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A THEOREM FOR DERIVING CONSEQUENCES OF THE AXIOM OF CHOICE

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I. *Introduction* This paper is addressed to the problem of proving results directly from the Axiom of Choice. A general theorem on mappings in partially-ordered sets will be proved, and proofs of Zorn's Lemma and the Well-Ordering Theorem will be given as corollaries to this theorem. The following concepts will be used.

A *partially-ordered set* is a set on which is defined a reflexive, transitive, anti-symmetric binary relation $\leqslant$. A *chain* is a totally-ordered subset of a partially-ordered set. If a partially-ordered set has a smallest and/or a greatest element, these will be represented respectively by 0 and 1. The least-upper-bound, if it has one, of a subset T of a partially-ordered set will be represented by $\bigsqcup T$. It should be noted that if every subset of a partially-ordered set X has a least-upper-bound, then $0 = \bigsqcup \phi$ and $1 = \bigsqcup X$ are in X, where ϕ is the void set.

A *choice function* on a set S is a function which assigns to each non-void subset T of S an element of T. The *Axiom of Choice* states that a choice function may be defined on any set.

The following additional notation will be employed. Set-theoretic inclusion will be represented by $\subseteq$, and strict inclusion by $\subset$. The power-set of a set S will be represented by $\mathsf{P}(S)$. If f is a function defined on a set S, then $f(T)$ will represent the set of images under f of the elements of T, for each subset T of S. In particular, $f(\phi) = \phi$. If $\mathcal{J}$ is a family of subsets of a set S, then $\bigcup \mathcal{J}$ and $\bigcap \mathcal{J}$ will represent respectively the union and intersection of the members of $\mathcal{J}$. In particular, $\bigcup \phi = \phi$ and $\bigcap \phi = S$. Finally, the difference of sets S and T will be represented by $S \backslash T$.

II. *The Main Theorem*

Theorem 1: *If X is a partially-ordered set in which each subset has a least-upper-bound, and g is a function from X into X which satisfies the following condition*:

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(i) $g(y) \leqslant x$ *implies* $g(y) \leqslant g(x)$, *for* x, y in X,

then there is a subset V of X which satisfies the following two conditions:

(ii) $\bigsqcup g(W)$ *is in* V, *for every* $W \subseteq V$,
(iii) $x \leqslant y$ *or* $g(y) \leqslant x$, *for* x, y *in* V.

Proof: Let $\mathcal{A}$ be the family of all subsets T of X for which $\bigsqcup g(W)$ is in T, whenever $W \subseteq T$. The family $\mathcal{A}$ contains X, and hence is non-void. Now let $V = \bigcap \mathcal{A}$. It follows immediately from the definition of V that V is in $\mathcal{A}$, and that if T is a subset of V which is also in $\mathcal{A}$, then $T = V$. The following two lemmas complete the proof of the theorem.

Lemma 1: *Suppose, for some fixed y in V, that $x < y$ implies $g(x) \leqslant y$, whenever x is in V. Then $x \leqslant y$ or $g(y) \leqslant x$, whenever x is in V.*

Proof: Let T be the set of all elements x of V for which $x \leqslant y$ or $g(y) \leqslant x$. We wish to show that T is in $\mathcal{A}$, and hence that $T = V$. Let W be an arbitrary subset of T. Now if $x < y$ for every x in W, then $g(x) \leqslant y$ for every x in W, and $\bigsqcup g(W) \leqslant y$, and $\bigsqcup g(W)$ is in T. Otherwise there is some x_0 in W such that $x_0 = y$ or $g(y) \leqslant x_0$. If $x_0 = y$, then $g(y) = g(x_0) \leqslant \bigsqcup g(W)$. If $g(y) \leqslant x_0$, then, by (i), we have $g(y) \leqslant g(x_0) \leqslant \bigsqcup g(W)$. Hence $\bigsqcup g(W)$ is again in T. Q.E.D.

Lemma 2: *If y is in V, then $x < y$ implies $g(x) \leqslant y$, whenever x is in V.*

Proof: Let T be the set of all elements y in V for which $x < y$ implies $g(x) \leqslant y$, whenever x is in V. We again wish to show that T is in $\mathcal{A}$. Let W be an arbitrary subset of T, and suppose that $x < \bigsqcup g(W)$, where x is in V. By Lemma 1 we have $x \leqslant y$ or $g(y) \leqslant x$, for every y in W. If $g(y) \leqslant x$ for every y in W, then $\bigsqcup g(W) \leqslant x$, which is impossible. Hence there is some y_0 in W such that $x \leqslant y_0$. If $x = y_0$, then $g(x) = g(y_0) \leqslant \bigsqcup g(W)$. On the other hand, if $x < y_0$, then $g(x) \leqslant y_0$, and by (i) we have $g(y) \leqslant g(y_0) \leqslant \bigsqcup g(W)$. In either case $\bigsqcup g(W)$ is in T. Q.E.D.

The following two propositions give some useful additional properties of V.

Proposition 1: *If x, y are in V, then either $g(y) \leqslant x$, or $x = y$, or $g(x) \leqslant y$.*

Proof: Suppose that $g(y) \nleqslant x$ and $x \neq y$. Then, by (iii), we have $x < y$. Hence $y \nleqslant x$, and, again by (iii), we have $g(x) \leqslant y$. Q.E.D.

Proposition 2: *If $g(x) \nleqslant x$, for every x in $X \backslash \{1\}$, then* $\bigsqcup g(V \backslash \{1\}) = 1$.

Proof: If $\bigsqcup g(V \backslash \{1\}) \neq 1$, then we have $g(\bigsqcup g(V \backslash \{1\})) \leqslant \bigsqcup g(V \backslash \{1\})$, which is impossible. Q.E.D.

III. *Zorn's Lemma* We will now use Theorem 1 to prove Zorn's Lemma, in the following form, from the Axiom of Choice.

Zorn's Lemma: *Any partially-ordered set K without maximal elements contains an unbounded chain.*

Proof: Let X be $\mathsf{P}(K)$, partially-ordered by the *inverse* of set-theoretic inclusion. Let f be a choice function on K, and let g be the function on $\mathsf{P}(K)$ defined by:

$$g(T) = \begin{cases} \{x \in K \mid f(T) < x\}\ , & T \neq \phi \\ \phi\ , & T = \phi\ . \end{cases}$$

Lemma 1: *The function g satisfies condition* (i) *of* Theorem 1.

Proof: Suppose that $g(T) \supseteq U$, where T, U are in $\mathsf{P}(K)$. We wish to show that $g(T) \supseteq g(U)$. This is clearly true if $U = \phi$. If $U \neq \phi$, then by assumption $f(U)$ is in $g(T)$, that is, $f(T) < f(U)$. Consequently we have $g(T) \supseteq g(U)$, by the transitivity and antisymmetry of $\leqslant$ in K. Q.E.D.

Now let $\mathscr{V}$ represent the sub-family V of $\mathsf{P}(K)$ which is provided by Theorem 1. The following two lemmas complete the proof of Zorn's Lemma.

Lemma 2: *$f(\mathscr{V} \backslash \{\phi\})$ is a chain in K.*

Proof: This lemma follows immediately from Proposition 1 and the reflexivity of $\leqslant$ in K. Q.E.D.

Lemma 3: *$f(\mathscr{V} \backslash \{\phi\})$ is unbounded.*

Proof: Since $\mathscr{V}$ satisfies the hypothesis of Proposition 2, we have $\bigcap g(\mathscr{V} \backslash \{\phi\}) = \phi$. Consequently any upper bound of $f(\mathscr{V} \backslash \{\phi\})$ must be in $f(\mathscr{V} \backslash \{\phi\})$. But for any $T \in \mathscr{V} \backslash \{\phi\}$ we have $g(T) = \phi$, since K has no maximal elements, and consequently we have $f(T) < f(g(T))$. Hence $f(T)$ is not an upper bound for $f(\mathscr{V} \backslash \{\phi\})$. Q.E.D.

IV. *Second Form of The Main Theorem* In this section and the next a slightly weaker form of the main theorem will be employed. If g is a mapping on a partially-ordered set X for which $x \leqslant g(x)$, for every x in X, then clearly g satisfies condition (i) of Theorem 1. It is also clear by Proposition 1 that the set V of Theorem 1 is totally ordered, if g satisfies this stronger condition.

Lemma 1: *If T is a subset of the set V of* Theorem 1, *then either T has a largest element, or else $\bigsqcup g(T) \leqslant \bigsqcup T$.*

Proof: If T has no largest element, then $\bigsqcup T \neq y$, for every y in T. Hence if y is in T, then there is some x in T such that $x \leqslant y$, so $g(y) \leqslant x \leqslant \bigsqcup T$. Consequently we have $\bigsqcup g(T) \leqslant \bigsqcup T$. Q.E.D.

Lemma 2: *If $x \leqslant g(x)$, for every x in X, where g is the function of* Theorem 1, *and T is a subset of V, then either T has a largest element or else $\bigsqcup g(T) = \bigsqcup T$.*

Proof: Clearly we have $\bigsqcup T \leqslant \bigsqcup g(T)$, so Lemma 2 follows immediately from Lemma 1. Q.E.D.

Corollary: *If in* Theorem 1 *we have* $x \leq g(x)$, *for every* x *in* X, *then* $\bigsqcup T$ *is in* V, *for any subset* T *of* V.

We may now state the following weaker form of Theorem 1:

Theorem 2: *If* X *is a partially-ordered set in which each subset has a least upper-bound, and* g *is a function on* X *such that* $x \leq g(x)$, *for every* x *in* X, *then there is a chain* V *in* X *such that:*

(a) $\bigsqcup T$ *is in* V, *for every* $T \subseteq V$,

(b) $g(V) \subseteq V$.

As a first application of Theorem 2 we will derive a variant of Zorn's Lemma from the Axiom of Choice. First we prove a fixed point theorem.

Theorem 3: *Suppose that* X *is a partially ordered set in which each subset has a least-upper-bound, and that* g *is a function on* X *such that* $x \leq g(x)$, *for every* $x \in X$. *If* Y *is a subset of* X *such that:*

(a) $\bigsqcup C$ *is in* Y, *for every chain* C *in* Y,

(b) $g(Y) \subseteq Y$,

then there is some y *in* Y *such that* $y = g(y)$.

Proof: Consider the element $\bigsqcup(V \cap Y)$ of Y, where V is the set V determined by Theorem 2. Since $V \cap Y$ is a chain in Y, it follows that $g(\bigsqcup(V \cap Y))$ is in $V \cap Y$, so we have $g(\bigsqcup(V \cap Y)) \leq \bigsqcup(V \cap Y)$. Q.E.D.

Zorn's Lemma: *If* X *is a partially-ordered set in which each subset has a least-upper-bound, and* Y *is a non-void subset of* X *such that* $\bigsqcup C$ *is in* Y, *whenever* C *is a non-void chain in* Y, *then* Y *is a maximal element.*

Proof: We wish to show that Theorem 4 is a consequence of the Axiom of Choice. Define a function g on X as follows: If x is a non-maximal element of Y, using the Axiom of Choice let $g(x)$ be an element of Y such that $x < g(x)$. Otherwise let $g(x) = x$. It is clear that g satisfies the hypotheses of Theorem 3, and that an element y of Y is maximal in Y if and only if $g(y) = y$. Consequently the result follows from Theorem 3. Q.E.D.

V. *The Well-Ordering Theorem* Theorem 2 will now be used, in conjunction with the Axiom of Choice, to prove the Well-Ordering Theorem. A binary relation on a set S (i.e., a subset of $S \times S$) is said to be a *well-order* relation on S if it is a total-order relation and every non-void subset of S has a smallest element with respect to it. We will call a binary relation on a set S a *quasi-well-order* on S if every non-void subset of S has a smallest element with respect to it. Such a relation need not be a partial-order. It follows immediately from the Axiom of Choice that a quasi-well-order can be defined on any set: if f is a choice function on a set S, then the relation $\bigcup_{\phi \neq T \subseteq S} [\{f(T)\} \times T]$ is clearly a quasi-well-order on S. Our intention is to "shrink" this relation down to one which is both a quasi-well-order and anti-symmetric.

Proposition 3: *An anti-symmetric quasi-well-order relation A on a set S is a well-order on S.*

Proof: We wish to show that A is a total-order. Since any one-element subset of S has a smallest element, A is reflexive. Since any two-element subset of S has a smallest element, any two elements of S are comparable. Since any three-element subset of S has a smallest element, it follows from the anti-symmetry of A that A is transitive. Q.E.D.

Well-Ordering Theorem: *There is a well-order relation on any set S.*

Proof: Let X be $\mathsf{P}(S)$, partially-ordered by the *inverse* of inclusion. Let f be a choice function on S, and let g be the function on $\mathsf{P}(S)$ defined by:

$$g(T) = \begin{cases} T\backslash\{f(T)\}\ , & T \neq \phi \\ \phi\ , & T = \phi\ . \end{cases}$$

Now let $\mathcal{V}$ represent the sub-family V of $\mathsf{P}(S)$ which is provided by Theorem 2, and let $A = \bigcup_{V \in \mathcal{V}\backslash\{\phi\}} [\{f(V)\} \times V]$. The following two lemmas complete the proof.

Lemma 1: *A is a quasi-well-order on S.*

Proof: Suppose that T is a non-void subset of S. We wish to find a W in $\mathcal{V}\backslash\{\phi\}$ such that $f(W)$ is in T and $T \subseteq W$. Let W be the intersection of all V in $\mathcal{V}$ for which $T \subseteq V$. W is in $\mathcal{V}$, by (a) of Theorem 2. If $f(W)$ were not in T, then we would have $T \subseteq g(W) \subseteq W$, which would contradict the definition of W. Q.E.D.

Lemma 2: *A is anti-symmetric.*

Proof: Suppose that $x \leqslant y$ and $y \leqslant x$, for x,y in S. Then there is some V in $\mathcal{V}$ such that $x = f(V)$ and y is in V, and there is some W in $\mathcal{V}$ such that $y = f(W)$ and x is in W. If $f(V) \neq f(W)$, then $f(V)$ is in $g(W)\backslash g(V)$, and $f(W)$ is in $g(V)\backslash g(W)$, which contradicts the comparability of $g(V)$ and $g(W)$. Hence $x = f(V) = f(W) = y$. Q.E.D.

VI. *Alternate Proof of the Well-Ordering Theorem* In this section an additional property of the set V of Theorem 1 will be proved and then used to give a different proof of the Well-Ordering Theorem. For each y in V, let $I_y = \{x \in V \mid x \leqslant y\}$.

Lemma: *Suppose, in* Theorem 1, *that $x \leqslant g(x)$, for every x in X. Then I_y is well-ordered by $\leqslant$, for every y in V.*

Proof: Let T be the set of all elements y of V for which I_y is well-ordered by $\leqslant$. We wish to show that T is in the family $\mathcal{J}$ of Theorem 1. Let W be an arbitrary subset of T. We wish to show that $I_{\bigsqcup g(W)}$ is well-ordered. Suppose that M is a non-void subset of $I_{\bigsqcup g(W)}$. If $M = \{\bigsqcup g(W)\}$, then clearly M has a smallest element. Otherwise there is some m in M such that $< \bigsqcup g(W)$, and, since V is totally-ordered, there must be some y in W such that $m \leqslant y$. Since I_y is well-ordered, the set of all elements of M smaller than m is contained in I_y and hence has a smallest element. This element is a smallest element for M, since V is totally-ordered. Q.E.D.

Proposition 4: *Suppose, in* Theorem 1, *that* $x \leqslant g(x)$, *for every* x *is* X. *Then* V *is well-ordered by* $\leqslant$.

Proof: Suppose that M is a non-void subset of V. Let y be an element in M. The set of all elements of M smaller than y is contained in I_y and hence has a smallest element, by the lemma. This element is a least element of M. Q.E.D.

To show that any set S can be well-ordered, we again use the X, f, g, and $\mathcal{V}$ of the proof of the Well-Ordering Theorem in the preceding section. Since $\mathcal{V}\backslash\{\phi\}$ is well-ordered, by Proposition 4, it suffices to show that the restriction $f|\mathcal{V}\backslash\{\phi\}$ of f to $\mathcal{V}\backslash\{\phi\}$ is a one-to-one correspondence between $\mathcal{V}\backslash\{\phi\}$ and S.

Proposition 5: *The function* f *maps* $\mathcal{V}\backslash\{\phi\}$ *into* S.

Proof: We will use Proposition 2. Suppose that x is an element of S which is not in $f(\mathcal{V}\backslash\{\phi\})$. Let $\mathcal{K}$ be the family of all sets V of $\mathcal{V}$ such that x is in V. If $\mathcal{U}$ is an arbitrary sub-family of $\mathcal{K}$, then, since x is not in $f(\mathcal{V}\backslash\{\phi\})$, x is in $g(V)$, for every V in $\mathcal{U}$, and consequently x is in $\bigcap g(U)$, so $\bigcap g(U)$ is in $\mathcal{K}$. Hence $\mathcal{K} = \mathcal{V}$, by the technique of Theorem 1, and we have

$$x \in \bigcap \mathcal{V} \subseteq \bigcap g(\mathcal{V}\backslash\{\phi\}) \neq \phi \quad ,$$

which contradicts Proposition 2. Q.E.D.

Proposition 6: *The restriction* $f|\mathcal{V}\backslash\{\phi\}$ *of* f *to* $\mathcal{V}\backslash\{\phi\}$ *is one-to-one.*

Proof: Suppose, for V, W in $\mathcal{V}\backslash\{\phi\}$, that $f(V) = f(W)$. Then we have $g(V) \not\supseteq W$ and $g(W) \not\supseteq V$, so by (iii) of Theorem 1 we have $V \supseteq V$ and $V \supseteq W$. Hence $V = W$. Q.E.D.

It is clear that the two well-order relations that have been defined on S, the second being the image under f of the well-order on $\mathcal{V}\backslash\{\phi\}$ are the same.

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