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A Deontic Counterpart of Lewis's S1

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Abstract In this paper we investigate nonnormal modal systems in the vicinity of the Lewis system S1. It might be claimed that Lewis's modal systems (S1, S2, S3, S4, and S5) are the starting point of modern modal logics. However, our interests in the Lewis systems and their relatives are not (merely) historical. They possess certain syntactical features and their frames certain structural properties that are of interest to us. Our starting point is not S1, but a weaker logic $S1^0$ (S1 without the schema [T]). We extend it to $S1^0D$, which can be considered as a deontic counterpart of the alethic S1. Soundness and completeness of these systems are then demonstrated within a prenormal idiom. We conclude with some philosophical remarks on the interpretation of our deontic logic.

1 Introduction

It is an unsettled question why we should study deontic logics. Some no doubt study them seeking illumination of the structure of moral reasoning, though such light as they shed shifts toward the infrared. Some study them simply as they would study any modal logic, purely for the technical interest. For such people the label "deontic" applies simply in virtue of the absence of the principle [T]. Our shared view is that in the matter of the relationship between deontic logic and moral discourse, these are early days, too early to settle the question. But there is intrinsic technical interest in better understanding the relationship between alethic and deontic systems, and such an improved understanding ought to influence any assignment to deontic logic of a broader conceptual role. One or two illustrations will preview later points. We obtain the so-called Standard Deontic Logic (SDL) by replacing [T] with [D] in the system KT of Kripke's classic paper. The corresponding difference between alethic accessibility and deontic accessibility is that alethic accessibility is reflexive, deontic accessibility serial. But as we insist in introductory logic courses, we can existentially generalize on either occurrence of x in Rxx. Certainly seriality ($\exists y Rxy$) is one

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generalization but so is converse seriality ($\exists y Ryx$). Why, if we are merely looking for a weakening of reflexivity, should we consider the one but not the other? On its deontic reading the former yields [D], $\Box p \to \neg\Box\neg p$, which seems to say that there are no moral conflicts; the second yields the rule of denecessitation ($\vdash \Box\alpha \Longrightarrow \vdash \alpha$), which seems to say that the only logically obligatory acts are theorems.

A second illustration concerns the very starting point of modal logics. The system K is nowadays regarded as the base Kripkean system since it is the system corresponding to the class of all relational frames. But it also corresponds post facto to an interpretive intuition: that the set of necessities constitutes a theory, that is, a deductively closed set. To that system, we add either an alethic principle or a deontic one depending upon whether we seek an alethic or a deontic system. But that intuition is a happy consequence of a convenient semantic idiom rather than a desideratum that compels the idiom. An algebraically inspired intuition might have suggested to some that

$$\begin{array}{ll} \text{[Con]} & \neg\Box\bot \\ \text{[N]} & \Box\top \\ \text{[RM]} & \dfrac{\vdash \alpha \to \beta}{\vdash \Box\alpha \to \Box\beta} \end{array}$$

ought to form the base system. Such a starting point for alethic modal logic would have the happy consequence that $\Box$ and $\Diamond$ have precisely the same logic. So this would yield a genuinely modal system rather than a system of necessity. For systems specifically of necessity or specifically of possibility, different extensions would be required. But this system is already deontic in character, and, moreover, a system in which obligation and permissibility have the same logic.

Now one could find grounds for rejecting all of the principles of either system except perhaps for the consistency principle [Con]. One might reject K because it permits no distinction between [Con] and [D]. One could reject [RM] as too strong, and one could reject [N] on the grounds that all obligations ought to be shirkable. Of course this last intuition is not adequately satisfied by the elimination of [N]. We would require a positive principle that states as much, a principle such as

$$\text{[Anti-N]}\ \neg\Box\top.$$

The point of these remarks is to justify the study of deontic logics in the region of S1. A first step toward doing so is to establish a base from which to explore generalizations of these systems and enlargements of the class of model-structures. If we needed an excuse for such explorations, they need not be that the system itself has plausible deontic interpretations. By such a standard, there would be no justification for studying SDL, all of whose basic principles are deontically implausible. However, S1 is a system in which an obvious alternative to the standard weakening of reflexivity is natural. And it is a system for which an intriguing reinterpretation of nonnormality presents itself, not perhaps for immediate deontic liquidity, but for experimental modification.

This is the point at which to end our excuse-making. All such explorations are experimental. It is the notion that the study of deontic logics has got beyond the stage of primary research that is suspect, and probably illusory.

2 Lewis Systems and Their Relatives

Lemmon in [6] provides alternative axiomatizations of the Lewis systems S1 – S4 with a propositional logic basis. Before introducing Lemmon's axiomatizations, we list the axioms that we need in the following:

$$
\begin{array}{ll}
[\mathrm{X}] & \Box(p \to q) \to (\Box(q \to r) \to \Box(p \to r)), \\
[\mathrm{K}] & \Box(p \to q) \to (\Box p \to \Box q), \\
[\mathrm{K!}] & \Box(p \to q) \to \Box(\Box p \to \Box q), \\
[\mathrm{M}] & \Box(p \wedge q) \to (\Box p \wedge \Box q), \\
[\mathrm{C}] & (\Box p \wedge \Box q) \to \Box(p \wedge q), \\
[\mathrm{D}] & \Box p \to \Diamond p, \\
[\mathrm{T}] & \Box p \to p\,.
\end{array}
$$

The following rules of inference are also required:

$$
\begin{array}{llllll}
[\mathrm{RN}] & \dfrac{\vdash \alpha}{\vdash \Box\alpha} & [\mathrm{RN}^{\mathrm{Ax}}] & \dfrac{\alpha \in \text{Axioms}}{\vdash \Box\alpha} & [\mathrm{RN}^{\mathrm{PL}}] & \dfrac{\alpha \in \mathrm{PL}}{\vdash \Box\alpha} \\[2ex]
[\mathrm{RdN}] & \dfrac{\vdash \Box\alpha}{\vdash \alpha} & & & & \\[2ex]
[\mathrm{RM}] & \dfrac{\vdash \alpha \to \beta}{\vdash \Box\alpha \to \Box\beta} & [{}^{\mathrm{S}}\mathrm{RM}] & \dfrac{\vdash \alpha \prec \beta}{\vdash \Box\alpha \prec \Box\beta} & & \\[2ex]
[\mathrm{RE}] & \dfrac{\vdash \alpha \leftrightarrow \beta}{\vdash \Box\alpha \leftrightarrow \Box\beta} & [{}^{\mathrm{S}}\mathrm{RE}] & \dfrac{\vdash \alpha = \beta}{\vdash \Box\alpha = \Box\beta} & [\mathrm{RE}^{\mathrm{PL}}] & \dfrac{\alpha \leftrightarrow \beta \in \mathrm{PL}}{\vdash \Box\alpha \leftrightarrow \Box\beta} \\[2ex]
[\mathrm{Eq}] & \dfrac{\vdash \alpha \leftrightarrow \beta}{\vdash \gamma \leftrightarrow \gamma[\alpha/\beta]} & [{}^{\mathrm{S}}\mathrm{Eq}] & \dfrac{\vdash \alpha = \beta}{\vdash \gamma = \gamma[\alpha/\beta]} & [\mathrm{Eq}^{\mathrm{PL}}] & \dfrac{\alpha \leftrightarrow \beta \in \mathrm{PL}}{\vdash \gamma \leftrightarrow \gamma[\alpha/\beta]} \\[2ex]
[\mathrm{RRE}] & \dfrac{\vdash \alpha \leftrightarrow \beta, \vdash \gamma}{\vdash \gamma[\alpha/\beta]} & [\mathrm{RRSE}] & \dfrac{\vdash \alpha = \beta, \vdash \gamma}{\vdash \gamma[\alpha/\beta]} & [\mathrm{RRTE}] & \dfrac{\alpha \leftrightarrow \beta \in \mathrm{PL}, \vdash \gamma}{\vdash \gamma[\alpha/\beta]} \\[2ex]
[\mathrm{MP}] & \dfrac{\vdash \alpha \to \beta, \vdash \alpha}{\vdash \beta} & [{}^{\mathrm{S}}\mathrm{MP}] & \dfrac{\vdash \alpha \prec \beta, \vdash \alpha}{\vdash \beta}\,. & &
\end{array}
$$

Notation 2.1

1. $\alpha \prec \beta$ abbreviates $\Box(\alpha \to \beta)$.
2. $\alpha = \beta$ abbreviates $(\alpha \prec \beta) \wedge (\beta \prec \alpha)$.
3. $\gamma[\alpha/\beta]$ is the well-formed formula resulting from replacing some (possibly zero) occurrence of α in γ with an occurrence of β.
4. "RN" stands for the rule of necessitation, "RdN" for the rule of denecessitation, "RM" for the rule of monotonicity, "RE" for the rule of extensionality, "Eq" for equivalents, "RRE" ("RRSE", "RRTE") for the rule of replacement of (strict, tautologous) equivalents, "MP" for modus ponens, the prefix "S" for strict (it means that the rule is the strict version of the named rule), and the suffixes "Ax" and "PL" for axioms and propositional logic (it means that the named rule is restricted to axioms or tautologies).

Definition 2.2 Lewis systems S1 to S5 are as follows: (Non-Lewis systems S0.5, S0.9, and T are included for comparison.)

S0.5	:	PL,	$[\mathrm{RN}^{\mathrm{PL}}]$,		[K],	[T]							
S0.9	:	PL,	$[\mathrm{RN}^{\mathrm{Ax}}]$,	$[{}^{\mathrm{S}}\mathrm{RE}]$,	[K],	[T]							
S1	:	PL,	$[\mathrm{RN}^{\mathrm{Ax}}]$,	$[{}^{\mathrm{S}}\mathrm{Eq}]$,	[X],	[T]							
S2	:	PL,	$[\mathrm{RN}^{\mathrm{Ax}}]$,	$[{}^{\mathrm{S}}\mathrm{RM}]$,	[K],	[T]							
S3	:	PL,	$[\mathrm{RN}^{\mathrm{Ax}}]$,		[K!],	[T]	T	:	PL,	[RN],	[K],	[T]	
S4	:	PL,	[RN],		[K!],	[T]	S4	:	PL,	[RN],	[K],	[T],	[4]
							S5	:	PL,	[RN],	[K],	[T],	[5]

Here PL means the set of all tautologies together with the rules of modus ponens [MP] and uniform substitution for propositional variables [US]. Note that when listed as part of a system, tautologies are considered as axioms of the system in question.

In the following we adopt the Chellas and Segerberg axiomatization of $\mathrm{S1}^0$ in [1] with some modifications.[1]

Definition 2.3 The systems $\mathrm{S1}^0$, $\mathrm{S1}^0\mathrm{D}$, and $\mathrm{S1}^0\mathrm{T}$ are as follows.

$\mathrm{S1}^0$	:	PL,	$[\mathrm{RN}^{\mathrm{Ax}}]$,	[RdN],	$[{}^{\mathrm{S}}\mathrm{Eq}]$,	[X]	
$\mathrm{S1}^0\mathrm{D}$	:	PL,	$[\mathrm{RN}^{\mathrm{Ax}}]$,	[RdN],	$[{}^{\mathrm{S}}\mathrm{Eq}]$,	[X],	[D]
$\mathrm{S1}^0\mathrm{T}$ (S1)	:	PL,	$[\mathrm{RN}^{\mathrm{Ax}}]$,		$[{}^{\mathrm{S}}\mathrm{Eq}]$,	[X],	[T]

By "1^0 system", we mean a system that includes PL and provides $[\mathrm{RN}^{\mathrm{Ax}}]$, [RdN], $[{}^{\mathrm{S}}\mathrm{Eq}]$, and [X]. Thus $\mathrm{S1}^0$ is the smallest 1^0 system. (Note that $\mathrm{S1}^0\mathrm{T}$ is just S1.)

3 Prenormal Idiom

Cresswell in [2] and [3] provides a semantic analysis for S1. The models he uses combine the Kripkean style binary relational models with the neighborhood models introduced by Montague and Scott. In Sections 4 and 5, we extend his method to analyze $\mathrm{S1}^0$ and its extension $\mathrm{S1}^0\mathrm{D}$. However, we do this within a "prenormal idiom" based on a recent recast and generalization of Cresswell's semantics for S1 by Chellas and Segerberg [1]. (Chellas and Segerberg develop the semantics to study a class of logics which they call "prenormal logics"; hence we call their semantics "prenormal idiom" although the name is not used by them.)

Definition 3.1 A prenormal frame $\mathfrak{F}$ is an ordered quintuple $\langle U, N, Q, R, S\rangle$ where

1. U (the universe of the frame) is a nonempty set of points;
2. N(the set of normal points) and Q (the set of nonnormal or queer points) are disjoint subsets of U that exhaust it (i.e., $N \cap Q = \varnothing$ and $N \cup Q = U$);
3. R is a binary relation in $N \times U$;
4. $S : Q \to \wp(\wp(U))$ is a neighborhood function subject to the condition that for every $x \in Q$, $U \notin S(x)$.

Definition 3.2 Let $\mathfrak{F} = \langle U, N, Q, R, S\rangle$ be a prenormal frame. A model $\mathfrak{M}$ on $\mathfrak{F}$ is an ordered pair $\langle \mathfrak{F}, V\rangle$ where $V : \mathtt{At} \to \wp(U)$ is a function which maps each atom (of the propositional modal language) to a set of points of U.

Definition 3.3 The prenormal idiom $\mathscr{P}$ is the ordered triple $\langle \mathtt{L}, \mathfrak{C}, \mathbb{T}\rangle$ where $\mathtt{L}$ is the propositional modal language, $\mathfrak{C}$ is the class of all prenormal frames, and $\mathbb{T}$ (truth in the idiom) is defined recursively on the set of well-formed formulas in accordance with the following truth conditions for $\Box\alpha$ (the truth conditions for atoms and propositional connectives are as usual):

$$\text{For } x \in N : \models_x^{\mathfrak{M}} \Box\alpha \iff (\forall y \in U, Rxy \Longrightarrow \models_y^{\mathfrak{M}} \alpha);$$
$$\text{For } x \in Q : \models_x^{\mathfrak{M}} \Box\alpha \iff \|\alpha\|^{\mathfrak{M}} \in S(x).$$

Notation 3.4 $\|\alpha\|^{\mathfrak{M}}$ is $\{x \in U | \models_x^{\mathfrak{M}} \alpha\}$, the truth set of α in $\mathfrak{M}$.

In this paper, we are interested in a restricted sense of validity: a formula is valid (in our restricted sense) on a prenormal frame if and only if it is true at every normal point in every model on that frame. Following Chellas and Segerberg in [1], we call validity relativized to normal points "Lewis-validity". Formally, we have the following definitions.

Definition 3.5 Let $\mathfrak{F} = \langle U, N, Q, R, S\rangle$ be a prenormal frame. Then a well-formed formula α is said to be Lewis-valid on $\mathfrak{F}$ ($\mathfrak{F} \models_{\text{Lew}} \alpha$) if and only if for every model $\mathfrak{M}$ on $\mathfrak{F}$ and $x \in N$, $\models_x^{\mathfrak{M}} \alpha$.

Definition 3.6 Let $\mathfrak{D}$ be a class of prenormal frames. Then a well-formed formula α is said to be Lewis-valid on $\mathfrak{D}(\mathfrak{D} \models_{\text{Lew}} \alpha)$ if and only if for every $\mathfrak{F} \in \mathfrak{D}$, $\mathfrak{F} \models_{\text{Lew}} \alpha$.

The notions of soundness, completeness, and determination of a system with respect to a class of prenormal frames can be defined in terms of Lewis-validity. As in the case of validity, we label them "Lewis-soundness", "Lewis-completeness", and "Lewis-determination".

4 A Semantics for $\mathbf{S1^0}$

In this section, we show that the system $S1^0$ is Lewis-determined with respect to a certain class of prenormal frames (which we call "1^0 frames").

Definition 4.1 A 1^0 frame is a prenormal frame $\mathfrak{F} = \langle U, N, Q, R, S\rangle$ which satisfies the following conditions:

1. $N \neq \varnothing$;
2. $\forall x \in U, \exists y \in N : Ryx$;
3. $\forall x \in Q, \forall a, b \subseteq U,\ a, b \in S(x) \Longrightarrow a \cup b \neq U$.

Theorem 4.2 *The system $S1^0$ is Lewis-sound with respect to the class $\mathfrak{C}_{1^0}$ of all 1^0 frames.*

Proof Clearly all the tautologies and [X] are true at every normal point in any model on any 1^0 frame. Thus they are Lewis-valid in $\mathfrak{C}_{1^0}$. For the next part of the proof, observe that they are also true at every queer point in any model on any 1^0 frame. The case for tautologies is obvious. For [X], note that for any 1^0 model $\mathfrak{M} = \langle U, N, Q, R, S, V\rangle$, $\|p \to q\|^{\mathfrak{M}} \cup \|q \to r\|^{\mathfrak{M}} = U$. Thus for any $x \in Q$, it is not the case that both $\|p \to q\|^{\mathfrak{M}} \in S(x)$ and $\|q \to r\|^{\mathfrak{M}} \in S(x)$ (by condition (3) in Definition 4.1). Thus [X] is trivially true at every queer point.

For [RN^{Ax}] Note that any axiom α (a tautology or [X]) is true at every point, normal or queer, in any model on any 1^0 frame. Thus its necessitation $\Box\alpha$ is true at every normal point in any such model. In other words, $\mathfrak{C}_{1^0} \models_{\text{Lew}} \Box\alpha$.

For [RdN] Assume that $\mathfrak{C}_{1^0} \models_{\mathrm{Lew}} \Box\alpha$, that is, $\Box\alpha$ is true at every normal point in any model on any 1^0 frame. Then $\mathfrak{C}_{1^0} \models_{\mathrm{Lew}} \alpha$ since every normal point in any such model has a normal predecessor (by condition (2) of Definition 4.1).

For [$^{\mathrm{S}}$Eq] Assume that $\mathfrak{C}_{1^0} \models_{\mathrm{Lew}} \alpha = \beta$ (and show that $\mathfrak{C}_{1^0} \models_{\mathrm{Lew}} \gamma = \gamma[\alpha/\beta]$). Let x be a point (normal or queer) in any model $\mathfrak{M}$ on any 1^0 frame. From the assumption, $\Box(\alpha \leftrightarrow \beta)$ is true at every normal point, and, by condition (2) of Definition 4.1, x has a normal predecessor. Thus $\models_x^{\mathfrak{M}} \alpha \leftrightarrow \beta$. Then the following hold:

$$\models_x^{\mathfrak{M}} \neg\alpha \leftrightarrow \neg\beta \qquad \models_x^{\mathfrak{M}} (\alpha \to \delta) \leftrightarrow (\beta \to \delta)$$
$$\models_x^{\mathfrak{M}} \Box\alpha \leftrightarrow \Box\beta \qquad \models_x^{\mathfrak{M}} (\delta \to \alpha) \leftrightarrow (\delta \to \beta).$$

Thus,

$$\mathfrak{C}_{1^0} \models_{\mathrm{Lew}} \neg\alpha = \neg\beta \qquad \mathfrak{C}_{1^0} \models_{\mathrm{Lew}} (\alpha \to \delta) = (\beta \to \delta)$$
$$\mathfrak{C}_{1^0} \models_{\mathrm{Lew}} \Box\alpha = \Box\beta \qquad \mathfrak{C}_{1^0} \models_{\mathrm{Lew}} (\delta \to \alpha) = (\delta \to \beta).$$

From the above we can conclude that $\mathfrak{C}_{1^0} \models_{\mathrm{Lew}} \gamma = \gamma[\alpha/\beta]$. □

In proving the completeness of 1^0 systems, we adopt the Henkin style of completeness proof. The strategy is as follows: for any 1^0 system L, we define a prenormal frame and model called the L-canonical frame and L-canonical model (the set of normal points of the frame and model is the set of all the maximal L-consistent sets of well-formed formulas), and show that

(A) a well-formed formula is true at a point in the canonical model if and only if it is a member of that point (the fundamental theorem for 1^0 systems), and
(B) the canonical frame is in a class $\mathfrak{C}$ of prenormal frames.

From the above results we can argue that any non-L-theorem is Lewis-invalid in class $\mathfrak{C}$ (since any such well-formed formula is absent from some normal point of the L-canonical model), and so any well-formed formula Lewis-valid in $\mathfrak{C}$ is an L-theorem. To demonstrate (A) and (B), we initially derive several theorems and rules of inference for a 1^0 system and demonstrate several general propositions about 1^0 canonical frames.

Lemma 4.3 *Let* L *be a system including* PL*. Then the following hold:*

1. *in the presence of* [RdN]*, if* L *has* [$^{\mathrm{S}}$Eq]*, then it also has* [RRSE]*;*
2. *in the presence of* [$\mathrm{RN}^{\mathrm{PL}}$]*, if* L *has* [RRSE]*, then it also has* [RRTE]*;*
3. *in the presence of* [RRTE]*, if* L *has* [X]*, then it also has* [K]*;*
4. *in the presence of* [$\mathrm{RN}^{\mathrm{PL}}$]*, if* L *has* [K]*, then it also has* [M] *and* [C]*.*

Proof We assume that in each case L has the rule(s) and/or schema stipulated.

For (1) To show that L has [RRSE], assume that $\vdash \alpha = \beta$ and $\vdash \gamma$. By [$^{\mathrm{S}}$Eq], $\vdash \gamma = \gamma[\alpha/\beta]$, and so by PL, $\vdash \Box(\gamma \to \gamma[\alpha/\beta])$. Then by [RdN], $\vdash \gamma \to \gamma[\alpha/\beta]$. Then by [MP], $\vdash \gamma[\alpha/\beta]$.

For (2) To show that L has [RRTE], assume that $\alpha \leftrightarrow \beta \in \mathrm{PL}$ and $\vdash \gamma$. Then by [$\mathrm{RN}^{\mathrm{PL}}$], $\vdash \alpha = \beta$. Then by [RRSE], $\vdash \gamma[\alpha/\beta]$.

For (3) To show that L has [K], note that $\vdash \Box(p \to q) \to (\Box(q \to r) \to \Box(p \to r))$. Then by [US], $\vdash \Box(\top \to p) \to (\Box(p \to q) \to \Box(\top \to q))$. But the well-formed formulas $(\top \to p)$ and p are tautologous equivalents, and so are $(\top \to q)$ and q. Thus by [RRTE], $\vdash \Box p \to (\Box(p \to q) \to \Box q)$.

For (4) By PL and $[\mathrm{RN}^{\mathrm{PL}}]$, $\vdash \Box(p \wedge q \to p)$. Then by [K], $\vdash \Box(p \wedge q) \to \Box p$. Similarly, $\vdash \Box(p \wedge q) \to \Box q$. Thus by PL, we have [M] $\Box(p \wedge q) \to \Box p \wedge \Box q$ as an L-theorem.

To obtain [C], note that by PL and $[\mathrm{RN}^{\mathrm{PL}}]$, $\vdash \Box(p \to (q \to p \wedge q))$. Then by successive use of [K], $\vdash \Box p \to (\Box q \to \Box(p \wedge q))$, that is, the schema [C] is an L-theorem. $\Box$

Since any 1^0 system includes PL and provides the rules [RdN], $[{}^{\mathrm{S}}\mathrm{Eq}]$, $[\mathrm{RN}^{\mathrm{Ax}}]$ (hence $[\mathrm{RN}^{\mathrm{PL}}]$), and the schema [X], we can help ourselves to all of the rules and schemata mentioned in Lemma 4.3 when proving rules and theorems for any such system.

Lemma 4.4 *A* 1^0 *system has the following rules:*

$$(1)\quad \frac{\vdash \Box\alpha}{\vdash \Box\top \prec \alpha}\,;$$

$$(2)\quad \frac{\vdash \Box\alpha}{\vdash \Box\top \prec \Box\alpha}\,;$$

$$(3)\quad \frac{\vdash \alpha}{\vdash \Box\top \prec \alpha}\,;$$

$$(4)\quad \frac{\vdash \alpha, \vdash \neg\Box\top \prec \alpha}{\vdash \Box\alpha}\,.$$

Proof For (1) Assume that $\vdash \Box\alpha$. Note that $\alpha \leftrightarrow (\top \to \alpha) \in \mathrm{PL}$. Thus by $[\mathrm{RN}^{\mathrm{PL}}]$, $\vdash \alpha = (\top \to \alpha)$. Then by $[{}^{\mathrm{S}}\mathrm{Eq}]$, $\vdash \Box\alpha = \Box(\top \to \alpha)$, and so $\vdash \Box(\Box\alpha \to \Box(\top \to \alpha))$. Then by [RdN], $\vdash \Box\alpha \to \Box(\top \to \alpha)$, and so by assumption together with [MP], $\vdash \Box(\top \to \alpha)$. But $\vdash \Box(\Box\top \to \top)$ (by applying $[\mathrm{RN}^{\mathrm{PL}}]$ to the tautology $\Box\top \to \top$). Thus by [X], $\vdash \Box(\Box\top \to \alpha)$, that is, $\vdash \Box\top \prec \alpha$.

For (2) Assume that $\vdash \Box\alpha$. Then $\vdash \Box(\top \to \alpha)$ (as we have shown when proving (1)). But $\vdash \Box(\alpha \to \top)$ (by applying $[\mathrm{RN}^{\mathrm{PL}}]$ to the tautology $\alpha \to \top$). Thus $\vdash \top = \alpha$, and so by $[{}^{\mathrm{S}}\mathrm{Eq}]$, $\vdash \Box\top = \Box\alpha$, whence we conclude that $\vdash \Box\top \prec \Box\alpha$.

For (3) Assume that $\vdash \alpha$, that is, α has a proof. We show by induction on the lines of the proof of α that $\vdash \Box\top \prec \alpha_k$, for every line α_k of the proof.

For α_1, it is an axiom. Thus by $[\mathrm{RN}^{\mathrm{Ax}}]$, $\vdash \Box\alpha_1$. Then by (1), $\vdash \Box\top \prec \alpha_1$.

For the induction step, the case of α_k being an axiom is the same as α_1. The other cases are α_k being obtained from previous line(s) by an application of one of the following rules: $[\mathrm{RN}^{\mathrm{Ax}}]$, [RdN], $[{}^{\mathrm{S}}\mathrm{Eq}]$, [MP]. In the following we show each case in turn.

Case 1 If $[\mathrm{RN}^{\mathrm{Ax}}]$, α_k is $\Box\alpha_i (i < k)$ and α_i is an axiom. Then by $[\mathrm{RN}^{\mathrm{Ax}}]$, $\vdash \Box\alpha_i$, and so by (2), $\vdash \Box\top \prec \Box\alpha_i$, that is, $\vdash \Box\top \prec \alpha_k$.

Case 2 If [RdN], there is a previous line $\Box\alpha_k$. By I.H., $\vdash \Box\top \prec \Box\alpha_k$. Then by [RdN], $\vdash \Box\top \to \Box\alpha_k$. Since $\vdash \Box\top$ (by $[\mathrm{RN}^{\mathrm{PL}}]$), $\vdash \Box\alpha_k$. Then by (1), $\vdash \Box\top \prec \alpha_k$.

Case 3 If $[{}^{\mathrm{S}}\mathrm{Eq}]$, α_k is $\beta = \beta[\gamma/\delta]$ where there is a previous line $\gamma = \delta$. Note that $\vdash \Box(\beta \leftrightarrow \beta)$ (by applying $[\mathrm{RN}^{\mathrm{PL}}]$ to the tautology $\beta \leftrightarrow \beta$). Then by (2), $\vdash \Box\top \prec \Box(\beta \leftrightarrow \beta)$, that is, $\vdash \Box\top \prec (\beta = \beta)$. Then by [RRSE], $\vdash (\Box\top \prec (\beta = \beta))[\gamma/\delta]$, and so $\vdash \Box\top \prec (\beta = \beta[\gamma/\delta])$, that is, $\vdash \Box\top \prec \alpha_k$.

Case 4 If [MP], there are two previous lines α_i and $\alpha_i \to \alpha_k$. From I.H., $\vdash \Box\top \prec \alpha_i$ and $\vdash \Box\top \prec (\alpha_i \to \alpha_k)$. Then $\vdash \Box\top \prec \alpha_k$. (Note that

$(\alpha \prec \beta) \wedge (\alpha \prec (\beta \to \gamma)) \to (\alpha \prec \gamma)$ is provable from the tautology $(p \to q) \wedge (p \to (q \to r)) \to (p \to r)$ by using [RN$^{\mathrm{PL}}$], [K], and [C].)

For (4) Assume that $\vdash \alpha$ and $\vdash \neg\Box\top \prec \alpha$. From [X], $\vdash \Box(\neg q \to p) \wedge \Box(p \to q) \to \Box(\neg q \to q)$. Then by [RRTE], $\vdash \Box(\neg p \to q) \wedge \Box(p \to q) \to \Box q$, that is, $\vdash (\neg p \prec q) \wedge (p \prec q) \to \Box q$. Then by [US], $\vdash (\neg\Box\top \prec \alpha) \wedge (\Box\top \prec \alpha) \to \Box\alpha$. Then by assumptions, together with (3) and [MP], we have $\vdash \Box\alpha$. □

Lemma 4.5 *A 1^0 system has the following theorem and rule:*

1. $\Diamond(p \wedge q) \to (\Diamond p \wedge \Diamond q)$;
2. $\dfrac{\vdash \Diamond\alpha \to \Box\beta}{\vdash \alpha \to \beta}$.

Proof For (1) $p \to p \vee q$ is a tautology. Thus by [RN$^{\mathrm{PL}}$] and [K], $\vdash \Box p \to \Box(p \vee q)$. Similarly, $\vdash \Box q \to \Box(p \vee q)$. Then by PL, $\vdash \Box p \vee \Box q \to \Box(p \vee q)$, and so $\vdash \Diamond\neg(p \vee q) \to (\Diamond\neg p \wedge \Diamond\neg q)$. Finally by [RRTE], $\vdash \Diamond(\neg p \wedge \neg q) \to (\Diamond\neg p \wedge \Diamond\neg q)$.

For (2) Assume that $\vdash \Diamond\alpha \to \Box\beta$. By (1), $\vdash \Diamond(\alpha \wedge \neg\beta) \to (\Diamond\alpha \wedge \Diamond\neg\beta)$. Then by PL and [RRTE]: $\vdash (\neg\Diamond\alpha \vee \neg\Diamond\neg\beta) \to \neg\Diamond\neg(\alpha \to \beta)$; $\vdash (\neg\Diamond\alpha \vee \Box\beta) \to \Box(\alpha \to \beta)$; $\vdash (\Diamond\alpha \to \Box\beta) \to \Box(\alpha \to \beta)$. Then by assumption and [MP], $\vdash \Box(\alpha \to \beta)$, and by [RdN], $\vdash \alpha \to \beta$. □

Lemma 4.6 *A 1^0 system has the following theorem:*

$$\Box\alpha \wedge \Box(\alpha \to \beta) \prec \Box\top.$$

Proof From [X] and [RN$^{\mathrm{Ax}}$], $\vdash \Box(\neg\alpha \to \alpha) \wedge \Box(\alpha \to (\alpha \to \beta)) \prec \Box(\neg\alpha \to (\alpha \to \beta))$. Then by [RRTE], $\vdash \Box\alpha \wedge \Box(\alpha \to \beta) \prec \Box\top$. □

We are now in a position to prove the fundamental theorem for 1^0 systems (after defining the canonical frames and models for such systems). However, to facilitate the presentation of the proofs of the fundamental theorem and subsequent completeness theorems, we will, after defining the canonical frames and models for 1^0 systems, demonstrate two more lemmas about their canonical frames.

Definition 4.7 Let L be a 1^0 system. Then the canonical frame for L is the prenormal frame $\mathfrak{F}_{\mathrm{L}} = \langle U_{\mathrm{L}}, N_{\mathrm{L}}, Q_{\mathrm{L}}, R_{\mathrm{L}}, S_{\mathrm{L}} \rangle$ where

1. $x \in N_{\mathrm{L}}$ if and only if x is a maximal L-consistent set of well-formed formulas;
2. $x \in Q_{\mathrm{L}}$ if and only if x is a maximal PL-consistent set of well-formed formulas satisfying both of the following conditions:
   (a) the set $\{\neg\Box\alpha | \alpha \notin x\}$ of well-formed formulas is L-consistent,
   (b) $\Box\top \notin x$;
3. $\forall x \in N_{\mathrm{L}}, \forall y \in U_{\mathrm{L}}, R_{\mathrm{L}}xy \iff \Box(x) \subseteq y$;
4. $\forall x \in Q_{\mathrm{L}}, \forall a \subseteq U_{\mathrm{L}},\ a \in S_{\mathrm{L}}(x) \iff \exists\alpha : a = |\alpha|_{\mathrm{L}}\ \&\ \Box\alpha \in x$.

Notation 4.8 $\Box(x)$ is $\{\alpha | \Box\alpha \in x\}$, and $|\alpha|_{\mathrm{L}}$ is $\{x \in U_{\mathrm{L}} | \alpha \in x\}$.

We note that $\mathfrak{F}_{\mathrm{L}}$ as defined above is indeed a prenormal frame since N_{L} and Q_{L} are disjoint ($\Box\top$, being a theorem of L, is in every normal point but it is not in any queer point.)

Definition 4.9 Let L be a 1^0 system. Then the canonical model for L is the prenormal model $\mathfrak{M}_{\mathrm{L}} = \langle \mathfrak{F}_{\mathrm{L}}, V_{\mathrm{L}} \rangle$ where

1. $\mathfrak{F}_{\mathrm{L}} = \langle U_{\mathrm{L}}, N_{\mathrm{L}}, Q_{\mathrm{L}}, R_{\mathrm{L}}, S_{\mathrm{L}} \rangle$ is the L-canonical frame;
2. for every $p \in \mathtt{At}$, $V_{\mathrm{L}}(p) = |p|_{\mathrm{L}}$.

Lemma 4.10 *Let $\mathfrak{F}_{\mathrm{L}} = \langle U_{\mathrm{L}}, N_{\mathrm{L}}, Q_{\mathrm{L}}, R_{\mathrm{L}}, S_{\mathrm{L}} \rangle$ be the canonical frame for a 1^0 system* L. *Then for every $x \in N_{\mathrm{L}}$ and well-formed formula α,*

$$\Box\alpha \notin x \implies \exists y \in U_{\mathrm{L}} : R_{\mathrm{L}}xy \,\&\, \alpha \notin y.$$

Proof Let x be a normal point. Assume that $\Box\alpha \notin x$. It is sufficient to show that the set $\Sigma = \Box(x) \cup \{\neg\alpha\}$ has an extension in N_{L} or Q_{L}. For reductio assume neither. That is, assume that

(1) Σ is not L-consistent and
(2) if y is a maximal PL-consistent extension of Σ, then y contains $\Box\top$.

(For the second point, note that if y is a maximal PL-consistent extension of Σ, then the set $\{\neg\Box\alpha | \alpha \notin y\}$ is already L-consistent since it is a subset of x.) From (1) and (2) above, we can infer the following:

$$\begin{aligned}
\exists\beta_1, \ldots, \beta_n \in \Box(x) \quad &: \quad \{\beta_1, \ldots, \beta_n, \neg\alpha\} \vdash_{\mathrm{L}} \bot \\
&\quad\;\; \vdash_{\mathrm{L}} \beta_1 \wedge \cdots \wedge \beta_n \to \alpha; \\
\exists\gamma_1, \ldots, \gamma_m \in \Box(x) \quad &: \quad \{\gamma_1, \ldots, \gamma_m, \neg\alpha, \neg\Box\top\} \vdash_{\mathrm{PL}} \bot \\
&\quad\;\; \vdash_{\mathrm{PL}} \neg\Box\top \to ((\gamma_1 \wedge \cdots \wedge \gamma_m) \to \alpha).
\end{aligned}$$

Let $\delta = \beta_1 \wedge \cdots \wedge \beta_n \wedge \gamma_1, \ldots, \gamma_m$. Then from the above, we have

(i) $\vdash_{\mathrm{L}} \delta \to \alpha$ and
(ii) $\vdash_{\mathrm{PL}} \neg\Box\top \to (\delta \to \alpha)$.

From (ii) and by [$\mathrm{RN}^{\mathrm{PL}}$], $\vdash_{\mathrm{L}} \neg\Box\top \prec (\delta \to \alpha)$. Then by (i) and Lemma 4.4(4), $\vdash_{\mathrm{L}} \Box(\delta \to \alpha)$. Then by [K], $\vdash_{\mathrm{L}} \Box\delta \to \Box\alpha$ and so $\Box\delta \to \Box\alpha \in x$ (note that every L-theorem is in x). However, $\Box\delta \in x$ since $\Box\beta_1, \ldots, \Box\beta_n, \Box\gamma_1, \ldots, \Box\gamma_m \in x$ and L provides [C]. Thus by the deductive closure of maximal L-consistent sets, $\Box\alpha \in x$. But this is contrary to the initial assumption that $\Box\alpha \notin x$. Thus by reductio, $\Box(x) \cup \{\neg\alpha\}$ has an extension in N_{L} or Q_{L}. □

Lemma 4.11 *Let* L *be a 1^0 system. Then for any well-formed formulas α and β,*

$$|\alpha|_{\mathrm{L}} = |\beta|_{\mathrm{L}} \implies \vdash_{\mathrm{L}} \alpha = \beta.$$

Proof Let α and β be arbitrary well-formed formulas. Assume that $|\alpha|_{\mathrm{L}} = |\beta|_{\mathrm{L}}$, that is, for any $x \in U_{\mathrm{L}}$, $\alpha \in x$ if and only if $\beta \in x$. In other words, $\alpha \leftrightarrow \beta \in x$, for every $x \in U_{\mathrm{L}}$. Assume, for reductio, that $\nvdash_{\mathrm{L}} \alpha = \beta$. Then there exists an $x' \in N_{\mathrm{L}}$ such that $\Box(\alpha \leftrightarrow \beta) \notin x'$. Then by Lemma 4.10, there exists a $y \in U_{\mathrm{L}}$ such that $R_{\mathrm{L}}x'y$ and $\alpha \leftrightarrow \beta \notin y$. But this is contrary to what can be inferred from the initial assumption. Thus, by reductio, $\vdash_{\mathrm{L}} \alpha = \beta$. □

Theorem 4.12 (Fundamental Theorem for 1^0 Systems) *Let $\mathfrak{M}_{\mathrm{L}} = \langle U_{\mathrm{L}}, N_{\mathrm{L}}, Q_{\mathrm{L}}, R_{\mathrm{L}}, S_{\mathrm{L}}, V_{\mathrm{L}} \rangle$ be the canonical model for a 1^0 system* L. *Then, for every $x \in U_{\mathrm{L}}$ and well-formed formula α,*

$$\models_x^{\mathfrak{M}_{\mathrm{L}}} \alpha \iff \alpha \in x.$$

Proof The proof is by induction on the structure of α. The only interesting case is the modal one. Assume that for every $x \in U_{\mathrm{L}}$, $\models_x^{\mathfrak{M}_{\mathrm{L}}} \alpha \iff \alpha \in x$ (I.H.), and show that for every $x \in U_{\mathrm{L}}$, $\models_x^{\mathfrak{M}_{\mathrm{L}}} \Box\alpha \iff \Box\alpha \in x$. We show only the difficult direction, namely, $\Longrightarrow$. Either $x \in N_{\mathrm{L}}$ or $x \in Q_{\mathrm{L}}$.

For $x \in N_{\mathrm{L}}$ Assume, for contraposition, that $\Box\alpha \notin x$. Then by Lemma 4.10, there exists a $y \in U_{\mathrm{L}}$ such that $R_{\mathrm{L}}xy$ and $\alpha \notin y$, that is, $\not\models_y^{\mathfrak{M}_{\mathrm{L}}} \alpha$ (I.H.). Thus $\not\models_x^{\mathfrak{M}_{\mathrm{L}}} \Box\alpha$.

For $x \in Q_{\mathrm{L}}$ Assume that $\models_x^{\mathfrak{M}_{\mathrm{L}}} \Box\alpha$. Then $\|\alpha\|^{\mathfrak{M}_{\mathrm{L}}} \in S_{\mathrm{L}}$, that is, $|\alpha|_{\mathrm{L}} \in S_{\mathrm{L}}$ (by I.H.). Then for some well-formed formula β, $|\alpha|_{\mathrm{L}} = |\beta|_{\mathrm{L}}$ & $\Box\beta \in x$. By Lemma 4.11, we can infer that $\vdash_{\mathrm{L}} \alpha = \beta$, and by [$^{\mathrm{S}}$Eq], $\vdash_{\mathrm{L}} \Box\alpha = \Box\beta$. Then the well-formed formula $\Box\alpha = \Box\beta$ is in every normal point. Moreover, it is evident that every queer point (say x') has a normal predecessor (since the set $\{\neg\Box\alpha | \alpha \notin x'\}$ is L-consistent and so has an extension y in N_{L} such that $\Box(y) \subseteq x'$). Thus $\Box\alpha \leftrightarrow \Box\beta$ is in every queer point. Thus $\Box\alpha$ is in x since $\Box\beta$ is. $\Box$

Theorem 4.13 *The canonical frame for the system* $\mathrm{S}1^0$ *is in the class* $\mathfrak{C}_{1^0}$ *of frames.*

Proof Let $\mathfrak{F}_{\mathrm{S}1^0} = \langle U_{\mathrm{S}1^0}, N_{\mathrm{S}1^0}, Q_{\mathrm{S}1^0}, R_{\mathrm{S}1^0}, S_{\mathrm{S}1^0}\rangle$ be the canonical frame for $\mathrm{S}1^0$. We show that it is a 1^0 frame, that is, it satisfies the conditions stipulated in Definition 4.1. Clearly $N_{\mathrm{S}1^0}$ is nonempty since $\mathrm{S}1^0$ is consistent. The more interesting conditions are the requirements

(1) that every point in $U_{\mathrm{S}1^0}$ has a normal predecessor and
(2) that no two neighborhoods of a queer point "cover" the universe.

For (1) If $x \in Q_{\mathrm{S}1^0}$, it is evident that it has a normal predecessor since the set $\{\neg\Box\alpha | \alpha \notin x\}$ is $\mathrm{S}1^0$-consistent (as we have already argued when proving Theorem 4.12). It remains to show that the same holds for every normal point. Thus let $x \in N_{\mathrm{S}1^0}$ be arbitrary. Assume, for reductio, that the set $\{\neg\Box\alpha | \alpha \notin x\}$ is not $\mathrm{S}1^0$-consistent. Then, for some $\beta_1, \ldots, \beta_n \notin x$,

$$\begin{aligned}
&\{\neg\Box\beta_1, \ldots, \neg\Box\beta_n\} \vdash_{\mathrm{S}1^0} \bot; \\
&\vdash_{\mathrm{S}1^0} (\neg\Box\beta_1 \wedge \cdots \wedge \neg\Box\beta_{n-1}) \to \Box\beta_n; \\
&\vdash_{\mathrm{S}1^0} (\Diamond\neg\beta_1 \wedge \cdots \wedge \Diamond\neg\beta_{n-1}) \to \Box\beta_n; \\
&\vdash_{\mathrm{S}1^0} \Diamond(\neg\beta_1 \wedge \cdots \wedge \neg\beta_{n-1}) \to \Box\beta_n && \text{by Lemma 4.5(1);} \\
&\vdash_{\mathrm{S}1^0} (\neg\beta_1 \wedge \cdots \wedge \neg\beta_{n-1}) \to \beta_n && \text{by Lemma 4.5(2).}
\end{aligned}$$

Then $(\neg\beta_1 \wedge \cdots \wedge \neg\beta_{n-1}) \to \beta_n \in x$. But $\neg\beta_1, \ldots, \neg\beta_{n-1} \in x$. Then $\beta_n \in x$, which is absurd since $\beta_n \notin x$. Thus, by reductio, the set $\{\neg\Box\alpha | \alpha \notin x\}$ is $\mathrm{S}1^0$-consistent, whence we can argue that x has a normal predecessor.

For (2) Let x be a queer point, a and b subsets of $U_{\mathrm{S}1^0}$. Assume that $a, b \in S_{\mathrm{S}1^0}(x)$. Further assume for reductio that $a \cup b = U_{\mathrm{S}1^0}$. Then for some well-formed formulas $\Box\alpha, \Box\beta \in x$, $a = |\alpha|_{\mathrm{S}1^0}$ and $b = |\beta|_{\mathrm{S}1^0}$. It is evident that $|\alpha \to \beta|_{\mathrm{S}1^0} \subseteq |\beta|_{\mathrm{S}1^0}$ (since $|\alpha|_{\mathrm{S}1^0} \cup |\beta|_{\mathrm{S}1^0} = U_{\mathrm{S}1^0}$), and so $|\alpha \to \beta|_{\mathrm{S}1^0} = |\beta|_{\mathrm{S}1^0}$. Then

$$\begin{aligned}
&\vdash_{\mathrm{S}1^0} (\alpha \to \beta) = \beta && \text{by Lemma 4.11,} \\
&\vdash_{\mathrm{S}1^0} \Box(\alpha \to \beta) = \Box\beta && \text{by } [^{\mathrm{S}}\mathrm{Eq}], \\
&\vdash_{\mathrm{S}1^0} \Box\beta \prec \Box(\alpha \to \beta).
\end{aligned}$$

Then $\Box\beta \to \Box(\alpha \to \beta) \in x$ (since every queer point has a normal predecessor). Then $\Box(\alpha \to \beta) \in x$ (since $\Box\beta \in x$). But $\Box\alpha \wedge \Box(\alpha \to \beta) \to \Box\top \in x$ (by Lemma 4.6 and x having a normal predecessor). Since $\Box\alpha, \Box(\alpha \to \beta) \in x$, we have $\Box\top \in x$, which is absurd. Thus, by reductio, $a \cup b \neq U_{\mathrm{S}1^0}$. $\Box$

Theorem 4.14 *The system* $S1^0$ *is Lewis-complete with respect to the class* $\mathfrak{C}_{1^0}$ *of frames.*

Proof The Lewis-completeness of the system $S1^0$ with respect to the class $\mathfrak{C}_{1^0}$ of frames follows directly from the Fundamental Theorem for $S1^0$ systems and the theorem that the canonical frame for $S1^0$ is in the class $\mathfrak{C}_{1^0}$ of frames (Theorems 4.12 and 4.13, respectively). □

Theorem 4.15 *The system* $S1^0$ *is Lewis-determined by the class* $\mathfrak{C}_{1^0}$ *of frames.*

Proof Since $S1^0$ is both Lewis-sound and Lewis-complete with respect to the class $\mathfrak{C}_{1^0}$ of frames, it is Lewis-determined by the class $\mathfrak{C}_{1^0}$ of frames. □

5 A Semantics for $S1^0D$

Definition 5.1 A 1^0D frame is a 1^0 frame whose relational component R satisfies the condition of seriality,

$$\forall x \in N, \exists y \in U : Rxy.$$

Theorem 5.2 *The system* $S1^0D$ *is Lewis-sound with respect to the class* $\mathfrak{C}_{1^0D}$ *of frames.*

Proof It suffices to note that [D] is true at every point, normal or queer, in any model on any 1^0D frame. □

Theorem 5.3 *The canonical frame for the system* $S1^0D$ *is in the class* $\mathfrak{C}_{1^0D}$ *of frames.*

Proof Let $\mathfrak{F}_{S1^0D} = \langle U_{S1^0D}, N_{S1^0D}, Q_{S1^0D}, R_{S1^0D}, S_{S1^0D}\rangle$ be the canonical frame for $S1^0D$. It suffices to show that R_{S1^0D} is serial. Let $x \in N_{S1^0D}$ be arbitrary. We show that the set $\Box(x)$ of well-formed formulas has an extension either in N_{S1^0D} or Q_{S1^0D}. For reductio, assume neither. That is to say, assume that (1) $\Box(x)$ is not $S1^0D$-consistent and (2) if y is a maximal PL-consistent extension of $\Box(x)$, then it contains $\Box\top$. (Note that the set $\{\neg\Box\alpha | \alpha \notin y\}$ is already $S1^0D$-consistent since it is included in x.) From the above, we can infer the following:

$$\begin{aligned}
\exists\alpha_1, \ldots, \alpha_n \in \Box(x) \quad &: \quad \{\alpha_1, \ldots, \alpha_n\} \vdash_{S1^0D} \bot \\
& \vdash_{S1^0D} \alpha_1 \wedge \cdots \wedge \alpha_{n-1} \to \neg\alpha_n; \\
\exists\beta_1, \ldots, \beta_m \in \Box(x) \quad &: \quad \{\beta_1, \ldots, \beta_m, \neg\Box\top\} \vdash_{PL} \bot \\
& \{\beta_1, \ldots, \beta_m, \neg\Box\top, \alpha_n\} \vdash_{PL} \bot \\
& \vdash_{PL} \neg\Box\top \to ((\beta_1 \wedge \cdots \wedge \beta_m) \to \neg\alpha_n).
\end{aligned}$$

Let $\gamma = \alpha_1 \wedge \cdots \wedge \alpha_{n-1} \wedge \beta_1 \wedge \cdots \wedge \beta_m$. Then from the above, we have

(i) $\vdash_{S1^0D} \gamma \to \neg\alpha_n$ and
(ii) $\vdash_{PL} \neg\Box\top \to (\gamma \to \neg\alpha_n)$.

From (ii) and by [RN^{PL}], $\vdash_{S1^0D} \neg\Box\top \prec (\gamma \to \neg\alpha_n)$. Then by (i) and Lemma 4.4(4), $\vdash_{S1^0D} \Box(\gamma \to \neg\alpha_n)$. Then by [K], $\vdash_{S1^0D} \Box\gamma \to \Box\neg\alpha_n$. But every $S1^0D$-theorem is in x. Thus $\Box\gamma \to \Box\neg\alpha_n \in x$. However $\Box\gamma \in x$. (Note that $\Box\alpha_1, \ldots, \Box\alpha_{n-1}, \Box\beta_1, \ldots, \Box\beta_m \in x$, and so is [C].) Thus by the deductive closure of maximal $S1^0D$-consistent sets, $\Box\neg\alpha_n$ is in x. But this is absurd, since $\alpha_n \in \Box(x)$, that is, $\Box\alpha_n \in x$ and so $\neg\Box\neg\alpha_n \in x$ (for [D] is a theorem of $S1^0D$ and it is in x). Thus, by reductio, $\Box(x)$ has an extension in $N_{\mathrm{S1^0D}}$ or $Q_{\mathrm{S1^0D}}$, whence we can conclude that x has a successor in $U_{\mathrm{S1^0D}}$. □

Theorem 5.4 *The system* $\mathrm{S1^0D}$ *is Lewis-complete with respect to the class* $\mathfrak{C}_{1^0\mathrm{D}}$ *of frames.*

Proof The Lewis-completeness of the system $\mathrm{S1^0D}$ with respect to the class $\mathfrak{C}_{1^0\mathrm{D}}$ of frames follows directly from the Fundamental Theorem for $\mathrm{S1^0}$ systems and the theorem that the canonical frame for $\mathrm{S1^0D}$ is in the class $\mathfrak{C}_{1^0\mathrm{D}}$ of frames. (Theorems 4.12 and 5.3, respectively). □

Theorem 5.5 *The system* $\mathrm{S1^0D}$ *is Lewis-determined by the class* $\mathfrak{C}_{1^0\mathrm{D}}$ *of frames.*

Proof Since $\mathrm{S1^0D}$ is both Lewis-sound and Lewis-complete with respect to the class $\mathfrak{C}_{1^0\mathrm{D}}$ of frames, it is Lewis-determined by the class $\mathfrak{C}_{1^0\mathrm{D}}$ of frames. □

6 Philosophical Remarks: From Alethic to Deontic Systems

> If we were to drop the condition that R be reflexive, this would be equivalent to abandoning the modal axiom $\Box A \supset A$. In this way we could obtain systems of the type required for deontic logic. (Kripke [5])

Thus states Kripke at the conclusion of his seminal paper. If we interpret $\Box\alpha$ as "It is a necessary truth that α", then the principle [T] $\Box p \rightarrow p$ and its substitutional instances would be unavoidable. We may call systems that have the principle [T] alethic systems. However as Kripke suggests in the passage quoted above, [T] would be inappropriate for some other readings of the modal operator $\Box$, for example, if we interpret it as "It is obligatory that . . . " (deontic logic) or "An agent believes that . . . " (doxastic logic). For such readings, [T] had better be dropped or replaced by some weaker principle such as [D] $\Box p \rightarrow \Diamond p$, though other weaker principles might do as well or better. We may call such systems "T-less" systems (or we can adopt the name "deontic logics" while bearing in mind that the term has a stricter meaning, namely, logics of obligation). In this paper, we have followed a similar path in choosing the systems $\mathrm{S1^0}$ and $\mathrm{S1^0D}$ for analysis. For comparison, we list in the following three trios of logics, including the normal system K, the regular system R, and their deontic and alethic extensions:

K	KD	KT
R	RD	RT
$\mathrm{S1^0}$	$\mathrm{S1^0D}$	$\mathrm{S1^0T}$ (or S1).

Notation 6.1 The system K can be axiomatized by PL, [RN], and [K], and the system R by PL, [RM], and [C]. The systems RD and RT are equivalent to Lemmon's D2 and E2 in [6].

Before we comment on the interpretation of the deontic systems KD, RD, and $\mathrm{S1^0D}$, we note a formal similarity in the above trios of systems. Syntactically [D] is weaker than [T] (in the sense that [D] is derivable from [T] but not the reverse in the absence of further axioms). This relation between [D] and [T] is reflected by a similar relation between the classes of frames to which the two principles correspond in the binary relational idiom: [D] corresponds to seriality, which is weaker than reflexivity to which [T] corresponds. However, the last trio brings out another point of interest: the systems $\mathrm{S1^0}$, $\mathrm{S1^0D}$, and $\mathrm{S1^0T}$ have the rule of denecessitation [RdN]

$\vdash \Box\alpha \implies \vdash \alpha$, which can be considered as another weak principle with which one could replace [T] in moving from alethic systems to a T-less system. Whether [RdN] could be given a plausible deontic reading (or some other reading) is another matter. At least it is formally interesting to note that the rule corresponds to converse seriality (every point has a predecessor), the converse of seriality (every point has a successor).[2] Both versions of seriality are weaker than reflexivity, a relation echoing the derivability of [RdN] and [D] from [T] in a formal system.

We do not give a detailed interpretation of the systems KD, RD, and $S1^0D$ here, but rather comment on a few points on their suitableness as a minimal deontic logic (the smallest logic that would preserve our intuitions about obligation). One of the (many) objections to the normal system KD as a deontic logic is its use of the rule of necessitation [RN] $\vdash \alpha \implies \vdash \Box\alpha$. What the rule effectively says (in deontic terms) is that it is a logical law that any law of logic is obligatory. But this seems to be contrary to our intuition that logic by itself cannot give rise to obligations and that only obligations can give rise to obligations. It is because of this objection that we consider nonnormal logics RD and $S1^0D$ rather than the normal KD as plausible candidates for the minimal deontic logic. Neither RD nor $S1^0D$ provides the rule [RN] (although the latter system has [RN^{Ax}], which is a weakened version of [RN]). Semantically, they avoid the rule [RN] by the presence of queer worlds at which there are no obligations whatsoever (in nonnormal frames) or no tautologous obligations (in prenormal frames). Whether this is a satisfactory way to drop the rule [RN] is controversial, for it may be objected that there are still logical obligations or tautologous obligations at normal worlds. However we will pursue this point no further here, for there is another objection to proposing RD and $S1^0D$ as candidates for the minimal deontic logic.

An objection to RD and $S1^0D$ (and also KD) as being the minimal deontic logic is related to their commitment to nonconflicting obligations (in virtue of [D]). Whether there are genuinely conflicting obligations or not will not be adjudicated here. However, if our project is to find the minimal deontic logic, the smallest logic that would preserve our intuitions about obligation, then RD and $S1^0D$ do not qualify as the minimal deontic logic (at least not without controversy). If so, we may ask whether [D] could be replaced by a weaker and less controversial principle such as [Con] $\neg\Box\bot$ (which is a weak version of the Kantian principle that ought implies can). Unfortunately the answer is no, for given the bases R and $S1^0$ of these systems, any extension of these bases that has [Con] also provides [D]. Schotch and Jennings point out in [7] that the minimal deontic logic should preserve important deontic distinctions such as [D] and [Con]. If this is correct, then the systems formed by adding [Con] to R or $S1^0$ would not be the minimal deontic logic we are looking for. What that logic is must remain an open question.

Notes

1. Chellas and Segerberg give two axiomatizations of $S1^0$, both of which are somewhat different from the one we provide here. For example, their first axiomatization, using the symbols of this paper, is PL, [RN^{Ax}], [$^S MP$], [RRSE], and [X]. It can easily be shown that it is equivalent to the axiomatization of $S1^0$ we adopt here. For the purpose of this paper, taking [RdN] instead of [$^S MP$] as primitive has the advantage of showing how $S1^0$ can be obtained from S1 by replacing [T] with its weaker rule counterpart [RdN].

2. The idea of rule correspondence (or defining a class of frames by a rule of inference) is based on Kapron's notion of modal sequent-axiomatic classes of frames in [4]. That the rule of denecessitation [RdN] corresponds to converse seriality has first been noticed by Schotch in communication with Jennings.

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