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The Expressive Unary Truth Functions of *n*-valued Logic

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Abstract The expressive truth functions of two-valued logic have all been identified. This paper begins the task of identifying the expressive truth functions of n-valued logic by characterizing the unary ones. These functions have distinctive algebraic, semantic, and closure-theoretic properties.

1 Introduction

Every consequence relation carries with it a notion of closure, a set being *closed* if and only if each consequence of its members is itself a member. If the consequence relation is not too odd, every intersection of closed sets will be closed and the closed sets will form a complete lattice. If, furthermore, the consequence relation is finitary, this lattice will have a minimal closed basis consisting of the irreducibles: the closed sets you can never reach by taking intersections of other closed sets. (Cf. Wójcicki [10], pp. 26–27.) If all these irreducibles are maximally consistent, the lattice is said to be *expressive* or is said to have a *Henkin basis*, the two properties being equivalent for finitary logics. (See Martin and Pollard [3], pp. 161–62, and Pollard and Martin [6], pp. 122–23. Cf. also Pollard and Martin [5], p. 113.) Now some truth functions force irreducibles to be maximally consistent: if a finitary logic expresses one of these functions, the corresponding lattice of closed sets will be expressive. Truth functions with this property earn the same epithet as the lattices: they too are said to be *expressive*. Pollard [4] shows that classical logic is blessed with a simple test for this sort of expressiveness. A classical truth function is expressive if and only if (1) it gives back **T** when you feed it nothing but **F**s and (2) there is at least one input that causes it to give back **F**. If we admit more than two truth-values, this neat scheme goes kerplooey. Not to worry: many-valued logics offer treats of their own. In the case of *unary* n-valued truth functions, the problem of identifying the expressive ones has an attractive solution applicable to logics with any finite number of truth-values.

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2 Preliminaries

A sentential logic with only one sentential variable might as well have *no* sentential variables. If there is only one thing we can place in the blanks of a matrix like $((\_\_ \wedge \neg\_\_) \rightarrow \_\_)$ (if there is to be no cross-referencing of blanks), then we can leave out the blanks: $((\wedge\neg) \rightarrow)$. Indeed, we can omit the language altogether and concentrate on compositions of the truth functions themselves. Our logic can be an algebra of truth functions. (Hardly a new idea: see Rasiowa [7] for a classic exposition.) It turns out that such algebras provide a mechanical test for the expressiveness of unary truth functions. Pick any n-membered set V ($n > 1$). The members of V are our truth-values. Some, but not all, of these values are singled out as "designated." The designated values are ways of being true. The undesignated values are ways of being untrue. A unary n-valued truth function is just a function $\alpha : V \rightarrow V$. The "mth power" of a function is the result of composing the function with itself $m - 1$ times. That is, $\alpha^{m+1} = \alpha \circ \alpha^m$ where α^0 is the identity function. Since taking the 0th power of a function is a form of composition, the identity function appears in every nonempty set that is closed under composition. Say that such a set is a *system*. That is, each system will satisfy the following conditions:

the identity function is a member;
if α and β are members, then so is $\alpha \circ \beta$.

Suppose A is a system. The *theories of A* or *A-theories* are the sets

$$D_k(A) = \{\alpha \in A : \alpha(k) \text{ is designated}\}.$$

$D_k(A)$ consists of the "sentences" that are true when our nonexistent sentential variable has value k. Somewhat less poetically, $D_k(A)$ consists of the functions in A that respond to input k by returning a designated value. A subset of A is *satisfiable* if and only if some A-theory contains it. An A-theory is *consistent* if and only if it is a proper subset of A. An A-theory is *maximally consistent* if and only if it is consistent and no consistent A-theory properly contains it. An A-theory is *reducible* if and only if it is the intersection of A-theories distinct from itself. Given a universe of discourse A, we let $\cap\varnothing = A$. So an inconsistent A-theory would be reducible, since such a theory would just be A itself.

We have moved from talking about closed sets generally (in §1) to speaking only of theories. Why? We obtain the closed sets of a system by taking intersections of its theories. (In the language of closure spaces, the theories form a closed basis.) So a closed set will be irreducible only if it is a theory. Since we are interested in the behavior of irreducibles, it is safe to confine our attention to theories.

3 Expressive Systems

A system is *expressive* if and only if all its irreducible theories are maximally consistent. If B is a set of functions, let $\mathbf{S}(B)$ be the set of all systems that contain B. Given any functions $\alpha_1, \ldots, \alpha_k$, let $[\alpha_1, \ldots, \alpha_k] = \cap\mathbf{S}(\{\alpha_1, \ldots, \alpha_k\})$. Pick any function $\alpha : V \rightarrow V$. Then the system $[\alpha]$ consists of all the powers of α. $D_1([\alpha]), \ldots, D_n([\alpha])$ are the theories contained in $\cap\mathbf{S}(\{\alpha\})$.

Lemma 3.1 *If every member of* $\mathbf{S}(\{\alpha\})$ *is expressive, then* $D_1([\alpha]), \ldots, D_n([\alpha])$ *are all consistent.*

Proof If j is undesignated, then $D_j([\alpha])$ is consistent since $\alpha^0 \notin D_j([\alpha])$. Pick a designated k and suppose $D_k([\alpha])$ is inconsistent. Pick any undesignated j and let β be the j-constant function. Note that any functions in $[\alpha, \beta]$ but not in $[\alpha]$ will be constant functions. The functions that do not belong to $D_k([\alpha, \beta])$ are all constant functions that yield an undesignated value. So $D_k([\alpha, \beta])$ is the only maximally consistent $[\alpha, \beta]$-theory and every $[\alpha, \beta]$-theory sitting immediately below it is an irreducible that is not maximally consistent. We know there are such $[\alpha, \beta]$-theories since we know that $D_k([\alpha, \beta])$ properly contains $D_j([\alpha, \beta])$ whenever j is undesignated. So $[\alpha, \beta]$ is not expressive. □

A truth-value j is α-*cyclic* if and only if $j = \alpha^m(j)$ for some $m > 0$. If $\alpha(j) = k$, $\alpha(k) = h$, and $\alpha(h) = j$, then j, k, and h belong to the cycle (jkh). This is the same cycle as (khj) and (hjk). In the following theorem and hereafter, we use '$\subset$' to represent *proper* containment.

Lemma 3.2 *If j is α-cyclic, then $D_j([\alpha]) \subset D_h([\alpha])$ only if $h \notin (j\ldots)$.*

Proof Suppose $D_j([\alpha]) \subset D_h([\alpha])$. Then $\alpha^m(h)$ is designated whenever $\alpha^m(j)$ is, but not vice versa. If $h \in (j\ldots)$, then $(j\ldots) = (h\ldots)$. But this would be absurd, since $(h\ldots)$ would feature more designated values than $(j\ldots)$. □

Lemma 3.3 *If every member of $\mathbf{S}(\{\alpha\})$ is expressive, then $D_j([\alpha]) \cup \{\alpha^0\}$ is unsatisfiable whenever j is both undesignated and α-cyclic.*

Proof Suppose $D_j([\alpha]) \subset D_k([\alpha])$ where k is designated but j is not. Let

$$\beta(h) = \begin{cases} k & \text{if } D_j([\alpha]) \subset D_h([\alpha]) \\ j & \text{otherwise.} \end{cases}$$

Suppose j is α-cyclic. Lemma 3.2 implies that $\beta(h) = j$ whenever $h \in (j\ldots)$. So applications of α and β to members of $(j\ldots)$ always take us to members of $(j\ldots)$. This means that $\delta(j) \in (j\ldots)$ whenever $\delta \in [\alpha, \beta]$. Note that $D_j([\alpha]) \subset D_h([\alpha])$ whenever $D_j([\alpha, \beta]) \subset D_h([\alpha, \beta])$. Furthermore, $\beta \in D_h([\alpha, \beta])$ whenever $D_j([\alpha]) \subset D_h([\alpha])$. So $\beta \in D_h([\alpha, \beta])$ whenever $D_j([\alpha, \beta]) \subset D_h([\alpha, \beta])$. Yet $\beta \notin D_j([\alpha, \beta])$. So the intersection of the theories that properly contain $D_j([\alpha, \beta])$ will itself properly contain $D_j([\alpha, \beta])$. So $D_j([\alpha, \beta])$ is irreducible. Now suppose $\gamma \in D_j([\alpha, \beta])$. We want to show that $\gamma \in D_k([\alpha, \beta])$. Suppose γ is of the form $\beta \circ \delta$ with $\delta \in [\alpha, \beta]$. Then $\gamma(j) = \beta(\delta(j)) = j$, since $\delta(j) \in (j\ldots)$. But this is impossible because j is undesignated and we just assumed that $\gamma(j)$ is designated. If $\gamma = \alpha^m$, then $\gamma \in D_j([\alpha]) \subset D_k([\alpha]) \subseteq D_k([\alpha, \beta])$. Suppose $\gamma = \alpha^m \circ \beta \circ \delta$ with $\delta \in [\alpha, \beta]$. Then $\gamma(j) = \alpha^m(\beta(\delta(j))) = \alpha^m(j)$. So $\alpha^m \in D_j([\alpha]) \subset D_k([\alpha])$. $\gamma(k)$ is either $\alpha^m(k)$ or $\alpha^m(j)$, both of which are designated. So $\gamma \in D_k([\alpha, \beta])$. We conclude that $D_j([\alpha, \beta]) \subset D_k([\alpha, \beta])$. If $D_k([\alpha, \beta])$ is consistent, then $[\alpha, \beta]$ is not expressive since $D_j([\alpha, \beta])$ is irreducible. If $D_k([\alpha, \beta])$ is inconsistent, then so is $D_k([\alpha])$ and we can apply Lemma 3.1. □

Lemma 3.4 *If j is α-cyclic, then so is $\alpha^p(j)$.*

Proof If $j = \alpha^m(j)$, then $\alpha^p(j) = \alpha^p(\alpha^m(j)) = \alpha^m(\alpha^p(j))$. □

Lemma 3.5 *If j and k are both α-cyclic, then $\alpha(j) = \alpha(k)$ only if $j = k$.*

Proof Suppose $j = \alpha^m(j)$, $k = \alpha^p(k)$, and $\alpha(j) = \alpha(k)$. Then $j = \alpha^m(j) = \alpha^m(k) = \alpha^m(\alpha^p(k)) = \alpha^p(\alpha^m(k)) = \alpha^p(j) = \alpha^p(k) = k$. □

Lemma 3.6 *Given any truth-value j, there is exactly one α-cyclic truth-value k such that $\alpha^m(j) = \alpha^m(k)$ for some m.*

Proof Since there are only finitely many truth-values, applications of α to j must eventually yield an α-cyclic value. So suppose $\alpha^m(j) = h = \alpha^p(h)$. Let $m = tp + r$ where $r < p$. Then, by Lemma 3.4, $\alpha^{p-r}(h)$ is α-cyclic. Furthermore, $\alpha^m(\alpha^{p-r}(h)) = \alpha^{tp+r}(\alpha^{p-r}(h)) = \alpha^{tp}(\alpha^p(h)) = h$. So, letting $\alpha^{p-r}(h) = k$, k is an α-cyclic truth-value such that $\alpha^m(j) = \alpha^m(k)$. Suppose i is α-cyclic and $\alpha^s(j) = \alpha^s(i)$. If $s \leq m$, then $\alpha^m(k) = \alpha^m(j) = \alpha^m(i)$. If $m \leq s$, then $\alpha^s(i) = \alpha^s(j) = \alpha^s(k)$. In either case, Lemmas 3.4 and 3.5 imply that $i = k$. □

Given any j, we let $\pi(j)$ be the unique k guaranteed by Lemma 3.6.

Lemma 3.7 $\alpha(\pi(j)) = \pi(\alpha(j))$.

Proof $\alpha(\pi(j))$ is α-cyclic, since $\pi(j)$ is. Furthermore, if $\alpha^m(j) = \alpha^m(\pi(j))$, then $\alpha^m(\alpha(j)) = \alpha(\alpha^m(j)) = \alpha(\alpha^m(\pi(j)) = \alpha^m(\alpha(\pi(j)))$. □

Lemma 3.8 *If every member of* $\mathbf{S}(\{\alpha\})$ *is expressive, then* $D_j([\alpha]) = D_{\pi(j)}([\alpha])$.

Proof As an inductive hypothesis, suppose, for each truth-value k, that $\alpha^m(k) = \alpha^m(\pi(k))$ only if $D_k([\alpha]) = D_{\pi(k)}([\alpha])$. Assume $\alpha^{m+1}(j) = \alpha^{m+1}(\pi(j))$. Then $\alpha^m(\alpha(j)) = \alpha^m(\alpha(\pi(j))) = \alpha^m(\pi(\alpha(j)))$. So $D_{\alpha(j)}([\alpha]) = D_{\pi(\alpha(j))}([\alpha]) = D_{\alpha(\pi(j))}([\alpha])$. If j and $\pi(j)$ are both designated or both undesignated, then $D_j([\alpha]) = D_{\pi(j)}([\alpha])$. If j is designated but $\pi(j)$ is not, then $D_{\pi(j)}([\alpha]) \subset D_j([\alpha])$ and Lemma 3.3 implies that some member of $\mathbf{S}(\{\alpha\})$ is not expressive. Suppose j is undesignated while $\pi(j)$ is designated. Let

$$\beta(h) = \begin{cases} \pi(j) & \text{if } h = j \text{ or } h = \pi(j) \\ j & \text{otherwise.} \end{cases}$$

Can $[\alpha, \beta]$ feature a function γ such that $\gamma(j)$ is designated but $\gamma(\pi(j))$ is not? Such a γ is not of the form $\delta \circ \beta$, since $\beta(j) = \beta(\pi(j))$. Neither is it of the form α^p, since $D_{\alpha(j)}([\alpha]) = D_{\alpha(\pi(j))}([\alpha])$. Consider $\beta \circ \alpha^p$ with $0 < p \leq m$. If $\alpha^p(j) = j$, then j is α-cyclic and Lemma 3.5 implies that $j = \pi(j)$. So, in fact, $\alpha^p(j) \neq j$, since we have assumed that j and $\pi(j)$ are distinct. Suppose $\alpha^p(j) = \pi(j)$. Then $\alpha^m(j)$ is α-cyclic, since $\alpha^p(j)$ is. So, by Lemma 3.5, $\alpha^m(j) = \alpha^m(\pi(j))$ and, hence, $D_j([\alpha]) = D_{\pi(j)}([\alpha])$, contrary to our earlier assumption. So $\alpha^p(j) \neq \pi(j)$. This means that $\beta(\alpha^p(j)) = j$. If $\beta(\alpha^p(\pi(j))) = j$, then γ is not of the form $\delta \circ \beta \circ \alpha^p$, since then $\gamma(j) = \gamma(\pi(j))$. On the other hand, if $\beta(\alpha^p(\pi(j))) = \pi(j)$, then $\beta \circ \alpha^p$ has taken us from j back to j and from $\pi(j)$ back to $\pi(j)$. Subsequent applications of α and β will only reproduce a case we have already reviewed. So there is no γ of the sort we contemplated. That is, $D_j([\alpha, \beta]) \subset D_{\pi(j)}([\alpha, \beta])$. β prevents $D_j([\alpha, \beta])$ from being a proper subset of any other theory. So $D_j([\alpha, \beta])$ is irreducible. If $D_{\pi(j)}([\alpha, \beta])$ is consistent, $[\alpha, \beta]$ is not expressive. If $D_{\pi(j)}([\alpha, \beta])$ is inconsistent, then so is $D_{\pi(j)}([\alpha])$ and we can apply Lemma 3.1. □

Theorem 3.9 *If every member of* $\mathbf{S}(\{\alpha\})$ *is expressive, then* $D_j([\alpha]) \cup \{\alpha^0\}$ *is unsatisfiable whenever j is undesignated.*

Proof Apply Lemmas 3.3 and 3.8. □

4 Expressive Functions

An *n-valued logic* is a quintuple $\langle V, D, L, CON, ^\circ \rangle$ where V is an n-membered set of truth-values, D (the set of designated values) is a nonempty proper subset of V, L is a nonempty set of variables, CON is a nonempty set of connectives, and $^\circ$ is a function that assigns an n-valued truth function to each member of CON. The formulas of such a logic are the members of L and any expressions $g(\varphi_1, \ldots, \varphi_m)$ where $\varphi_1, \ldots, \varphi_m$ are formulas and g is an m-ary member of CON. An *interpretation* is any homomorphism that assigns members of V to formulas. That is, if $\mathbf{M}$ is an interpretation, then $\mathbf{M}(g(\varphi_1, \ldots, \varphi_m)) = g^\circ(\mathbf{M}(\varphi_1), \ldots, \mathbf{M}(\varphi_m))$. If α is a unary truth function and $\psi(P)$ is a formula, then $\psi(P)$ *expresses* α if and only if $\mathbf{M}(\psi(P)) = \alpha(\mathbf{M}(P))$ for every interpretation $\mathbf{M}$. A logic *expresses* a truth function if and only if one of its formulas does. Given any interpretation $\mathbf{M}$, we let the *theory* $D_\mathbf{M}$ be the set of formulas assigned a designated value by $\mathbf{M}$. $D_\mathbf{M}$ is *consistent* if and only if there are formulas that do not belong to $D_\mathbf{M}$. A set of formulas is *satisfiable* if and only if some theory contains it. A theory is *maximally consistent* if and only if it is consistent and no consistent theory properly contains it. A theory is *reducible* if and only if it is the intersection of theories distinct from itself. A logic is *expressive* if and only if all its irreducible theories are maximally consistent. A truth function $\alpha : V \to V$ is *expressive* with respect to a set of designated values D if and only if every logic $\langle V, D, L, CON, ^\circ \rangle$ that expresses α is expressive. A function α *identifies* values j and k if and only if $D_j([\alpha]) = D_k([\alpha])$. When we say that α *identifies no values* we mean, of course, that it identifies no distinct values: that is, $D_j([\alpha]) = D_k([\alpha])$ only if $j = k$. A *derangement* is a permutation with no fixed points. That is, if α is a derangement, then for no k do we have $\alpha(k) = k$.

Theorem 4.1 *Every expressive function that identifies no values is a derangement.*

Proof Suppose α is expressive. The systems of §3 behave like n-valued logics with just one variable. Each of these logics that expresses α is expressive. So every member of $\mathbf{S}(\{\alpha\})$ is expressive. Suppose α identifies no values. Then Lemma 3.8 implies that every truth-value is α-cyclic. So Lemma 3.5 implies that α is a permutation. If $\alpha(h) = h$, then $D_h([\alpha])$ is either inconsistent or empty (depending on whether h is designated or not). Lemma 3.1 rules out the former. Theorem 3.9 rules out the latter, since $\{\alpha^0\}$ is always satisfiable. □

Theorem 4.2 *If $D_j([\alpha]) \cup \{\alpha^0\}$ is unsatisfiable whenever j is undesignated and $\mathbf{M}$ is any interpretation in a logic that expresses α, then $D_\mathbf{M}$ is maximally consistent.*

Proof Let $\mathbf{M}$ be any interpretation in a logic where $\neg$ expresses α. Given any formula ψ, let $\neg^0\psi = \psi$ and $\neg^{m+1}\psi = \neg\neg^m\psi$. Suppose $\varphi \notin D_\mathbf{M}$. Let $\mathbf{M}(\varphi) = j$ where j is undesignated. Then $\mathbf{M}(\neg^m\varphi) = \alpha^m(j)$. So $\{\neg^m\varphi : \alpha^m \in D_j([\alpha])\} \subseteq D_\mathbf{M}$. Suppose $D_\mathbf{M} \cup \{\varphi\} \subseteq D_\mathbf{N}$. Let $\mathbf{N}(\varphi) = k$ where k is designated. Note that $\mathbf{N}(\neg^m\varphi) = \alpha^m(k)$. So $\alpha^m(k)$ is designated whenever $\alpha^m \in D_j([\alpha])$. But then $D_j([\alpha]) \subseteq D_k([\alpha])$. $\alpha^0 \in D_k([\alpha])$ since k is designated. So $D_j([\alpha]) \cup \{\alpha^0\} \subseteq D_k([\alpha])$. We conclude that $D_\mathbf{M} \cup \{\varphi\}$ is satisfiable only if $D_j([\alpha]) \cup \{\alpha^0\}$ is. Suppose the latter set is unsatisfiable whenever j is undesignated. Then $D_\mathbf{M} \cup \{\varphi\}$ is unsatisfiable whenever $\varphi \notin D_\mathbf{M}$. So $D_\mathbf{M}$ is maximally consistent if it is consistent. Suppose $D_\mathbf{M}$ is inconsistent. Let $\mathbf{M}(\varphi) = k$. Then $\alpha^m(k)$ is always designated, since

$\alpha^m(k) = \mathbf{M}(\neg^m \varphi)$. So $D_k([\alpha])$ is inconsistent and every subset of $[\alpha]$ is satisfiable, contrary to our earlier assumption. □

If $D_j([\alpha]) \cup \{\alpha^0\}$ is unsatisfiable whenever j is undesignated, then logics that express α are expressive because all their theories are maximally consistent. Even better: if $D_j([\alpha]) \cup \{\alpha^0\}$ is unsatisfiable whenever j is undesignated, then we can apply an algorithm to write down sequents that guarantee the maximal consistency of every theory in every logic that expresses α. Let '$A \Rightarrow B$' mean that some member of B is designated whenever every member of A is. That is, B intersects every theory that contains A. The sequents we associate with α are

$$D_j([\alpha]) \cup \{\alpha^0\} \Rightarrow \varnothing$$

for each undesignated j and

$$\varnothing \Rightarrow B \cup \{\alpha^0\}$$

for each B that features one element from each theory $D_j([\alpha])$ with j undesignated. Here is an example. Let β be the function whose truth table is given below. $[\beta]$ has only three members.

j	$\beta^0(j)$	$\beta^1(j)$	$\beta^2(j)$
1^*	1^*	3	2
2	2	1^*	3
3	3	2	1^*

Letting 1^* be our only designated value, our theories are

$$\begin{aligned} D_{1^*}([\beta]) &= \{\beta^0\}, \\ D_2([\beta]) &= \{\beta^1\}, \\ D_3([\beta]) &= \{\beta^2\}. \end{aligned}$$

The characteristic sequents are

$$\begin{aligned} &\{\beta^0, \beta^1\} \Rightarrow \varnothing, \\ &\{\beta^0, \beta^2\} \Rightarrow \varnothing, \\ &\varnothing \Rightarrow \{\beta^0, \beta^1, \beta^2\}. \end{aligned}$$

Pick any three-valued logic where $\neg$ expresses β. No interpretation will assign a designated value to both φ and $\neg\varphi$; no interpretation will assign a designated value to both φ and $\neg\neg\varphi$; but every interpretation will assign a designated value to φ, $\neg\varphi$, or $\neg\neg\varphi$. In the sequent notation,

$$\begin{aligned} &\{\varphi, \neg\varphi\} \Rightarrow \varnothing, \\ &\{\varphi, \neg\neg\varphi\} \Rightarrow \varnothing, \\ &\varnothing \Rightarrow \{\varphi, \neg\varphi, \neg\neg\varphi\}. \end{aligned}$$

Now suppose $\varphi \notin D_{\mathbf{M}}$. Then the third of our sequents guarantees that either $\neg\varphi$ or $\neg\neg\varphi$ belongs to $D_{\mathbf{M}}$. In either case, one of the two other sequents guarantees that $D_{\mathbf{M}} \cup \{\varphi\}$ is unsatisfiable. So $D_{\mathbf{M}}$ is maximally consistent.

Theorem 4.3 *The following are equivalent:*

1. *α is expressive;*
2. *$D_j([\alpha]) \cup \{\alpha^0\}$ is unsatisfiable whenever j is undesignated;*
3. *$D_1([\alpha]), \ldots, D_n([\alpha])$ are all maximally consistent.*

Proof Apply Theorems 3.9 and 4.2. □

Theorem 4.3 provides us with a mechanical test for expressiveness. For example, let α be the six-valued derangement with cycles (12*) and (34*5*6*). $[\alpha]$ has four members.

j	$\alpha^0(j)$	$\alpha^1(j)$	$\alpha^2(j)$	$\alpha^3(j)$
1	1	2*	1	2*
2*	2*	1	2*	1
3	3	4*	5*	6*
4*	4*	5*	6*	3
5*	5*	6*	3	4*
6*	6*	3	4*	5*

Letting 2*, 4*, 5*, and 6* be our designated values, our theories are

$$\begin{aligned} D_1([\alpha]) &= \{\alpha^1, \alpha^3\}, \\ D_{2^*}([\alpha]) &= \{\alpha^0, \alpha^2\}, \\ D_3([\alpha]) &= \{\alpha^1, \alpha^2, \alpha^3\}, \\ D_{4^*}([\alpha]) &= \{\alpha^0, \alpha^1, \alpha^2\}, \\ D_{5^*}([\alpha]) &= \{\alpha^0, \alpha^1, \alpha^3\}, \\ D_{6^*}([\alpha]) &= \{\alpha^0, \alpha^2, \alpha^3\}. \end{aligned}$$

Two theories are reducible: $D_1([\alpha])$ and $D_{2^*}([\alpha])$. Since the remaining theories are all maximally consistent, this system is expressive. But we know that α itself is *not* expressive, since $D_1([\alpha])$ and $D_{2^*}([\alpha])$ are not maximally consistent. We can, in fact, use the technique of Lemma 3.3 to create new theories $D_1([\alpha, \beta])$, $D_3([\alpha, \beta])$, and $D_{5^*}([\alpha, \beta])$ such that $D_1([\alpha, \beta]) \subset D_3([\alpha, \beta]) \cap D_{5^*}([\alpha, \beta])$, thus replacing the reducible $D_1([\alpha])$ with the irreducible, but still not maximally consistent, $D_1([\alpha, \beta])$.

5 Logics with a Single Designated (or Undesignated) Value

Every permutation is a product of disjoint cycles. A permutation consisting of a single cycle of length n is a *cyclic negation*.

Theorem 5.1 *If there is exactly one designated value or exactly one undesignated value, then a function that identifies no values is expressive if and only if it is a cyclic negation.*

Proof Suppose α is expressive and identifies no values. By Theorem 4.1, α is a derangement. If α consists of more cycles than there are designated values, then some of the theories $D_1([\alpha]), \ldots, D_n([\alpha])$ are empty. If α consists of more cycles than there are undesignated values, then some of the theories $D_1([\alpha]), \ldots, D_n([\alpha])$ are inconsistent. But Theorem 4.3 guarantees that none of these theories are either empty or inconsistent. On the other hand, if α is a cyclic negation, then, by Lemma 3.2, $D_1([\alpha]), \ldots, D_n([\alpha])$ are all maximally consistent. So Theorem 4.3 guarantees that α is expressive. □

6 Two-valued Logics

The next four sections survey some examples of expressive functions. We begin with classical logic. There is one derangement of two truth-values. It is the cyclic negation characterized by the following sequents.

$$\{P, \neg P\} \Rightarrow \varnothing$$
$$\varnothing \Rightarrow \{P, \neg P\}$$

These are the principle *ex falso quodlibet* and the law of excluded middle. Classical negation is the only expressive unary truth function of two-valued logic.

7 Three-valued Logics

There are two derangements of three truth-values. Each is a cyclic negation. If there is just one designated value, then the cyclic negations satisfy the following sequents.

$$\{P, \neg P\} \Rightarrow \varnothing$$
$$\{P, \neg\neg P\} \Rightarrow \varnothing$$
$$\varnothing \Rightarrow \{P, \neg P, \neg\neg P\}$$

We have already seen that these sequents force theories to be maximally consistent. If there is just one undesignated value, the cyclic negations satisfy the duals of the above sequents.

$$\varnothing \Rightarrow \{P, \neg P\}$$
$$\varnothing \Rightarrow \{P, \neg\neg P\}$$
$$\{P, \neg P, \neg\neg P\} \Rightarrow \varnothing$$

These sequents also guarantee that every theory will be maximally consistent. For suppose $\varphi \notin D_{\mathbf{M}}$. Then our first two sequents guarantee that both $\neg\varphi$ and $\neg\neg\varphi$ belong to $D_{\mathbf{M}}$. So the third sequent guarantees that $D_{\mathbf{M}} \cup \{\varphi\}$ is unsatisfiable. The cyclic negations are the three-valued expressive functions that do not identify values. Whether we have one or two designated values, there will be two three-valued functions that identify truth-values and, because they behave like classical negation, are expressive.

8 Four-valued Logics

There are 256 unary truth functions in a four-valued logic. Among those that identify values, some will behave like expressive three-valued functions, while others will behave like classical negation. Which ones behave one way or the other will depend on our choice of designated values. If we consider only the functions that do not identify values, our search for expressive functions can be confined to the nine derangements. Six are cyclic negations. All six of these will be expressive no matter what values are designated. If there is just one designated value, the cyclic negations satisfy the following sequents.

$$\{P, \neg P\} \Rightarrow \varnothing$$
$$\{P, \neg\neg P\} \Rightarrow \varnothing$$
$$\{P, \neg\neg\neg P\} \Rightarrow \varnothing$$
$$\varnothing(\{P, \neg P, \neg\neg P, \neg\neg\neg P\}$$
$$\Rightarrow$$

The dual sequents characterize the cyclic negations when there is just one undesignated value. Each set of sequents guarantees that every theory will be maximally consistent. If there are exactly two designated values, two of the six cyclic negations will identify values and will, in fact, behave like classical negation. The remaining four will satisfy the following sequents.

$$\{P, \neg\neg P\} \Rightarrow \varnothing$$
$$\varnothing \Rightarrow \{P, \neg\neg P\}$$

The three derangements that are not cyclic negations are products of two disjoint cycles with two values in each cycle. All of these functions identify values. If there is just one designated or just one undesignated value, these functions will not be expressive. If there are exactly two designated values, two of the three derangements will behave like classical negation, while the third will yield an inconsistent and an empty theory. (So the first two will be expressive, while the third is not).

9 Five-valued Logics

There are 3,125 unary truth functions in a five-valued logic; 44 are derangements; 24 of the derangements are cyclic negations and are expressive no matter what values are designated. When there is just one designated or just one undesignated value, the cyclic negations satisfy generalizations of excluded middle and *ex falso quodlibet* such as those we have already reviewed. If there are exactly two designated values, there are two possibilities. Half the cyclic negations will satisfy the following sequents.

$$\{P, \neg\neg P\} \Rightarrow \varnothing$$
$$\{P, \neg\neg\neg P\} \Rightarrow \varnothing$$
$$\varnothing \Rightarrow \{P, \neg\neg P, \neg\neg\neg P\}$$

The other half will satisfy these:

$$\{P, \neg P\} \Rightarrow \varnothing$$
$$\{P, \neg\neg\neg\neg P\} \Rightarrow \varnothing$$
$$\varnothing \Rightarrow \{P, \neg P, \neg\neg\neg\neg P\}.$$

If there are exactly two undesignated values, half the cyclic negations will satisfy the duals of the sequents in the first set, while the other half will satisfy the duals of the sequents in the second set. The 20 derangements that are not cyclic negations are products of two cycles with two values in one cycle and three in the other. If such a derangement is to be expressive, there must be a designated and an undesignated value in each cycle. If there are exactly two designated values and each cycle features one of them, the derangement will satisfy the following sequents.

$$\{P, \neg P\} \Rightarrow \varnothing$$
$$\{P, \neg\neg\neg\neg\neg\neg P\} \Rightarrow \varnothing$$
$$\varnothing \Rightarrow \{P, \neg P, \neg\neg\neg\neg\neg\neg P\}$$

If there are exactly two undesignated values and each cycle features one of them, the derangement will satisfy the duals of these sequents.

10 Closure Operators

We now begin to explore the closure-theoretic properties of expressive functions. Suppose $\alpha : V \to V$ is expressive with respect to D and identifies no values. Let $\langle V, D, L, CON, ^\circ \rangle$ be an n-valued logic in which $\neg$ expresses α. Let S be the set of formulas of this logic. Suppose p is the first natural number such that $\alpha^p = \alpha^m$ for some $m < p$. If $\varphi \in S$ and $h \in V$, let

$$\Delta_h(\varphi) = \{\neg^m \varphi : m < p \text{ and } \alpha^m \in D_h([\alpha])\}.$$

Think of the members of $\Delta_h(\varphi)$ as those "negations" of φ that are designated when φ has value h. Together they announce that φ has value h.

Lemma 10.1 $\Delta_h(\varphi) \subseteq D_{\mathbf{M}}$ *if and only if* $\mathbf{M}(\varphi) = h$.

Proof Since α is expressive and identifies no values, the following are equivalent.

$\Delta_h(\varphi) \subseteq D_{\mathbf{M}}$.

$\mathbf{M}(\neg^m \varphi)$ is designated whenever $\alpha^m \in D_h([\alpha])$.

$\alpha^m(\mathbf{M}(\varphi))$ is designated whenever $\alpha^m \in D_h([\alpha])$.

$D_h([\alpha]) \subseteq D_{\mathbf{M}(\varphi)}([\alpha])$.

$D_h([\alpha]) = D_{\mathbf{M}(\varphi)}([\alpha])$ (since each theory is maximally consistent).

$\mathbf{M}(\varphi) = h$. □

A *closure operator* on S is any function K that assigns subsets of S to subsets of S and satisfies the following condition whenever A and B are subsets of S:

$$A \subseteq K(B) \text{ if and only if } K(A) \subseteq K(B).$$

$K(A)$ is the *closure* of A. A *finitary* closure operator K satisfies the following condition: $\varphi \in K(A)$ only if φ belongs to the closure of some finite subset of A. If K_1 and K_2 are closure operators on S, we say that $K_1 \leq K_2$ just in case $K_1(A) \subseteq K_2(A)$ whenever $A \subseteq S$. If $K_1 \leq K_2$ whenever K_2 is a closure operator on S, then we say that K_1 is the *smallest* closure operator on S. Given any subset A of S, let $Cl(A)$ be the intersection of the theories that contain A.

Theorem 10.2 *Cl is the smallest finitary closure operator on S that satisfies the following four conditions.*

Cl(1) : $\bigcap_{h \in V} Cl(A \cup \Delta_h(\varphi)) \subseteq Cl(A)$ *whenever* $\varphi \in S$ *and* $A \subseteq S$.

Cl(2) : $Cl(\Delta_j(\varphi) \cup \Delta_k(\varphi)) = S$ *whenever* $\varphi \in S$ *and* $j \neq k$.

Cl(3) : $Cl(\Delta_j(\varphi) \cup \{\varphi\}) = S$ *whenever* $\varphi \in S$ *and* j *is undesignated.*

Cl(4) : $\Delta_h(g(\varphi_1, \ldots, \varphi_m)) \subseteq Cl(\Delta_j(\varphi_1) \cup \cdots \cup \Delta_k(\varphi_m))$ *whenever* $\varphi_1, \ldots, \varphi_m \in S$, $g \in CON$, *and* $g^\circ(j, \ldots, k) = h$.

Sketch of Proof Since the proof uses well-publicized techniques, a sketch should suffice. (Cf. Beall and van Fraassen [1], pp. 182–85.) It is easy to confirm that Cl is a closure operator. Furthermore, Cl is known to be finitary. (See van Fraassen [8], pp. 142–44; Weaver [9]; and Woodruff [11].) Does Cl really satisfy Cl(1)–Cl(4)?

Cl(1) If $A \subseteq D_{\mathbf{M}}$ and $\mathbf{M}(\varphi) = h$, then $Cl(A \cup \Delta_h(\varphi)) \subseteq D_{\mathbf{M}}$. So if a formula belongs to $\bigcap_{h \in V} Cl(A \cup \Delta_h(\varphi))$, it belongs to every theory that contains A.

Cl(2) Suppose $j \neq k$. Then $\Delta_j(\varphi) \cup \Delta_k(\varphi)$ is unsatisfiable, since no interpretation assigns both j and k to φ. So $Cl(\Delta_j(\varphi) \cup \Delta_k(\varphi)) = \cap\varnothing = S$.

Cl(3) $\Delta_j(\varphi) \cup \{\varphi\}$ is satisfiable only if j is designated.

Cl(4) Suppose $(\Delta_j(\varphi_1)\cup\cdots\cup\Delta_k(\varphi_m)) \subseteq D_{\mathbf{M}}$. Then $\mathbf{M}(\varphi_1) = j, \ldots, \mathbf{M}(\varphi_m) = k$. So $\mathbf{M}(g(\varphi_1, \ldots, \varphi_m)) = g^\circ(j, \ldots, k)$. But then $\Delta_h(g(\varphi_1, \ldots, \varphi_m)) \subseteq D_{\mathbf{M}}$ if $g^\circ(j, \ldots, k) = h$.

Now suppose K is a finitary closure operator on S that satisfies Cl(1)–Cl(4). We want to show that $Cl \leq K$. If $A \subseteq S$, say that A is a *K-maxiset* if and only if $K(A) \neq S$ but $K(A \cup \{\varphi\}) = S$ whenever $\varphi \notin A$. The following fact about K-maxisets will be useful.

Lemma 10.3 *If A is a K-maxiset and $\varphi \in S$, then $\Delta_h(\varphi) \subseteq A$ for exactly one value h.*

Proof Since K satisfies Cl(2), $\Delta_h(\varphi) \subseteq A$ for at most one value h. Suppose, on the other hand, there is no such h. Then $K(A \cup \Delta_h(\varphi)) = S$ for each h. But Cl(1) then implies that $K(A) = S$. □

Now pick any K-maxiset A. Lemma 10.3 justifies the following definition of the function $\mathbf{M}$:

$$\mathbf{M}(\varphi) = h \text{ if and only if } \Delta_h(\varphi) \subseteq A.$$

We want to show that $\mathbf{M}$ is an interpretation. Suppose $\mathbf{M}(\varphi_1) = j, \ldots, \mathbf{M}(\varphi_m) = k$. Then $(\Delta_j(\varphi_1) \cup \cdots \cup \Delta_k(\varphi_m)) \subseteq A$. Suppose $g^\circ(j, \ldots, k) = h$. Then, by Cl(4), $\Delta_h(g(\varphi_1, \ldots, \varphi_m)) \subseteq A$. So $\mathbf{M}(g(\varphi_1, \ldots, \varphi_m)) = h = g^\circ(\mathbf{M}(\varphi_1), \ldots, \mathbf{M}(\varphi_m))$, as desired. Since $\mathbf{M}$ is an interpretation, it makes sense to talk about the theory $D_{\mathbf{M}}$. Cl(3) allows us to show that $D_{\mathbf{M}} = A$. More generally, every K-maxiset is a theory of $\langle V, D, L, CON, ^\circ \rangle$. Let B be any subset of S and suppose $\varphi \in Cl(B)$. Then φ belongs to every theory that contains B. So φ belongs to every K-maxiset that contains B. Since K is finitary, Cl(1) and Cl(3) allow us to show that $K(B)$ is the intersection of the K-maxisets that contain B. So $\varphi \in K(B)$. We conclude that $Cl \leq K$. □

Suppose we have a deductive system in the language of $\langle V, D, L, CON, ^\circ \rangle$. If A is a set of sentences in this language, let $K(A)$ be the smallest deductively closed set that contains A. If K satisfies Cl(1)–Cl(4), then our deductive system is complete with respect to $\langle V, D, L, CON, ^\circ \rangle$: if φ belongs to every theory that contains A, then φ is derivable from members of A.

11 What is Denial?

The question "What is negation?" has inspired a substantial literature. (See Gabbay and Wansing [2], for example.) A natural response to this question is to identify conditions under which one sentence can reasonably be regarded as the negation of another. It may not be clear whether the results of this paper contribute to this enterprise. It is clear, however, that those results help us answer a slightly different question. Say that one denies a sentence when one attributes to it some form of untruth. One might wonder how many forms of denial are supplied by a given language. That is, how many ways of attributing untruth are available? Suppose $\neg$ expresses an expressive function that identifies no values. Then $\neg$ provides at least as many ways of denying a sentence as there are undesignated values, for we can attribute the value h to a sentence φ by affirming every member of $\Delta_h(\varphi)$. If there is only one designated value, then $\neg$ expresses a cyclic negation and each set $\Delta_h(\varphi)$ is a singleton of the form $\{\neg^m\varphi\}$ (with $m = 0$ if h is designated). If m is greater than 0 but less than

n (the number of truth-values), then an affirmation of $\neg^m \varphi$ denies φ by attributing to it an undesignated value. (Does this help us to decide whether the "cyclic negation" $\neg$ really is a negation? Perhaps a connective in a logic really is a negation if it does a good job of representing the behavior of "not" in some form of English-language discourse. Since determining this would require an empirical linguistic inquiry, we put the question on hold and move on.) Now suppose $\neg$ expresses the five-valued function with cycles (1*2) and (3*45). Then

$$\begin{aligned}
\Delta_{1^*}(\varphi) &= \{\neg^0\varphi, \neg^2\varphi, \neg^4\varphi\},\\
\Delta_2(\varphi) &= \{\neg^1\varphi, \neg^3\varphi, \neg^5\varphi\},\\
\Delta_{3^*}(\varphi) &= \{\neg^0\varphi, \neg^3\varphi\},\\
\Delta_4(\varphi) &= \{\neg^2\varphi, \neg^5\varphi\},\\
\Delta_5(\varphi) &= \{\neg^1\varphi, \neg^4\varphi\}.
\end{aligned}$$

In this setting, the most informative way to deny φ is to attribute to it one of our three undesignated values. For example, we could attribute 4 to φ by affirming $\neg^2\varphi$ and $\neg^5\varphi$. The next most informative way to deny φ is to assert that it has one of two undesignated values. For example, we could assert that φ has either value 2 or value 5 by affirming $\neg^1\varphi$, since $\{\neg^1\varphi\} = \Delta_2(\varphi) \cap \Delta_5(\varphi)$. We cannot issue a less informative denial using just $\neg$, since $\Delta_2(\varphi) \cap \Delta_4(\varphi) \cap \Delta_5(\varphi) = \varnothing$. (Here is a difference between affirmation and denial: φ always belongs to $\bigcap_{h \in D} \Delta_h(\varphi)$. Does this represent some necessary truth about language? Could there be a language in which one denies a sentence by enunciating that very sentence?)

12 Overview

To determine whether the unary n-valued truth function α is expressive (with respect to some choice of designated values) we need only inspect a very simple logic: an n-valued logic with just one sentential variable, whose only connective expresses α. α is expressive if and only if the theories of this logic are all maximally consistent. Let us consider the two directions of this biconditional one at a time.

Left-right: The definition of expressiveness guarantees that if α is expressive, then the irreducible theories of any logic expressing α will be maximally consistent. What about the reducible theories? If there were any, they would certainly not be maximally consistent. It turns out, though, that no logic expressing a unary expressive truth function will have any reducible theories. This is a special feature of *unary* expressive truth functions. By way of contrast, consider material implication. This binary truth function is expressive. But, in a sentential logic whose only connective expresses material implication, every sentence will be true when all the sentential variables are true. So, in such a logic, the set of all sentences is a theory and is reducible since we identify it with $\cap\varnothing$.

Right-left: Consider again the logic with just one sentential variable and just one connective, the latter expressing α. Suppose all the theories of this logic are maximally consistent. Then this logic itself is expressive. What happens, though, when we add variables and connectives? Might we acquire an irreducible that is not maximally consistent? As we have seen, the answer is "No." Add as many variables as you wish. Add any connectives that take your fancy. The result can only be another expressive logic.

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