) \to \neg\Box_f(A \to B)$	AL
6.	$\Box(A \to B) \to (\Diamond_f B \to \Diamond_f A)$	4, 5, PL

The proof of A2 is the following.

1.	$\neg A \land \neg B \to \neg(A \lor B)$	PL
2.	$\neg\Box_f\neg A \land \neg\Box_f\neg B \to \neg\Box_f\neg(A \lor B)$	1, RK$\Box_f$
3.	$\Diamond_f A \land \Diamond_f B \to \Diamond_f(A \lor B)$	2, EXC, REP

Now we show that $\mathbf{HK}_{\boldsymbol{f}} \subseteq \mathbf{H}$. AL is easily proved by $\neg\Diamond\neg A \to \Diamond_f\neg A$ (A3) and EXC. The proof of K$\Box_f$ is the following.

1.	$\Diamond_f\neg(A \to B) \land \Diamond_f\neg A \to \Diamond_f(\neg(A \to B) \lor \neg A)$	A2
2.	$(\neg(A \to B) \lor \neg A) \leftrightarrow (B \to \neg A)$	PL
3.	$\Diamond_f\neg(A \to B) \land \Diamond_f\neg A \to \Diamond_f(B \to \neg A)$	1, 2, REP
4.	$\Box(\neg B \to (B \to \neg A)) \to (\Diamond_f(B \to \neg A) \to \Diamond_f\neg B)$	A1
5.	$\neg B \to (B \to \neg A)$	PL
6.	$\Box(\neg B \to (B \to \neg A))$	5, RN$\Box$
7.	$\Diamond_f(B \to \neg A) \to \Diamond_f\neg B$	4, 6, PL
8.	$\Diamond_f\neg(A \to B) \land \Diamond_f\neg A \to \Diamond_f\neg B$	3, 7, PL
9.	$\Diamond_f\neg(A \to B) \to (\Diamond_f\neg A \to \Diamond_f\neg B)$	8, PL
10.	$\neg\Box_f(A \to B) \to (\neg\Box_f A \to \neg\Box_f B)$	9, EXC, REP

$\Box$

Let us see how we got these proofs. In K$_f$, axioms A1, A2, and A3 capture three basic properties of the finite in modal logic (with respect to the set of worlds which are accessible from a fixed one). Via our map $A \mapsto A^2$, these axioms may be analyzed in K^f. Clearly, axiom A3 corresponds to AL, whereas A1 and A2 correspond (modulo AL and PL) to two well-known theorems of normal systems: K$\Diamond^f$ = $\Box^f(A \to B) \to (\Diamond^f A \to \Diamond^f B)$ and C$\Diamond^f$ = $\Diamond^f(A \lor B) \to (\Diamond^f A \lor \Diamond^f B)$. Of course, K$\Diamond^f$ and C$\Diamond^f$ are theorems of HK^f, since HK^f contains a copy of K with respect to these symbols. Thus, by virtue of Theorem 7.2, our proofs in HK$_f$ of A1 and A2 were obtained by standard proofs in HK^f of K$\Diamond^f$ and C$\Diamond^f$. This approach was also successfully applied in finding the proof of K$\Box_f$ in H.

Note

1. Notice that the map $A \mapsto A^2$ is not surjective (think of $\Diamond^f p$), but we can proceed as follows: let B be a formula of K^f; define B' by replacing each occurrence of $\Diamond^f$ and $\Box^f$ in B with $\neg\neg\Diamond^f$ and $\neg\neg\Box^f$, respectively. Then $B \leftrightarrow B'$ is valid in K^f and B' is in the range of $A \mapsto A^2$.

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