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Some Open Questions for Superatomic Boolean Algebras

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Abstract In connection with some known results on uncountable cardinal sequences for superatomic Boolean algebras, we shall describe some open questions for superatomic Boolean algebras concerning singular cardinals.

1 Superatomic Boolean Algebras

A *superatomic Boolean algebra* is a Boolean algebra in which every subalgebra is atomic. Suppose that B is a Boolean algebra. It is a well-known fact that B is superatomic if and only if its Stone space $S(B)$ is scattered. For every ordinal α, the α-*derivative* of $S(B)$ is defined by induction on α as follows. $S(B)^0 = S(B)$; if $\alpha = \beta + 1$, $S(B)^\alpha$ is the set of accumulation points of $S(B)^\beta$; and if α is a limit, $S(B)^\alpha = \bigcap\{ S(B)^\beta : \beta < \alpha \}$. Then, $S(B)$ is scattered if and only if $S(B)^\alpha = \varnothing$ for some α. This process can be transferred to the Boolean algebra B, obtaining in this way an increasing sequence of ideals I_α which are defined by transfinite induction as follows. We put $I_0 = \{0\}$; if $\alpha = \beta + 1$, $I_\alpha =$ the ideal generated by $I_\beta \cup \{ b \in B : b/I_\beta$ is an atom in $B/I_\beta \}$; and if α is a limit, $I_\alpha = \bigcup\{ I_\beta : \beta < \alpha \}$. Then B is superatomic if and only if there is an ordinal α such that $B = I_\alpha$. As usual, we abbreviate 'superatomic Boolean algebra' as 'sBA'.

2 Open Questions

Suppose that B is an sBA. We define *the height* of B by $\mathrm{ht}(B) =$ the least ordinal α such that B/I_α is finite. For every $\alpha < \mathrm{ht}(B)$, we denote the cardinality of the set of atoms of B/I_α by '$\mathrm{wd}_\alpha(B)$'. The *cardinal sequence* of B is then defined by $\mathrm{CS}(B) = \langle \mathrm{wd}_\alpha(B) : \alpha < \mathrm{ht}(B) \rangle$. If κ is an infinite cardinal and α is a nonzero

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ordinal, we say that B is a (κ, α)-sBA, if $\mathrm{ht}(B) = \alpha$ and $\mathrm{wd}_\beta(B) \leq \kappa$ for every $\beta < \alpha$.

The countable sequences of cardinals that arise as cardinal sequences of superatomic Boolean algebras were characterized by La Grange on the basis of ZFC set theory (see Koppelberg [6]). However, the situation becomes more complicated when we want to gain insight into uncountable cardinal sequences. In [3], it was shown by Juhász and Weiss that there is an (ω, α)-sBA for any $\alpha < \omega_2$. This result is, in a sense, the best possible, since it is known that the existence of an (ω, ω_2)-sBA is independent of ZFC (see Baumgartner and Shelah [1] and Just [4]). Yet it is not known whether there exists an (ω_1, ω_2)-sBA. Nevertheless, it was proved in Koepke and Martínez [5] that under $V = L$, there is a (κ, κ^+)-sBA for every regular cardinal κ. Also, it was shown in Martínez [7] that if κ is an infinite cardinal such that $\kappa^{<\kappa} = \kappa$, then there is a cardinal-preserving partial order that forces the existence of a (κ, α)-sBA for every $\alpha < \kappa^{++}$. It is not known whether these results can be extended to singular cardinals. So the following question appears to be open.

Question 2.1 *Let κ be a specific singular cardinal, for example, $\kappa = \aleph_\omega$. Is it consistent with* ZFC *that there exists a* (κ, κ^+)-sBA *?*

Another interesting class of superatomic Boolean algebras with an uncountable cardinal sequence is the class of the so called thin-thick Boolean algebras. Suppose that B is an sBA. Let κ be an uncountable cardinal. We say that B is *κ-thin-thick* if $\mathrm{ht}(B) = \kappa + 1$, $\mathrm{wd}_\alpha(B) \leq \kappa$ for every $\alpha < \kappa$, and $\mathrm{wd}_\kappa(B) \geq \kappa^+$. And we say that B is *κ-very thin-thick* if $\mathrm{ht}(B) = \kappa^+ + 1$, $\mathrm{wd}_\alpha(B) \leq \kappa$ for every $\alpha < \kappa^+$, and $\mathrm{wd}_{\kappa^+}(B) \geq \kappa^{++}$. It was shown by Baumgartner in [1] that the consistency of the existence of an inaccessible cardinal implies the consistency of the nonexistence of an ω_1-thin-thick sBA. However, it was shown by Weese in [9] that GCH implies the existence of a κ-thin-thick sBA for every infinite cardinal κ. In contrast with this result, it can be easily checked that under GCH we have that, for any infinite cardinal κ, there is no κ-very thin-thick sBA. Nevertheless, it was proved in [5] that if $\kappa^{<\kappa} = \kappa$ and there is a simplified $(\kappa^+, 1)$-morass, then there is a cardinal-preserving partial order that forces the existence of a κ-very thin-thick sBA. However, we do not know whether the cardinality assumption "$\kappa^{<\kappa} = \kappa$" can be omitted in this theorem. Thus the following problem is open.

Question 2.2 *Let κ be a specific singular cardinal. Is it consistent with* ZFC *that there exists a κ-very thin-thick superatomic Boolean algebra ?*

Also, the following general question seems to have some interest.

Question 2.3 *For a specific singular cardinal κ, what are the cardinal sequences $\theta = \langle \kappa_\alpha : \alpha < \kappa \rangle$ such that it is consistent with* ZFC *that there is a superatomic Boolean algebra B with* $\mathrm{CS}(B) = \theta$*?*

With respect to Question 2.3, we hope to prove in a future paper that if GCH holds and $\theta = \langle \kappa_\alpha : \alpha < \kappa \rangle$ is such that $\kappa_\alpha \geq \kappa$ for each $\alpha < \kappa$, then there is a cardinal-preserving partial order that forces the existence of an sBA B with $\mathrm{CS}(B) = \theta$.

On the other hand, in [8], Ruyle studied the notion of a PCF structure—a refinement of the notion of partial order introduced by Baumgartner in [1]—in which some conditions are added in order to reflect the fundamental properties of the PCF operator on $\{\omega_n : n \geq 1\}$. Then every PCF structure T has associated with it a superatomic Boolean algebra $B = B(T)$ which satisfies that $|B| = |T|$ and $\mathrm{wd}_\alpha(B) \leq |\alpha + \omega|$ for every $\alpha < \mathrm{ht}(B)$ (see [8]). The interest of the notion of a PCF structure lies in the fact that in the proof of Shelah's theorem that $2^{\aleph_\omega} < \aleph_{\omega_4}$ if $\aleph_\omega$ is a strong limit cardinal, it is shown by means of a combinatorial argument that there is no PCF structure of size $\geq \omega_4$ (see Burke and Magidor [2] and [8]). Then one could improve Shelah's bound on $2^{\aleph_\omega}$ to $\aleph_{\omega_3}$ by showing that in ZFC there is no PCF structure of size ω_3. In [8], it was proved by Ruyle that it is consistent with ZFC that there is a PCF structure T such that $B(T)$ is an (ω, ω_2)-sBA, and so we cannot hope to improve Shelah's bound on $2^{\aleph_\omega}$ to $\aleph_{\omega_2}$, at least by using the original argument given by Shelah. In [8], it was also proved that for any ordinal $\alpha < \omega_2$ an (ω, α)-sBA can be constructed in ZFC from a PCF structure. However, the following question remains open.

Question 2.4 *Is it consistent with* ZFC *that there is a PCF structure whose associated superatomic Boolean algebra is an* (ω_2, ω_3)-sBA *?*

If we could answer Question 2.4 in the affirmative, we could not hope to use PCF theory to improve Shelah's bound on $2^{\aleph_\omega}$ to $\aleph_{\omega_3}$.

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