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A Tail Club Guessing Ideal Can Be Saturated without Being a Restriction of the Nonstationary Ideal

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Abstract We outline the proof of the consistency that there exists a saturated tail club guessing ideal on ω_1 which is not a restriction of the nonstationary ideal. A new class of forcing notions and the forcing axiom for the class are introduced for this purpose.

1 Introduction

The notion of club guessing sequences was introduced by Shelah. See Kojman and Shelah [8] and Shelah [9] for some of the earliest appearances of the notion. It has proven to be a very nice tool in set theory. For example, its application to PCF theory yields surprising upper bounds of powers of singular cardinals. There are two flavors of club guessing sequences, tail club guessing sequences and fully club guessing sequences. Shelah also defined an ideal associated with each of the club guessing sequences. These ideals, called club guessing ideals, reflect the behavior of the corresponding sequences. Since these ideals are defined naturally from relatively simple parameters, their properties are of some interest. In particular, if one of these ideals is precipitous, then a generic embedding using the ideal leads to a nontrivial example of outside club guessing phenomena (studied in Džamonja and Shelah [2]).

One of the most important properties of ideals is the saturation. It has already been proven that a tail club guessing ideal can be saturated. Woodin proved in [11] from $\mathrm{ZF} + \mathrm{AD}$ that a tail club guessing ideal on ω_1 can be saturated, and Foreman and Komjáth showed in [4] that, from an almost huge cardinal, a tail club guessing ideal on any given uncountable regular cardinal below the almost huge cardinal can be saturated. However, in both results, the tail club guessing ideal is a restriction of the nonstationary ideal.

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Addressing this issue, the author proved in [7] that from a supercompact cardinal a tail club guessing ideal on ω_1 can be saturated without being a restriction of the nonstationary ideal. The goal of this article is to outline the proof of this result.

Our strategy parallels the proof of Foreman, Magidor, and Shelah in [3] that Martin's Maximum implies that the nonstationary ideal on ω_1 is saturated. However, since the Proper Forcing Axiom already implies that there is no club guessing sequence, we need to find a forcing axiom which is weak enough to be consistent with a tail club guessing sequence but strong enough to imply the saturation of the tail club guessing ideal. We also need a new notion which plays the role of $\mathcal{P}_\kappa(\lambda)$ in [3]. These objects are of intrinsic interest.

2 Definitions

Most of our notation is standard. The following may need to be mentioned, however. 'Lim' stands for the class of all limit ordinals. When X and Y are sets of ordinals, we say that X is *almost contained* in Y (denoted by '$X \subseteq^* Y$') if and only if for some $\zeta < \sup X$, $X \setminus \zeta \subseteq Y$. When X is a set and κ is a cardinal, $[X]^\kappa$ is the set of all subsets of X whose cardinality is κ. 'NS_κ' denotes the nonstationary ideal on an uncountable regular cardinal κ.

Definition 2.1 Let κ be an uncountable regular cardinal and S a stationary subset of κ which consists of limit ordinals. A tail club guessing sequence on S is a sequence $\vec{C} = \langle C_\gamma : \gamma \in S\rangle$ such that

1. each C_γ is an unbounded subset of γ, and
2. for every club subset D of κ, there exists a $\gamma \in S$ such that $C_\gamma \subseteq^* D$.

By 'a tail club guessing sequence on κ' we simply mean 'a tail club guessing sequence on $\kappa \cap \mathrm{Lim}$'.

It is easy to see that if $\vec{C}$ is a tail club guessing sequence on κ, then there exists a stationary set of γ such that $C_\gamma \subseteq^* D$. Shelah showed that when $\kappa \geq \aleph_2$, there always exists a tail club guessing sequence. This remarkable result has been used in PCF theory and model theory. In particular, the upper bounds of powers of singular cardinals established by Shelah rely on the existence of club guessing sequences.

Associated with club guessing sequences are club guessing filters and ideals.

Definition 2.2 Suppose that $\vec{C} = \langle C_\gamma : \gamma \in S\rangle$ is a tail club guessing sequence. Then the tail club guessing filter $\mathrm{TCG}(\vec{C})$ associated with $\vec{C}$ is defined as the filter on κ generated by the sets of the form $\{\gamma \in S : C_\gamma \subseteq^* D\}$ for a club subset D of κ. The tail club guessing ideal is the dual ideal of the tail club guessing filter.

In this paper, if F is a filter, the dual ideal of F is denoted by '$\breve{F}$'. It is shown by Shelah that $\mathrm{TCG}(\vec{C})$ is a κ-complete normal filter.

Recall that an ideal I on κ is saturated if and only if $\mathcal{P}(\kappa)/I$ is κ^+-cc. Saturated tail club guessing ideals have already been constructed by Foreman and Komjáth in [4] and by Woodin in [11]. However, it was not known whether there is a saturated tail club guessing ideal which is not a restriction of the nonstationary ideal. Our main theorem resolves this question.

Theorem 2.3 *It is consistent relative to the existence of a supercompact cardinal that there exists a tail club guessing sequence $\vec{C}$ on ω_1 such that $\mathrm{TCG}(\vec{C})$ is saturated but is not a restriction of NS_{ω_1}.*

The restriction on the order type of C_γ can be weakened, but we do not know if a similar argument can be applied to all tail club guessing sequences on ω_1. Moreover, we can reduce the large cardinal assumption to the existence of a Woodin cardinal. A complete proof will appear in [7].

In the following sections, we shall sketch the proof of Theorem 2.3. Fix a tail club guessing sequence $\vec{C} = \langle C_\gamma : \gamma \in \omega_1 \cap \mathrm{Lim} \rangle$ on ω_1 such that $\mathrm{ot}(C_\gamma) = \omega$ for every $\gamma \in \omega_1 \cap \mathrm{Lim}$. We will write 'TCG' for $\mathrm{TCG}(\vec{C})$.

3 Forcing Axioms

In [3], Foreman, Magidor, and Shelah showed that Martin's Maximum, denoted by 'MM', implies that the nonstationary ideal on ω_1 is saturated. The similarity of the nonstationary ideal and tail club guessing ideals suggests a forcing axiom approach to our question. However, it is well known that there is a proper poset which kills all club guessing sequences on ω_1, and hence the proper forcing axiom, denoted by 'PFA', implies that there is no club guessing sequence. Thus we need a weaker forcing axiom which is consistent with the existence of a tail club guessing sequence.

It is easy to see that ω-semiproper forcing preserves all club guessing sequences whose components are of order type ω. Hence MA(ω-proper) is a forcing axiom consistent with the existence of a tail club guessing sequence where MA(Γ) denotes Martin's Axiom for all posets with property Γ. Nonetheless, this is too weak to make the tail club guessing ideal saturated. Thus we need to find the right forcing axiom between these axioms. We shall first define the class of TCG-closed posets to state the forcing axiom to be used. Then, by modifying the proof in [3], we can obtain Lemma 3.3.

Definition 3.1 Let P be a poset and M a countable set. A decreasing sequence $\langle p_n : n < \omega \rangle$ in P is called an (N, P)-semigeneric sequence if and only if for every $n < \omega$, $p_n \in N$ and, for every P-name for a countable ordinal $\dot{\alpha}$ lying in N, there exists an $n < \omega$ such that p_n decides $\dot{\alpha}$.

Definition 3.2 A poset Q is said to be TCG-closed if and only if there exists a maximal antichain $\langle A_\alpha : \alpha < \eta \rangle$ in $\mathcal{P}(\omega_1)/\mathrm{TCG}$ such that if for some sufficiently large regular cardinal λ, N is a countable elementary substructure of $\langle H(\lambda), \in, \triangle \rangle$ with $\vec{C}, Q, \langle A_\alpha : \alpha < \eta \rangle \in N$, and $N \cap \omega_1 \in \bigcup_{\alpha \in N \cap \eta} A_\alpha$, then for every (N, Q)-semigeneric sequence $\langle q_n : n < \omega \rangle$, there exists a $q \in Q$ such that $q \leq q_n$ for every $n < \omega$.

Lemma 3.3 MA(*semiproper and* TCG-*closed*) *implies that* TCG *is saturated.*

Sketch of Proof Suppose that $\langle A_\alpha : \alpha < \eta \rangle$ is a maximal antichain in $\mathcal{P}(\omega_1)/\mathrm{TCG}$ where $\eta \geq \omega_2$. We shall define a two-stage iteration $P * \dot{Q}$ as follows. Let $P = \mathrm{Coll}(\omega_1, \eta)$. Let $G \subseteq P$ be generic. Set $f = \bigcup G$ and $X = \triangledown_{\alpha < \omega_1} A_{f(\alpha)}$. Define a poset Q by: $q \in Q$ if and only if q is a closed bounded subset of ω_1 such that for every $\gamma \in \omega_1 \cap \mathrm{Lim}$, $C_\gamma \subseteq^* q$ implies $\gamma \in X$. Q is ordered by extension. Let $\dot{Q}$ be a P-name for Q. We can show that $P * \dot{Q}$ is semiproper and TCG-closed. By MA(semiproper and TCG-closed), there exists a filter $G \subseteq P * \dot{Q}$ such that $f = \bigcup \{p : (p, \dot{q}) \in G \text{ for some } \dot{q}\}$ is a function on ω_1 and $D = \{\gamma < \omega_1 : p \Vdash \text{`}\gamma \in \dot{q}\text{' for some } (p, \dot{q}) \in G\}$ is club. Then, by definition, if $C_\gamma \subseteq^* D$ then $\gamma \in X$. It follows that X belongs to $\mathrm{TCG}(\vec{C})$. Let $\alpha \in \eta \setminus f''\omega_1$.

Then, since A_α is $\mathrm{TCG}(\vec{C})$-positive, so is $A_\alpha \cap X$. However, by the definition of X, $A_\alpha \cap X \in \mathrm{T\check{C}G}(\vec{C})$. It is a contradiction. □

We might have considered MA(TCG^+-preserving) which resembles MM more than our forcing axiom. However, this turns out to be inconsistent, which is proved in [7]. Though our axiom is strong enough for our purpose, the following question remains open.

Question 3.4 *What is the strongest forcing axiom consistent with the existence of a club guessing sequence on ω_1?*

4 $\vec{C}$-Semiproper Forcing

Now we would like to show the consistency of MA(semiproper and TCG-closed). We shall begin with a model having a supercompact cardinal. In [3], to prove the consistency of MM, Foreman, Magidor, and Shelah iteratively force with semiproper posets given by the Laver function. Although the iteration seemingly takes care of only semiproper posets, the resulting model satisfies MM. In [10], Shelah showed that the semiproper forcing axiom implies MM directly.

We follow the same strategy. However, instead of using semiproper posets, we shall use $\vec{C}$-semiproper posets, defined as follows.

Definition 4.1 A poset P is said to be $\vec{C}$-semiproper if and only if there exists a structure $\mathfrak{A}$ expanding $\langle H(\lambda), \in, \triangle\rangle$ such that for every ω-tower $\langle N_n : n < \omega\rangle$ of countable elementary substructures of $\mathfrak{A}$ with $\vec{C}, P \in N_0$ and every $p \in P \cap N_0$, if $C_\delta \setminus \zeta = \{N_n \cap \omega_1 : n < \omega\}$ for some $\zeta < \delta$, then there exists a $q \le p$ such that q is (N_n, P)-semigeneric for every $n < \omega$.

This is clearly a weakening of ω-semiproperness. The difference is that the tower $\langle N_n : n < \omega\rangle$ must satisfy another requirement that $C_\delta \setminus \zeta = \{N_n \cap \omega_1 : n < \omega\}$. Notice that while we are dealing with a tail club guessing ideal, an initial segment of the tower cannot be ignored. This is needed in the current version of the proof. We also note that $\vec{C}$-semiproperness does not imply semiproperness.

As in the case of MM, it is necessary to use the Revised Countable Support (RCS) iteration. The preservation theorem for $\vec{C}$-semiproper posets, expressed as follows, can be proved in the same way as the preservation theorem for semiproper posets.

Theorem 4.2 ([7]) *If $\langle P_\alpha, \dot{Q}_\beta : \beta < \alpha \le \eta\rangle$ is an* RCS *iteration such that for any $\alpha < \eta$, $P_\alpha \Vdash$ '$\dot{Q}_\alpha$ is $\vec{C}$-semiproper' and $P_{\alpha+1} \Vdash$ '$|P_\alpha| \le \aleph_1$', then P_η is $\vec{C}$-semiproper.*

Now we shall define an RCS-iteration $\langle P_\alpha, \dot{Q}_\beta : \beta < \alpha \le \kappa\rangle$ as follows. Let $f : \kappa \to V_\kappa$ be the Laver function. Suppose that we have defined P_α. If $f(\alpha)$ is not a P_α-name for a semiproper $\vec{C}$-semiproper poset, then let $\dot{Q}_\alpha$ be a P_α-name for the trivial poset. Otherwise, let $\dot{Q}_\alpha$ be a P_α-name for $f(\alpha) * \mathrm{Coll}(\omega_1, |P_\alpha * f(\alpha)|)$. Then by the preservation theorem for semiproper posets, P_κ is semiproper. By Theorem 4.2, P_κ is $\vec{C}$-semiproper.

Note that in V^{P_κ}, $\mathrm{T\check{C}G}(\vec{C}) \restriction S$ is not equal to $\mathrm{NS}_{\omega_1} \restriction S$ for every stationary subset S of ω_1. To see this, suppose otherwise. Then S is added by some initial segment P_α of P_κ. It is easy to see that a poset $\mathrm{Add}(\omega_1)$ to add one Cohen subset of ω_1 appears at some stage after α. It is known that $\mathrm{Add}(\omega_1)$ forces that every club guessing sequence $\vec{C}'$ on some stationary subset of ω_1 in the ground model $\mathrm{T\check{C}G}(\vec{C}')$ is not a restriction of NS_{ω_1}. Moreover, since every tail of P_κ is semiproper, and hence preserves stationary subsets of ω_1, this property is preserved. Therefore, the claim was proved.

5 $\tilde{S}(\mathfrak{A})$

The final step in proving our main theorem—Theorem 2.3—is to show that the model obtained in the last section satisfies MA(semiproper and TCG-closed). Again, we need to find the equivalent in our context to the nonstationary ideal on $\mathcal{P}_\kappa(\lambda)$.

The author introduced the following notion in [6].

Definition 5.1 Suppose that X is a transitive set such that $\langle X, \in\rangle$ satisfies ZF except the power set axiom, and $\vec{C} \in X$. Let $\mathfrak{A}$ be a structure on X expanding $\langle X, \in, \triangle, \vec{C}\rangle$. We define a subset $\tilde{S}(\mathfrak{A})$ of $[X]^{\aleph_0}$ by the following: $N \in \tilde{S}(\mathfrak{A})$ if and only if there exists a sequence $\langle N_\beta : \tilde{\beta} \leq \beta < \zeta\rangle$ such that

1. $\bigcup_{\tilde{\beta} \leq \beta < \zeta} N_\beta = N$;
2. for every β with $\tilde{\beta} \leq \beta < \zeta$, $\langle N_\gamma : \tilde{\beta} \leq \gamma \leq \beta\rangle \in N_{\beta+1}$;
3. $\langle N_\beta : \tilde{\beta} \leq \beta < \zeta\rangle$ is an increasing continuous sequence;
4. $\tilde{\beta} \subseteq N_{\tilde{\beta}}$;
5. for every β with $\tilde{\beta} \leq \beta < \zeta$, N_β is an elementary substructure of $\mathfrak{A}$; and
6. $C_{N \cap \omega_1} \setminus \tilde{\beta} \subseteq \{\beta \in [\tilde{\beta}, \zeta) : N_\beta \cap \omega_1 = \beta\}$.

The most typical example of X is $H(\lambda)$ for an uncountable regular cardinal λ. Let $\mathcal{F}(X)$ be the filter on $[X]^{\aleph_0}$ generated by all sets of the form $\tilde{S}(\mathfrak{A})$ for a structure $\mathfrak{A}$ expanding $\langle X, \in, \triangle, \vec{C}\rangle$. When $\mathfrak{A} = \langle X, \in, \triangle, \vec{C}\rangle$, we write $\tilde{S}(X)$ instead of $\tilde{S}(\mathfrak{A})$.

There is an equivalent definition of $\vec{C}$-semiproperness in terms of $\tilde{S}$-sequences. In addition, $\tilde{S}$ reflects the properties of $\vec{C}$ very well. Actually, in our proof, $\mathcal{F}(H(\lambda))$ plays the role of a club subset of $\mathcal{P}_\kappa(H(\lambda))$ in the proof of Foreman, Magidor, and Shelah. Though more technical difficulties must be overcome, it can be shown that $\mathcal{F}(H(\lambda))$ shares essential properties of the club filter of $\mathcal{P}_\kappa(H(\lambda))$. This lemma is an example.

Lemma 5.2 *Let $\mathfrak{A}$ be a structure expanding $\langle X, \in, \triangle\rangle$. Then for any $T \in \mathrm{TCG}(\vec{C})^+$, $\tilde{S}(\mathfrak{A}) \cap \{N : N \cap \omega_1 \in T\}$ is stationary.*

A careful translation of the argument of Foreman, Magidor, and Shelah in [3] proves Lemma 5.3 below and hence completes the proof that the tail club guessing ideal is saturated. An essential component of this translation is the fact that the filters $\mathcal{F}(H(\lambda))$ form a tower.

Lemma 5.3 *The model obtained in the last section satisfies* MA(*semiproper and* TCG-*closed*).

As we have mentioned at the last part of Section 4, $\mathrm{T\check{C}G}(\vec{C})$ is not a restriction of NS_{ω_1}. Thus the previous lemma completes the proof of the main theorem.

We conclude this section by pointing out that no restriction of NS_{ω_1} is saturated in the obtained model. For this, suppose that $\mathrm{NS}_{\omega_1} \restriction S$ for some stationary subset S of ω_1 is saturated. Now recall the following theorem proved by Baumgartner, Taylor, and Wagon in [1].

Theorem 5.4 ([1]) *If $I \subseteq J$ are κ-complete normal ideals on an uncountable regular cardinal κ and I is saturated, then $J \restriction X = I \restriction X$ for some J-positive subset X of κ.*

Since $\mathrm{NS}_{\omega_1} \restriction S \subseteq \mathrm{T\check{C}G} \restriction S$, by this theorem, there exists a TCG-positive set T such that $\mathrm{NS}_{\omega_1} \restriction T = \mathrm{T\check{C}G} \restriction T$. This contradicts the fact we proved in the last paragraph of Section 4.

6 Remarks and Open Questions

We are interested in the similarity and the difference between tail club guessing ideals and the nonstationary ideal. Our model provides one of the significant differences between them, that is, even when the tail club guessing ideal associated with $\vec{C}$ is saturated, the nonstationary ideal is not necessarily saturated. Nonetheless, we do not know whether an analogous result can be obtained for the precipitousness.

Question 6.1 *If a tail club guessing ideal on an uncountable regular cardinal κ is precipitous, then is the nonstationary ideal on κ always precipitous? Or vice versa?*

While the consistency of a precipitous tail club guessing ideal on any uncountable regular cardinal μ relative to a Woodin cardinal above μ is proved by the author in [6], Goldring showed in [5] that the nonstationary ideal on μ is also precipitous in the same model.

As a more general project, we wonder which theorems and arguments regarding the nonstationary ideals can be translated into the context of tail club guessing ideals. We hope that in the course of the development of this project, an essential difference between these ideals will be found.

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