Notre Dame Journal of Formal Logic
Volume 45, Number 1, 2004

Wittgensteinian Predicate Logic

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Abstract We investigate a first-order predicate logic based on Wittgenstein's suggestion to express identity of object by identity of sign and difference of objects by difference of signs. Hintikka has shown that predicate logic can indeed be set up in such a way; we show that it can be done nicely. More specifically, we provide a perspicuous cut-free sequent calculus, as well as a Hilbert-type calculus, for Wittgensteinian predicate logic and prove soundness and completeness theorems.

> Roughly speaking: to say of *two* things that they are identical is nonsense, and to say of *one* thing that it is identical with itself is to say nothing. (*Tractatus* 5.5303)

1 Introduction

In the *Tractatus* [5], Wittgenstein suggests (5.53–5.535) that a proper logical notation has no place for a symbol of identity. Identity of object is to be expressed, according to Wittgenstein, by means of identity of sign, difference of objects by means of difference of signs. Anything worthy of expression, he intimates, can be expressed with the help of this convention, and without using an identity symbol. By way of example, he proposes to write $F(a, a)$ instead of $F(a, b) \wedge a = b$ and $F(a, b)$ instead of $F(a, b) \wedge a \neq b$, it being understood that, in his notation, (typographically) different free variables take different values (5.531). What is standardly written as $\exists x \exists y (F(x, y) \wedge x = y)$ should rather be formulated as $\exists x F(x, x)$; the standard notation $\exists x \exists y (F(x, y) \wedge x \neq y)$ is to be replaced by $\exists x \exists y F(x, y)$, and what is

Received July 14, 2003; accepted February 5, 2004; printed September 1, 2004
2000 Mathematics Subject Classification: Primary, 03A05, 03B60, 03F05
Keywords: Wittgenstein, variables, cut-free calculi

commonly expressed by $\exists x \exists y F(x, y)$ is to be written as $\exists x \exists y F(x, y) \vee \exists x F(x, x)$ (5.532).

Hintikka [3] has shown that such a reformulation of predicate logic is possible in principle, and indeed without loss of expressive power. The formal system he proposes is, however, of a rather nonstandard nature, and while he asserts that his calculus is complete, no proof of this fact is offered.

In the present paper, we provide a natural, cut-free sequent calculus for Wittgensteinian predicate logic and prove its completeness with respect to the intended semantics; a Hilbert-type axiomatization is also given.[1] Along the way, we rehearse and slightly generalize Hintikka's results on the mutual interpretability of standard and Wittgensteinian logic.[2]

2 Terminology and Basic Notions

The language $\mathcal{L}$ of first-order logic without equality, FOL, has as primitive symbols infinitely many free individual variables $a, b, a_1, \ldots$; infinitely many bound individual variables $x, y, x_1, \ldots$; for each $n < \omega$, infinitely many n-ary predicate symbols $R^n, S^n, R^n_1, \ldots$;[3] the 0-ary propositional connective $\perp$, the binary propositional connective $\to$, and the first-order universal quantifier $\forall$. The language $\mathcal{L}^=$ of first-order logic with equality, FOL$^=$, has the binary predicate symbol $=$ in addition to the primitive symbols of $\mathcal{L}$. The formulas of $\mathcal{L}$ and of $\mathcal{L}^=$ are defined inductively as usual; we use $\neg A$ as an abbreviation for $A \to \perp$, $\top$ for $\perp \to \perp$, and $s \neq t$ for $\neg s = t$; further propositional connectives and the existential quantifier may be assumed to have their standard definitions. The set of free variables occurring in the formula A is denoted by $\mathsf{FV}(A)$. We use Γ, Δ, Λ to denote finite sets of formulas. A sequent is an ordered pair of finite formula sets; we write $\Gamma : \Delta$ instead of $\langle \Gamma, \Delta \rangle$; also Γ, Δ instead of $\Gamma \cup \Delta$; Γ, A instead of $\Gamma, \{A\}$; $\mathsf{FV}(\Gamma)$ instead of $\bigcup\{\mathsf{FV}(A) | A \in \Gamma\}$, and so on.

As usual, a structure is a tuple $\mathcal{U} = \langle U, \langle R^n_{\mathcal{U}} \rangle \rangle$, where U, the domain or universe of $\mathcal{U}$, is a nonempty set, and for each n-ary predicate symbol R^n of $\mathcal{L}$, $R^n_{\mathcal{U}}$ is an n-ary relation over U. A $\mathcal{U}$-assignment is a function mapping the free variables into U. We assume the standard notions of an assignment σ satisfying a formula A in a structure $\mathcal{U}$, $\mathcal{U} \models A[\sigma]$, of validity of A in $\mathcal{U}$, $\mathcal{U} \models A$, and of a formula's logical truth, $\models A$; in the case of $\mathcal{L}^=$, we let the equality symbol be interpreted as true identity. We extend these notions to sequents by regarding $\Gamma : \Delta$ as an alternative notation for the formula $\bigwedge \Gamma \to \bigvee \Delta$, where it is understood that the empty conjunction is $\top$ and the empty disjunction is $\perp$.

$\mathcal{L}$ is the language underlying Wittgensteinian predicate logic, or W-logic for short. As mentioned in the introduction, the fundamental semantic idea behind W-logic is that distinct free variables must be assigned distinct values. We thus define satisfaction of an $\mathcal{L}$-formula A by an assignment σ in an $\mathcal{L}$-structure $\mathcal{U}$ with respect to W-logic, $\mathcal{U} \Vdash A[\sigma]$, only for the case that σ is 1-1 on $\mathsf{FV}(A)$. Consequently, the quantifier clause in the definition of $\mathcal{U} \Vdash A[\sigma]$ reads: $\mathcal{U} \Vdash \forall x F(x)[\sigma]$ if and only if for all a-variants τ of σ that are 1-1 on $\mathsf{FV}(F(a))$, $\mathcal{U} \Vdash F(a)[\tau]$, where a is a fresh (i.e., not occurring in $\forall x F(x)$) free variable.[4]

$\mathcal{U} \Vdash A$ means $\mathcal{U} \Vdash A[\sigma]$ for all σ that are 1-1 on $\mathsf{FV}(A)$; so we vacuously have $\mathcal{U} \Vdash A$ if the cardinality of $\mathsf{FV}(A)$ exceeds that of U. $\Vdash A$ means $\mathcal{U} \Vdash A$ for all $\mathcal{L}$-structures $\mathcal{U}$. As before, these semantic notions are extended to sequents by identifying the sequent $\Gamma : \Delta$ with the formula $\bigwedge \Gamma \to \bigvee \Delta$; for instance, $\mathcal{U} \Vdash \Gamma : \Delta$ if and only if $\mathcal{U} \Vdash \Gamma : \Delta[\sigma]$ for all σ that are 1-1 on $\mathsf{FV}(\Gamma, \Delta)$.

3 Translations

W-logic translates straightforwardly into FOL$^=$: Inductively define a mapping ψ on the $\mathcal{L}$-formulas as follows. For atomic formulas P (including $\bot$), $\psi(P)$ is just P; $\psi(A \to B)$ is $\psi(A) \to \psi(B)$, and $\psi(\forall x F(x))$ is

$$\forall x \left(\bigwedge \{x \neq b | b \in \mathsf{FV}(\forall x F(x))\} \to \psi(F(x)) \right),$$

where $\psi(F(x))$ is the result of substituting x for a in $\psi(F(a))$, a being a fresh free variable.[5] It is then routine to show that, for all $\mathcal{L}$-formulas A, all $\mathcal{L}$-structures $\mathcal{U}$, and all $\mathcal{U}$-assignments σ that are 1-1 on $\mathsf{FV}(A)$, $\mathcal{U} \Vdash A[\sigma]$ if and only if $\mathcal{U} \models \psi(A)[\sigma]$. As a corollary, we obtain that $\mathcal{U} \Vdash A$ if and only if $\mathcal{U} \models \bigwedge\{a \neq b | a, b \in \mathsf{FV}(A), a \text{ distinct from } b\} \to \psi(A)$. In particular, for sentences (closed formulas) A, $\mathcal{U} \Vdash A$ if and only if $\mathcal{U} \models \psi(A)$. Hence, in one important sense, whatever is expressible in W-logic can also be expressed in FOL$^=$.

What is perhaps more remarkable is that FOL$^=$ can also be translated into W-logic. Let $\top(a)$ be the formula $Ra \to Ra$, where R is an arbitrarily chosen unary predicate symbol of $\mathcal{L}$, and let $\bot(a, b)$ be the formula $\neg(Sab \to Sab)$, where S is an arbitrary binary predicate symbol of $\mathcal{L}$.[6] Let φ be the function mapping $\mathcal{L}^=$-formulas to $\mathcal{L}$-formulas that is defined inductively as follows: Where P is an atomic formula (including $\bot$), but not an equation, $\varphi(P)$ is just P; $\varphi(a = b)$ is $\bot(a, b)$ if a and b are distinct free variables, and $\varphi(a = a)$ is $\top(a)$. $\varphi(A \to B)$ is just $\varphi(A) \to \varphi(B)$. Finally, $\varphi(\forall x F(x))$ is

$$\forall x \varphi(F(x)) \wedge \bigwedge \{\varphi(F(b)) | b \in \mathsf{FV}(\forall x F(x))\},$$

where again $\varphi(F(x))$ is the result of substituting x for a in $\varphi(F(a))$, for some fresh free variable a.

It is routine to show that, for all $\mathcal{L}^=$-formulas A, all structures $\mathcal{U}$, and $\mathcal{U}$-assignments σ that are 1-1 on $\mathsf{FV}(A)$, $\mathcal{U} \models A[\sigma]$ if and only if $\mathcal{U} \Vdash \varphi(A)[\sigma]$; hence $\mathcal{U} \models \bigwedge\{a \neq b | a, b \in \mathsf{FV}(A), a \text{ distinct from } b\} \to A$ if and only if $\mathcal{U} \Vdash \varphi(A)$, and so for $\mathcal{L}^=$-sentences A, $\mathcal{U} \models A$ if and only if $\mathcal{U} \Vdash \varphi(A)$. Thus, everything expressible in FOL$^=$ can also be expressed in W-logic: The two logics are of the same expressive power. Indeed, as the translation results show, for $\mathcal{L}^=$-sentences (respectively, $\mathcal{L}$-sentences) A, we have $\models A \leftrightarrow \psi(\varphi(A))$ (respectively, $\Vdash A \leftrightarrow \varphi(\psi(A))$).[7]

4 A Cut-free Sequent Calculus for W-Logic

In this section, we develop a Gentzen-style sequent calculus for W-logic. As usual, derivations will be labeled finite trees such that logical axioms are assigned to terminal nodes, and whenever $c_1, \ldots, c_n$ are the immediate successors of a node c, the sequents associated with $c_1, \ldots, c_n$ are the premises of an inference rule whose conclusion is the sequent associated with c.

It will be obvious that all logical axioms (§4.1) are W-logical truths; similarly, it will be immediately clear that the propositional rules of inference (§4.2) are sound (i.e., whenever all premises of such a rule are W-logical truths, so is its conclusion).

As one would expect, the nonstandard interpretation of the variables in W-logic becomes manifest in the quantifier rules (§4.3), for which we provide soundness arguments.

If we write $\vdash_{\mathrm{W}} \Gamma : \Delta$ for the derivability of $\Gamma : \Delta$ by means of (4.1) – (4.3), we will then have established the following soundness theorem.

Theorem 4.1 *If* $\vdash_{\mathrm{W}} \Gamma : \Delta$, *then* $\Vdash \Gamma : \Delta$.

But now for the specification of our calculus.

4.1 Logical axioms Logical axioms are all sequents of the form $\Gamma, P : P, \Delta$, where P is atomic, or of the form $\Gamma, \bot : \Delta$. It is clear from our semantic definitions that all logical axioms are W-logical truths.

4.2 Propositional inference rules The first propositional rule, implication introduction in the succedent ($\to$ S), allows the derivation of $\Gamma : A \to B, \Delta$ from the premise $\Gamma, A : B, \Delta$.

The other propositional rule is implication introduction in the antecedent ($\to$ A): It allows the derivation of $\Gamma, A \to B : \Delta$ from the two premises $\Gamma : A, \Delta$ and $\Gamma, B : \Delta$. The soundness of both rules is obvious.

4.3 Quantificational rules of inference An application of Wittgensteinian universal quantifier introduction in the succedent ($\forall \mathsf{S}^{\mathrm{w}}$) has the following premises:

1. $\Gamma : F(a), \Delta$, where $a \notin \mathsf{FV}(\Gamma, \forall x\, F(x), \Delta)$;
2. for every $b \in \mathsf{FV}(\Gamma, \Delta) \setminus \mathsf{FV}(\forall x\, F(x))$, the sequent $\Gamma : F(b), \Delta$.

Its conclusion then is $\Gamma : \forall x\, F(x), \Delta$.

To see that this rule is sound, suppose that $\mathcal{U} \nVdash \Gamma : \forall x\, F(x), \Delta[\sigma]$ for some σ that is 1-1 on $\mathsf{FV}(\Gamma, \forall x\, F(x), \Delta)$. In particular then, $\mathcal{U} \nVdash \forall x\, F(x)[\sigma]$. So there is an element u of U not in $\sigma[\mathsf{FV}(\forall x\, F(x))]$ such that, whenever c is a free variable not occurring in $\forall x\, F(x)$, $\mathcal{U} \nVdash F(c)[\sigma\{c := u\}]$, where $\sigma\{c := u\}$ is the c-variant of σ that maps c to u. If u is not in $\sigma[\mathsf{FV}(\Gamma, \Delta)]$ either, then $\sigma\{a := u\}$ is 1-1 on $\mathsf{FV}(\Gamma, F(a), \Delta)$ and $\mathcal{U} \nVdash \Gamma : F(a), \Delta[\sigma\{a := u\}]$, because a does not occur in Γ, Δ. But if u is among $\sigma[\mathsf{FV}(\Gamma, \Delta)]$, say $u = \sigma(b)$, where $b \in \mathsf{FV}(\Gamma, \Delta) \setminus \mathsf{FV}(\forall x\, F(x))$, then $\mathcal{U} \nVdash \Gamma : F(b), \Delta[\sigma]$ and σ is 1-1 on $\mathsf{FV}(\Gamma, F(b), \Delta)$.

The rule ($\forall \mathsf{A}^{\mathrm{w}}$), Wittgensteinian universal quantifier introduction in the antecedent, allows the derivation of $\Gamma, \forall x\, F(x) : \Delta$ from the premise $\Gamma, F(a) : \Delta$, provided that

1. $a \notin \mathsf{FV}(\forall x\, F(x))$ and
2. either $a \in \mathsf{FV}(\Gamma, \Delta)$ or $\mathsf{FV}(\Gamma, \forall x\, F(x), \Delta) = \varnothing$.

To see that the rule is sound, suppose that $\mathcal{U} \nVdash \Gamma, \forall x\, F(x) : \Delta[\sigma]$ for some σ that is 1-1 on $\mathsf{FV}(\Gamma, \forall x\, F(x), \Delta)$. Hence $\mathcal{U} \Vdash \forall x\, F(x)[\sigma]$. First suppose that $a \in \mathsf{FV}(\Gamma, \Delta)$. Then σ is 1-1 on $\mathsf{FV}(\Gamma, F(a), \Delta)$. Also, $\mathcal{U} \Vdash F(a)[\sigma]$ because $a \notin \mathsf{FV}(\forall x\, F(x))$. Hence $\mathcal{U} \nVdash \Gamma, F(a) : \Delta[\sigma]$. Now suppose $\mathsf{FV}(\Gamma, \forall x\, F(x), \Delta) = \varnothing$. Then any assignment is 1-1 on $\mathsf{FV}(\Gamma, F(a), \Delta)$; in particular, σ is. Since $\mathcal{U} \Vdash F(a)[\sigma]$, we also have $\mathcal{U} \nVdash \Gamma, F(a) : \Delta[\sigma]$.

The quantifier rules may not look too perspicuous at this point, so it is perhaps in order to comment briefly on them. First, the proliferation of premises in the case of ($\forall \mathsf{S}^{\mathrm{w}}$) is due to the fact that the free variable a in the first premise has as its scope the whole sequent $\Gamma :\ F(a), \Delta$, so that its range is restricted to those objects not already assigned to a member of $\mathsf{FV}(\Gamma, \forall x\, F(x), \Delta)$. The range of the bound variable x in $\forall x\, F(x)$, however, is only restricted to objects not already assigned to a member of $\mathsf{FV}(\forall x\, F(x))$, so that we need to ensure $F(b)$ for those members b of $\mathsf{FV}(\Gamma, \Delta)$ that do not also occur in $\mathsf{FV}(\forall x\, F(x))$ in order to be able to establish the conclusion.

Second, the variable conditions in ($\forall \mathsf{A}^{\mathrm{w}}$) arise as follows. Obviously, the free variable a must not occur in $\forall x\, F(x)$, for otherwise $F(a)$ cannot be inferred from $\forall x\, F(x)$ on our Wittgensteinian semantics. The condition '$a \in \mathsf{FV}(\Gamma, \Delta)$ or $\mathsf{FV}(\Gamma, \forall x\, F(x), \Delta) = \varnothing$' is perhaps less obvious. Let us consider the first disjunct first. It is designed in order to prevent a loss of free variables in passing from the premise to the conclusion; this is problematic because models of cardinality $n < \omega$ vacuously verify any sequent containing more than n free variables. Consider, for instance, the sequent $Rab \vee \neg Rab : \exists x \exists y (Rxy \vee \neg Rxy)$. This is vacuously valid in 1-element structures, and obviously valid in all structures with more than 1 element, so it is a W-logical truth. Uncritical application of ($\forall \mathsf{A}^{\mathrm{w}}$) would yield the sequent

$$\forall x (Rxb \vee \neg Rxb) : \exists x \exists y (Rxy \vee \neg Rxy),$$

which, however, fails to hold in 1-element structures. Hence we must, in general, make sure that no free variables are lost during an application of this rule. However, and this brings us to the second disjunct in the variable condition, if the conclusion contains no free variables at all, the rule is still sound, as shown in the soundness proof. It is also indispensable to allow for this case, for without it, it would be impossible to derive the W-logically valid sequent $\forall x\, Rx, \forall x \neg Rx : \varnothing$, as one easily sees.

Incidentally, similar considerations arise with respect to the cut rule, allowing the inference from the premises $\Gamma : A, \Delta$ and $\Gamma, A : \Delta$ to the conclusion $\Gamma : \Delta$, which is sound (and hence, by the completeness theorem of the next section, admissible for our calculus) only under the following condition on variables: *Either* $\mathsf{FV}(A) \subseteq \mathsf{FV}(\Gamma, \Delta)$, *or* Γ, Δ contains no free variables at all, and A contains only one free variable. It is clear that, under this restriction, the rule is sound; and the following example shows that the second disjunct is the only permissible relaxation of the "no loss of free variables" maxim: While we have

$$\Vdash Rab \vee \neg Rab : \exists x \exists y (Rxy \vee \neg Rxy)$$

and

$$\Vdash \varnothing : Rab \vee \neg Rab, \exists x \exists y (Rxy \vee \neg Rxy),$$

we also have

$$\nVdash \varnothing : \exists x \exists y (Rxy \vee \neg Rxy).$$

The same considerations are relevant to the rule of modus ponens for the Hilbert-type system of Section 6.[8]

5 Completeness

Suppose that $\Vdash \Gamma : \Delta$. By the results of Section 3, it follows that $\models \psi[\Gamma] : \psi[\Delta]$, $\{a = b | a, b \in \mathsf{FV}(\Gamma, \Delta), a \text{ distinct from } b\}$. By the completeness theorem for $\mathrm{FOL}^{=}$, we then have

$$\vdash \psi[\Gamma] : \psi[\Delta], \{a = b | a, b \in \mathsf{FV}(\Gamma, \Delta), a \text{ distinct from } b\}$$

for some suitable notion $\vdash$ of derivability in FOL$^{=}$ (such a notion will be introduced below). If we can show that this entails $\models_{\mathrm{w}} \Gamma : \Delta$, we will have established the following completeness theorem for W-logic.

Theorem 5.1 *If* $\Vdash \Gamma : \Delta$*, then* $\models_{\mathrm{w}} \Gamma : \Delta$.

Before embarking on the proof, we must specify a suitable complete calculus for FOL$^{=}$. The following will do.[9]

Logical axioms are all $\mathcal{L}^{=}$-sequents of the form $\Gamma, P : P, \Delta$, where P is atomic, or of the form $\Gamma, \bot : \Delta$. The propositional rules are $(\rightarrow \mathsf{S})$ and $(\rightarrow \mathsf{A})$.

The rule $(\forall\mathsf{S})$ allows the derivation of $\Gamma : \forall x\, F(x), \Delta$ from $\Gamma : F(a), \Delta$, provided that $a \notin \mathsf{FV}(\Gamma, \forall x\, F(x), \Delta)$; and $(\forall\mathsf{A})$ permits the derivation of $\Gamma, \forall x\, F(x) : \Delta$ from $\Gamma, F(a) : \Delta$.

There are two additional rules governing the inferential behavior of the equality symbol: The rule $(= \mathsf{Refl})$ licenses the inference from $\Gamma, a = a : \Delta$ to $\Gamma : \Delta$; and the rule $(= \mathsf{Cong})$ permits the derivation of $\Gamma, Rb_1 \ldots b_n, b_1 = a_1, \ldots, b_n = a_n : \Delta$ from $\Gamma, Ra_1 \ldots a_n : \Delta$, where R is an n-ary predicate symbol, possibly the equality symbol itself (in which case, of course, $n = 2$).

The *order* of a derivation tree of this calculus is the maximum of the lengths of paths through the tree. We write $\models_{n} \Gamma : \Delta$ if there is a derivation of $\Gamma : \Delta$ of order at most n; so we have $\models_{n} \Gamma : \Delta$ for all $n < \omega$ if $\Gamma : \Delta$ is a logical axiom, and if $\models_{n} \Gamma_i : \Delta_i$ (for $i = 0$ or $i = 0, 1$) and $\Gamma_i : \Delta_i$ is (are) the premise(s) of an inference rule whose conclusion is $\Gamma : \Delta$, then $\models_{n+1} \Gamma : \Delta$. Clearly $\Gamma : \Delta$ is derivable, $\vdash \Gamma : \Delta$ if and only if, for some $n < \omega$, $\models_{n} \Gamma : \Delta$.

Our cut-free calculus for FOL$^{=}$ has the following useful proof-theoretic properties:

substitution rule If $\models_{n} \Gamma(a) : \Delta(a)$, and a occurs only as indicated, then $\models_{n} \Gamma(b) : \Delta(b)$ for any free variable b.

structural rule If $\models_{n} \Gamma : \Delta$ and $\Gamma \subseteq \Gamma_0$, $\Delta \subseteq \Delta_0 \cup \{\bot\}$, then $\models_{n} \Gamma_0 : \Delta_0$.

invertibility of ($\rightarrow$ S) If $\models_{n} \Gamma : A \rightarrow B, \Delta$, then $\models_{n} \Gamma, A : B, \Delta$.

invertibility of ($\rightarrow$ A) If $\models_{n} \Gamma, A \rightarrow B : \Delta$, then $\models_{n} \Gamma : A, \Delta$ and $\models_{n} \Gamma, B : \Delta$.

The desired result follows from this slightly more general lemma.

Lemma 5.2 *Let* $\Gamma : \Delta$ *be an* $\mathcal{L}$*-sequent. Let* Λ_0 *be a finite set of equations of the form* $a = a$*, and let* Λ_1 *be a finite set of equations of the form* $a = b$*, where* a *and* b *are distinct free variables occurring in* $\Gamma : \Delta$*. If* $\models_{m} \Lambda_0, \psi[\Gamma] : \psi[\Delta], \Lambda_1$*, then also* $\models_{\mathrm{w}} \Gamma : \Delta$.

Proof We proceed by induction on m. It is easy to see that in case $m = 0$, $\Gamma : \Delta$ must be a logical axiom.

For the induction step, suppose given a FOL$^{=}$-derivation of the sequent Λ_0, $\psi[\Gamma] : \psi[\Delta], \Lambda_1$ of order at most $m + 1$. We distinguish six cases, according to the last inference rule applied in this derivation.

Case 1 The case of $(\rightarrow \mathsf{S})$ is straightforwardly dealt with by invoking the induction hypothesis and the rule $(\rightarrow \mathsf{S})$.

Case 2 The case of $(\rightarrow \mathsf{A})$ is almost as straightforward, except that we need to invoke the structural rule (importing the main formula into the premises' antecedents, if not already present) in order for Λ_1 not to be too large for an application of the induction hypothesis.

Case 3 The case of ($=$ Refl) is trivial.

Case 4 Now suppose the last rule applied was ($=$ Cong). Hence $\Lambda_0, \psi[\Gamma]$ can also be represented as $\Gamma_0, Rb_1 \dots b_n, b_1 = a_1, \dots, b_n = a_n$. Since $\psi[\Gamma]$ cannot contain equations, all formulas $b_i = a_i$ must belong to Λ_0, and hence, for all i, a_i is the very same variable as b_i. Suppose first that R is the equality symbol. Then $n = 2$, b_1 is the same variable as b_2, and the premise of the application of ($=$ Cong) is identical with its conclusion, so $\vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$ by the induction hypothesis. Now suppose R is not the equality symbol. Then $Rb_1 \dots b_n$ is in $\psi[\Gamma]$, and Γ_0 is a subset of $\Lambda_0, \psi[\Gamma]$, so we may apply the structural rule to expand the premise of the ($=$ Cong)-inference to $\Lambda_0, \psi[\Gamma] : \psi[\Delta], \Lambda_1$, and $\vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$ follows by the induction hypothesis.

Case 5 Now suppose the last inference was an application of ($\forall$S). Then $\Lambda_0, \psi[\Gamma] : \psi[\Delta], \Lambda_1$ has the form $\Lambda_0, \psi[\Gamma] : \forall x G(x), \Delta_0$, so that, in particular, $\forall x G(x) \in \psi[\Delta]$ and $\Lambda_1 \subseteq \Delta_0$. The formula $\forall x G(x)$ must then be of the form $\psi(\forall x F(x))$, that is, $\forall x\, (x \neq b_1 \to \dots \to x \neq b_n \to \psi(F(x)))$, where $b_1, \dots, b_n$ are the n free variables occurring in $\forall x F(x)$. The premise of the ($\forall$S)-inference is therefore

$$\Lambda_0, \psi[\Gamma] : a \neq b_1 \to \dots \to a \neq b_n \to \psi(F(a)), \Delta_0,$$

where a does not occur in $\Lambda_0, \psi[\Gamma], \forall x F(x), \Delta_0$. By n applications of ($\to$ S)-inversion, we obtain $\vdash_{\overline{m}} \Lambda_0, \psi[\Gamma], a \neq b_1, \dots, a \neq b_n : \psi(F(a)), \Delta_0$, and by n applications of ($\to$ A)-inversion, we then also have

$$\vdash_{\overline{m}} \Lambda_0, \psi[\Gamma] : \psi(F(a)), \Delta_0, a = b_1, \dots, a = b_n.$$

By the structural rule, we obtain

$$(*) \qquad \vdash_{\overline{m}} \Lambda_0, \psi[\Gamma] : \psi(F(a)), \psi[\Delta], \Lambda_1, a = b_1, \dots, a = b_n.$$

From $(*)$, by the substitution rule, for each $b \in \mathsf{FV}(\Gamma, \Delta) \setminus \mathsf{FV}(\forall x F(x))$:

$$(**) \qquad \vdash_{\overline{m}} \Lambda_0, \psi[\Gamma] : \psi(F(b)), \psi[\Delta], \Lambda_1, b = b_1, \dots, b = b_n.$$

From $(*)$ and $(**)$ we obtain, with the help of the induction hypothesis, $\vdash_{\overline{\mathrm{w}}} \Gamma : F(a), \Delta$, where $a \notin \mathsf{FV}(\Gamma, \forall x F(x), \Delta)$, and $\vdash_{\overline{\mathrm{w}}} \Gamma : F(b), \Delta$ for each $b \in \mathsf{FV}(\Gamma, \Delta) \setminus \mathsf{FV}(\forall x F(x))$. Hence, by ($\forall \mathsf{S}^{\mathrm{w}}$), $\vdash_{\overline{\mathrm{w}}} \Gamma : \forall x F(x), \Delta$. This is the desired result, because $\forall x F(x)$ must be a member of Δ anyway, that is, $\vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$.

Case 6 Finally, suppose the last inference was an application of ($\forall$A). Then $\Lambda_0, \psi[\Gamma] : \psi[\Delta], \Lambda_1$ can be written as $\Gamma_0, \forall x G(x) : \psi[\Delta], \Lambda_1$, and $\forall x G(x)$ must be $\psi(\forall x F(x))$ for some formula $\forall x F(x) \in \Gamma$, that is, $\forall x G(x)$ is $\forall x\, (x \neq b_1 \to \dots \to x \neq b_n \to \psi(F(x)))$, where $b_1, \dots, b_n$ are the n free variables of $\forall x F(x)$. So the premise of the ($\forall$A)-inference is

$$\Gamma_0, a \neq b_1 \to \dots \to a \neq b_n \to \psi(F(a)) : \psi[\Delta], \Lambda_1$$

for some free variable a. By the structural rule, we then have

$$\vdash_{\overline{m}} \Lambda_0, \psi[\Gamma], a \neq b_1 \to \dots \to a \neq b_n \to \psi(F(a)) : \psi[\Delta], \Lambda_1.$$

Various applications of the inversion rules for ($\to$ S) and ($\to$ A) yield

$$(+) \qquad \vdash_{\overline{m}} \Lambda_0, \psi[\Gamma], \psi(F(a)) : \psi[\Delta], \Lambda_1$$

and

(++) $\vdash_{\overline{m}} \Lambda_0, \psi[\Gamma], a = b_i : \psi[\Delta], \Lambda_1$ for each $i = 1, \ldots, n$.

Suppose first that $a \in \mathsf{FV}(\forall x F(x))$. Then it follows from (+) that $\vdash_{\overline{m}} a = a$, $\Lambda_0, \psi[\Gamma] : \psi[\Delta], \Lambda_1$, and the induction hypothesis immediately yields $\vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$. So without loss of generality we may assume that $a \notin \mathsf{FV}(\forall x F(x))$. By the induction hypothesis, $\vdash_{\overline{\mathrm{w}}} \Gamma, F(a) : \Delta$, so if $a \in \mathsf{FV}(\Gamma, \Delta)$ or if $\mathsf{FV}(\Gamma, \forall x F(x), \Delta) = \varnothing$, $\vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$ follows by an application of $(\forall \mathsf{A}^{\mathrm{w}})$. Let us thus assume that $a \notin \mathsf{FV}(\Gamma, \Delta)$ and that $\mathsf{FV}(\Gamma, \forall x F(x), \Delta) \neq \varnothing$. If there is a free variable, say b_1, in $\mathsf{FV}(\forall x F(x))$, then (++), the substitution and structural rules imply $\vdash_{\overline{m}} \Lambda_0, \psi[\Gamma], b_1 = b_1 : \psi[\Delta]$, Λ_1, so the induction hypothesis immediately yields $\vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$. Now suppose that $\mathsf{FV}(\forall x F(x))$ is empty. Then there is a free variable $b \in \mathsf{FV}(\Gamma, \Delta)$. From (+) and the substitution rule we obtain $\vdash_{\overline{m}} \Lambda_0', \psi[\Gamma], \psi(F(b)) : \psi[\Delta], \Lambda_1$, where Λ_0' results from Λ_0 by replacing $a = a$, if it occurs at all, with $b = b$. By the induction hypothesis, $\vdash_{\overline{\mathrm{w}}} \Gamma, F(b) : \Delta$, where $b \notin \mathsf{FV}(\forall x F(x))$ and $b \in \mathsf{FV}(\Gamma, \Delta)$, and by an application of $(\forall \mathsf{A}^{\mathrm{w}})$, $\vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$ follows. □

The completeness theorem may be strengthened so as to cover derivations from a (possibly infinite) set of nonlogical axioms: For a given set T of $\mathcal{L}$-sentences, we add to our calculus for W-logic the rule (T), according to which the sequent $\Gamma : \Delta$ may be inferred from $\Gamma, A : \Delta$, provided that A is an element of T. Derivability of a sequent $\Gamma : \Delta$ in W-logic with the help of the (T)-rule will be denoted $\mathsf{T} \vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$.

It is easy to verify that the extended calculus is sound in the following sense. If $\mathsf{T} \vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$, then $\mathcal{U} \Vdash \Gamma : \Delta$ for all $\mathcal{U} \Vdash \mathsf{T}$ (i.e., all $\mathcal{U}$ with $\mathcal{U} \Vdash A$ for all $A \in \mathsf{T}$). Now suppose that $\mathcal{U} \Vdash \Gamma : \Delta$ for all $\mathcal{U} \Vdash \mathsf{T}$, that is, $\Gamma : \Delta$ is a W-logical consequence of T. As before,

$$\psi[\Gamma] : \psi[\Delta], \{a = b | a, b \in \mathsf{FV}(\Gamma, \Delta), a \text{ distinct from } b\}$$

is then a logical consequence of $\psi[\mathsf{T}]$ in the sense of $\mathrm{FOL}^{=}$, and hence, by the compactness theorem, it is a logical consequence of a finite subset $\psi[\mathsf{T}']$ of $\psi[\mathsf{T}]$. Hence

$$\vdash \psi[\mathsf{T}'], \psi[\Gamma] : \psi[\Delta], \{a = b | a, b \in \mathsf{FV}(\Gamma, \Delta), a \text{ distinct from } b\};$$

so by the lemma, $\vdash_{\overline{\mathrm{w}}} \mathsf{T}', \Gamma : \Delta$, and by finitely many applications of the (T)-rule, $\mathsf{T} \vdash_{\overline{\mathrm{w}}} \Gamma : \Delta$.

6 A Hilbert-type Axiomatization

In this section we briefly discuss a Hilbert-type calculus for W-logic. Derivability of an $\mathcal{L}$-formula A from a set T of sentences in this Hilbert system, $\mathsf{T} \vdash_{\overline{\mathrm{wh}}} A$, is defined inductively as follows.

Whenever A is an $\mathcal{L}$-instance of a propositional tautology (in the usual sense), then $\mathsf{T} \vdash_{\overline{\mathrm{wh}}} A$ (propositional axioms).

Whenever $\forall x F(x)$ is an $\mathcal{L}$-formula in which the free variable a does not occur, $\mathsf{T} \vdash_{\overline{\mathrm{wh}}} \forall x F(x) \to F(a)$ (quantifier axioms).

For all members A of T, $\mathsf{T} \vdash_{\overline{\mathrm{wh}}} A$ (nonlogical axioms of T).

If $\mathsf{T} \vdash_{\overline{\mathrm{wh}}} A \to B$ and $\mathsf{T} \vdash_{\overline{\mathrm{wh}}} A$, then $\mathsf{T} \vdash_{\overline{\mathrm{wh}}} B$, provided that *either* $\mathsf{FV}(A) \subseteq \mathsf{FV}(B)$, *or* $\mathsf{FV}(B) = \varnothing$ and $|\mathsf{FV}(A)| = 1$ (modus ponens).

If $\mathsf{T} \vdash_{\overline{\text{wh}}} B \to F(a)$, where $a \notin \mathsf{FV}(B, \forall x\, F(x))$, and if, moreover, for each $b \in \mathsf{FV}(B) \setminus \mathsf{FV}(\forall x\, F(x))$, $\mathsf{T} \vdash_{\overline{\text{wh}}} B \to F(b)$, then we also have $\mathsf{T} \vdash_{\overline{\text{wh}}} B \to \forall x\, F(x)$ (quantifier rule).

The (easy) proof that this calculus is sound with respect to W-logic is left to the reader. For completeness, it suffices to show that the sequent calculus of Section 4 can be embedded into the Hilbert system. This can be done roughly as follows.

Given a sequent $\Gamma : \Delta$ derivable in the sequent calculus, one shows, by induction along the definition of sequent-style derivations, that "the" formula[10] $\bigwedge \Gamma \to \bigvee \Delta$ is derivable in the Hilbert calculus. This, too, is routine, keeping in mind that, whenever A and B are tautologically equivalent, and $\mathsf{T} \vdash_{\overline{\text{wh}}} A$ and $\mathsf{FV}(A) \subseteq \mathsf{FV}(B)$, then $\mathsf{T} \vdash_{\overline{\text{wh}}} B$.

Notes

1. The primary purpose of the present note is a systematic one. It is not intended as a contribution to Wittgenstein scholarship, an area in which we do not feel competent. It seems that there are a number of interesting exegetical issues that come up vis-à-vis the technical results below, but we are confident that others can do a better job in clearing these up.

2. We should mention that we focus on what Hintikka calls the "weakly exclusive" interpretation of the variables. As Hintikka points out, this appears to be the interpretation intended by Wittgenstein: On Hintikka's "strongly exclusive" reading, the paraphrase Wittgenstein suggests in the first sentence of 5.5321 does not have the right truth conditions. Floyd ([2], p. 163) discusses a further interpretation, but it seems that on this additional reading, Wittgenstein must be taken to depart radically from compositional semantics altogether.

3. Nothing is lost if we consider languages with only a subset of these relation symbols present, as long as there is at least one predicate symbol of positive arity (otherwise W-logic collapses into propositional logic).

4. In thus restricting the space of relevant variable assignments, Wittgensteinian semantics is a special case of van Benthem's ([4], pp. 177–78) dependency models. It is interesting to note in this respect that Wittgensteinian predicate logic is *not* decidable (cf. note 7).

5. For the purposes of Section 5, it is more convenient to take $\psi(\forall x\, F(x))$ to be the formula
$$\forall x\, (x \neq b_1 \to \cdots \to x \neq b_n \to \psi(F(x)))\,,$$
where $b_1, \ldots, b_n$ are the n free variables occurring in $\forall x\, F(x)$, and implications are to be associated to the right. We shall assume this definition in Section 5.

6. Obviously $\top(a)$ and $\bot(a, b)$ can be given adequate definitions as long as there is at least one predicate symbol of positive arity present in the language.

7. The translations given in this section are essentially due to Hintikka. Unlike Hintikka, we include treatment of nonclosed formulas, which—it seems to us—makes the results somewhat more transparent. Given these translations, it follows from the completeness theorem for $\mathrm{FOL}^{=}$ that the logical truths of W-logic are positively decidable: To check whether a sentence A is a logical truth of W-logic, simply generate all logical truths of

FOL$^=$ by means of a standard deductive system and check whether $\psi(A)$ occurs among them. So in a sense we already have a completeness theorem for W-logic; however, the FOL$^=$-derivations appealed to will in general involve formulas in which the equality symbol occurs, and rules pertaining to the equality symbol. Since it is our goal to show that the equality symbol is eliminable from logical notation altogether, we want to develop a complete calculus for W-logic that does not involve the equality symbol at all, which is what we do in Section 4. It should be noted that our translations also show that W-logic inherits the undecidability of FOL$^=$, for if W-logic were decidable, in order to decide the logical truth of a FOL$^=$-sentence A, we should only have to decide the W-logical truth of $\varphi(A)$ (and by Hintikka's analogous results for the strongly exclusive interpretation, the same is true in that case).

8. Carnap ([1], p. 50) objects to Wittgenstein's proposal that it "leads to certain complications" with respect to the rule of substitution. His argument is roughly this: $P(c, c)$ should not, on Wittgenstein's proposal, logically follow from $P(a, b)$, when a, b, and c are distinct free variables. But according to Carnap it is "not possible to see" how, if in a first step $P(c, b)$ is derived from $P(a, b)$, one could then prevent the derivation of $P(c, c)$ from $P(c, b)$ by a second application of the substitution rule, and he suggests that the introduction of an additional notation, such as ${}^{c,b}P(c, b)$, is necessary to prevent this. It seems clear, however, that this objection is without substance: It is the fact that c occurs in $P(c, b)$ that prevents the substitution of c for b—an entirely transparent syntactic criterion.

9. That this sequent calculus is indeed complete with respect to FOL$^=$ can be seen by first proving admissibility of the cut rule (that is, if both $\Gamma : A, \Delta$ and $\Gamma, A : \Delta$ are derivable, then so is $\Gamma : \Delta$—also known as Gentzen's *Hauptsatz*), and then embedding a standard Hilbert-style system into the sequent calculus (the cut rule taking care of *modus ponens*).

10. Strictly speaking, there are, in general, many such formulas; however, they are all tautologically equivalent and contain the same free variables, so that, in our Hilbert system, we may choose any one of them.

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Acknowledgments

Thanks to Aldo Antonelli, Juliet Floyd, Norbert Gratzl, Sam Hillier, Tom Ricketts, and Peter Schroeder-Heister for various discussions, and to Albert Visser for many valuable comments. I am further grateful to an anonymous referee for this journal who provided a number of useful suggestions, not all of which I have been able to incorporate in this paper.

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