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SOME COMPLETE CALCULI OF INDIVIDUALS

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1. *Introduction.** Goodman proposed and partially analyzed (in [2] and [3]) a notion of an individual which we seek to explicate in this paper. The necessary truths peculiar to this notion receive axiomatic treatment in formal systems (calculi of individuals) whose semantics is developed purely within set theory. These systems are semantically sound and complete and differ, in respects to be mentioned, from the calculus of individuals developed by Leonard and Goodman (in [4]).

For the convenience of the reader, a brief reconstruction of what is taken to be Goodman's notion of an individual is presently given. Goodman (notably in [3]) is understood to hold that the general theory of individuals differs from the general theory of classes, even though individuals may themselves be classes. The general theory of individuals is characterized chiefly by the adoption of a principle of individuation which may be informally rendered as follows:

(1) *Individuals are identical just in case they have the same ultimate constituents.*

Ultimate constituents (also called 'atoms') appear to be *R-minimal elements* relative to a so-called 'generating relation' R (that is, they are elements of the field of R to which nothing bears the relation R). The notion of a 'generating relation' is not defined, but only exemplified (in [3]) by the ancestral of membership and by the relation of being a proper part, a relation which is given axiomatic treatment in [4]. It appears further that the theory of individuals is characterized by some principle of sum-formation, a principle governing the formation of individual wholes on the basis of ultimate constituents. Such principles will be discussed below.

2. *Part-whole relations and universes of individuals.* In order to formulate a principle of individuation akin to (1) as well as further principles requisite to explicating the notion of an individual, some auxiliary set

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theoretical concepts are needed. The set theoretic apparatus used in this context is that of a Zermelo-Fraenkel set theory without individuals. All standard notation of that theory, whether symbolic or English, is understood to have its usual meaning. In particular,

$$\mathfrak{D}R,\ \mathfrak{R}R,\ \mathfrak{F}R,\ R_S,\ \mathfrak{I}_S,\ \subseteq_S \text{ and } \mathfrak{P}S$$

represent, respectively, the domain, the range, the field, and the restriction to a set S of (a relation) R, identity restricted to S, inclusion restricted to S, and the power set of S.

The common designatum of all improper descriptions is the empty set, 0. An object x is an *R-least element of S* just in case x is in S, and no element of S which is distinct from x bears R to x. An *R-least element* is an R-least element of the field of R. The supremum, relative to a partial ordering R, of S (in symbols, $sup_R S$) is the unique object y satisfying the two conditions that (a) every element of S bears R to y, and (b) for all z, if every element of S bears R to z then y itself bears R to z. Such a supremum is said to exist just in case there is a unique object y satisfying these two conditions. If R is a part-whole relation, in a sense to be clarified below, then an R-least element of a set S is intuitively regarded as an 'ultimate constituent' or 'atom', in Goodman's sense, relative to R and S; and the supremum relative to R and S is taken to be a sum or whole having the elements of S for its parts (in the sense of R).

The further auxiliary notion of a part-whole relation is introduced by the following definition:

Definition 1. A partial ordering R is a *part-whole relation* just in case there exists an infinite set A of *R-least* elements satisfying the following conditions:

(1) for every non-empty subset S of A, the supremum, relative to R, of S exists,

(2) for every non-empty subset S of A and for all x in A, if x bears R to $sup_R S$ then x is in S, and

(3) the field of R is the set of all suprema, relative to R, of non-empty subsets of A.

In Goodman's terminology, the field of a part-whole relation is taken to comprise an infinite set of 'ultimate constituents' in the sense of that relation, and all possible sums or wholes 'generated' from these 'ultimate constituents' relative to the given relation. Condition (2) of the above definition provides that all 'atoms' of a part-whole relation are 'discrete' in the following sense: no atom can be a proper part of a sum of two or more atoms all different from the given one.

Arbitrary part-whole relations can be represented by the relation of set-inclusion, as indicated by the following lemma:

Lemma 1. *R is a part-whole relation just in case for some infinite set S, R is isomorphic to* $\subseteq_{\mathfrak{P}S-\{0\}}$.

Outline of the proof. Under the assumption that R is a part-whole relation, let S be the set of R-least elements. The isomorphism is given by a function f whose domain is the field of R, which maps each R-least element into its singleton, and which is such that for each x in its domain, $f(x)$ is the union of all $f(y)$, where y is an R-least element bearing R to x. The implication in the other direction is an immediate consequence of the fact that $\subseteq_{\mathfrak{P}S-\{0\}}$ is a part-whole relation, provided that S is infinite.

Next, the notion of a 'universe of individuals' is to be specified. A universe of individuals shall be a subset of the field of a part-whole relation satisfying a principle of individuation akin to (1) and a principle of summation appropriate to Goodman-type individuals. An individual will then be indirectly characterized as any element of such a universe of individuals. Given a part-whole relation, each universe of individuals delimited within its field is intuitively regarded as a 'possible world' of individuals which can be parts or wholes in the sense of that relation.

Suppose that R is a part-whole relation and that U is a subset of R. Then the following principle of individuation, which may be regarded as a formal counterpart of (1), appears to be a suitable condition on U if U is to be a universe of individuals in the sense intended:

(2) *for all x and y in U, $x = y$ just in case the following condition obtains: for every R-least element z of U, zRx if and only if zRy.*

Thus, informally, individuals in a given universe are to be the same just in case they have the same atomic parts in that given universe.

Let us turn to principles of summation which might appear to be suitable conditions on every subset U of the field of a part-whole relation R, if U is to be a universe of individuals in R:

(3) *for every x in U there exists a set S of R-least elements of U such that $x = sup_R S$.*

Informally: every individual in U is a sum of atoms in U.

(4) *for every non-empty subset S of U, $sup_R S$ is again in U.*[1]

Informally: the sum of any individuals is always in turn an individual.

Although Goodman adopts a closure principle of summation akin to (4), there are natural part-whole relations with respect to which the principle seems counter-intuitive. For example, a part-whole relation obtaining between physical objects may be specified so as to satisfy the condition that every part of a physical whole is to be within some small spatial distance from some other part of that same whole. Then the sum, relative to this part-whole relation, of two distant physical atoms fails to exist.

The 'division principle' (3), on the other hand, seems plausible as a general condition on all universes of individuals. In addition to implying the principle of individuation (2)—as indicated by Corollary 1 below—(3) has

1. A syntactical analogue of (4) is a theorem of the Leonard-Goodman calculus of individuals (in [4]); a syntactical analogue of (3) appears in Goodman's system of qualia (in [2]; theorems 7.12 and 7.18), but not in the calculus of individuals.

the intuitive content that each individual is a *least* whole composed of its atomic parts. This latter condition is formulated more precisely in Lemma 2 below. We embody (3) in the following definition:

Definition 2. Suppose that R is a part-whole relation; then U is a *universe of individuals for* R just in case the following conditions are satisfied:

(1) U is a non-empty subset of the field of R, and

(2) for every x in U there exists a set S of R-least elements of U such that $x = sup_R S$.

Lemma 2. *Suppose that R is a part-whole relation; then U is a universe of individuals for R just in case U is a non-empty subset of the field of R, and for all x in U and for all y in the field of R the following condition obtains:*

(5) *if every R-least element of U which bears R to x bears R to y, then x itself bears R to y.*

Proof. Assume the hypothesis.

(A) Suppose that U is a universe of individuals for R and assume the antecedent of (5) for some x and y. For some set S of R-least elements of U, $x = sup_R S$. Since x is an *upper* bound of S and by the antecedent of (5), every element of S bears R to y. Then it is sufficient to note that x is the *least* upper bound of S.

(B) Under the appropriate assumption for the proof of necessity, let x be an arbitrary element of U. Let S be the set of all R-least elements z of U such that zRx. By the assumptions and Definition 1, the supremum of S exists. Since every element of S bears R to x, so does, by definition, $sup_R S$. But every R-least element of U which bears R to x bears R to $sup_R S$. Hence, by (5), x bears R to $sup_R S$. Thus, $x = sup_R S$, by antisymmetry.

Corollary 1. *Suppose that R is a part-whole relation and U is a universe of individuals for R; then for all x and y in U, $x = y$ just in case the following condition obtains: for every R-least element z of U, zRx if and only if zRy.*

Corollary 1, which expresses that the appropriate principle of individuation obtains in universes of individuals, is an immediate consequence of Lemma 2.

It shall be our aim to represent universes of individuals as fields of partial orderings satisfying the additional requirement of 'atomicity'. This notion, which closely resembles condition (5) of Lemma 2, is introduced by the following definition:

Definition 3. A relation R is *atomistic* just in case for all x and y in the field of R the following condition obtains:

(6) *if every R-least element which bears R to x bears R to y, then x itself bears R to y.*

Lemma 3. *Suppose that R is an atomistic partial ordering and that f is a function whose domain is the field of R, and such that, for all x in that field,*

$$f(x) = \bigcup\{\{y\} : y \text{ is an } R\text{-least element and } yRx\};$$

then R is isomorphic, by f, to $\subseteq_{\Re f}$.

Lemma 3, whose proof presents no difficulties, aids in showing that non-empty fields of atomistic partial orderings are universes of individuals, a fact more precisely expressed by the following theorem:

Theorem 1. *Let R be any relation; then there exists a part-whole relation R' such that the field of R is a universe of individuals for R' if and only if R satisfies the following conditions:*

(1) *the field of R is non-empty,*

(2) *R is a partial ordering, and*

(3) *R is atomistic.*

Proof. (A) If the field of a relation R is a universe of individuals for some part-whole relation R', then the conditions (1) and (2) of the theorem are immediate consequences of the definitions while condition (3) follows from Lemma 2. (B) Suppose that a relation R satisfies the conditions (1)-(3). Let B be an infinite set whose power set is disjoint from the field of R and which has at least the same power as that field. Let f be a 1-1 function from the field of R into B. Let g be a function whose domain is the field of R and such that, for all x in that field,

$$g(x) = \bigcup\{\{f(y)\} : y \text{ is an } R\text{-least element and } yRx\}.$$

Thus, due to the characterization of g and f and by Lemma 3, R is isomorphic, by g, to $\subseteq_{\Re g}$. Also, g maps R-least elements onto the singletons of their correlates by f. Let $S = \wp B - \{0\}$. Thus, by Lemma 1, $\subseteq_S$ is a part-whole relation. We note that for all $x \epsilon \Re g$ and for all $y \epsilon S$, if every least element, by $\subseteq$, of $\Re g$ which is included in x is included in y, then x itself is included in y. For if x and y satisfy this hypothesis, then the singleton of every element of x is a least element, by $\subseteq$, of $\Re g$ which is included in x, and hence included in y. Thus, by Lemma 2, $\Re g$ is a universe of individuals for $\subseteq_S$.

Let $h = g \cup \mathfrak{I}_{S-\Re g}$. h is a 1-1 function whose range is S. Let R' be a relation such that, for all $x, y \epsilon \mathfrak{D}h$, $xR'y$ if and only if $h(x) = h(y)$. R' is a part-whole relation and the field of R is a universe of individuals for R'.

3. *Common syntax of the systems to follow.* The discussion so far has taken place in a language comprising informal set theory, a theory not likely to be acceptable to a nominalist. For this reason it may be of interest to characterize formal axiomatic systems which a nominalist can endorse and whose theorems may be regarded as necessary truths peculiar to individuals. Several such systems, called 'calculi of individuals', shall presently be examined.

The common syntactical structure of all calculi of individuals to follow is that of the lower predicate calculus without identity, unless explicit exception is made. The common *vocabulary* of these systems comprises (a) variables (lower case Latin letters with or without subscripts), (b) parentheses, (c) the five sentential connectives '$\sim$', '$\rightarrow$', '$\wedge$', '$\vee$', '$\leftrightarrow$' (which are the respective symbols of negation, conditional, conjunction, disjunction, and biconditional), (d) the universal quantifier '$\bigwedge$', and the existential quantifier '$\bigvee$', (e) the primitive two-place predicate '$\leqslant$', (f) the defined two-place predicate '$<$', and (g) the defined one-place predicate '*At*'. Ordinary quotation marks or display are used as Quine uses quasi-quotes. If α and β are variables, then "$\alpha \leqslant \beta$" (read: "α is a part of β"), "$\alpha < \beta$" (read "α is a proper part of β"), and "$At\alpha$" (read: "α is an atom") are *atomic formulas* of all systems to follow. The *formulas* and *sentences* are characterized on the basis of the given atomic formulas in a manner usual in the lower predicate calculus. The *logical axioms* underlying all systems to be considered are those of a standard first-order predicate calculus appropriate to the given vocabulary. For *inference rules* we employ modus ponens and universal generalization. The *definitional axioms* common to all calculi of individuals to follow are the following:

(D1) $x < y \leftrightarrow (x \leqslant y \wedge \sim(y \leqslant x))$.

Informally: x is a proper part of y just in case x is a part of y but y is not a part of x.

(D2) $Atx \leftrightarrow \sim\bigvee y(y < x)$.

Informally: x is an atom just in case x has no proper parts.

4. *The system* **LGCI**. **LGCI** is a formal system closely similar to the Leonard-Goodman calculus of individuals (in [4]; also in [2]), but using a different primitive and avoiding reference to classes (made in [4], but presumably objectionable to a nominalist) by introducing an axiom-schema.

In addition to the syntactical features mentioned in section 3, the *vocabulary* of **LGCI** comprises also the identity symbol '$=$' as a primitive, and the defined two-place predicate '$\circ$'. If α and β are variables, "$\alpha = \beta$" and "$\alpha \circ \beta$" (read: "α overlaps β") are *atomic formulas* additional to those mentioned in section 3. All axioms of the identity calculus appropriate to the present vocabulary are *logical axioms* of **LGCI**. There is an additional *definitional axiom*:

(D3) $x \circ y \leftrightarrow \bigvee z(z \leqslant x \wedge z \leqslant y)$.

Informally: individuals overlap just in case they have a common part.

For all formulas ϕ of **LGCI** such that the variable 'z' does not occur in ϕ, the following are *proper axioms* of **LGCI**:

(AS1) $\bigvee x\phi \rightarrow \bigvee z\bigwedge y(y \circ z \leftrightarrow \bigvee x(\phi \wedge y \circ x))$.

Informally: If some individuals satisfy the condition ϕ then there exists an individual which overlaps all and only those individuals which overlap something satisfying ϕ. Or, in closer accord with the intended intuitive content: if some individuals satisfy the condition ϕ then the sum of all individuals satisfying ϕ exists.

(A2) $(x \leqslant y \wedge y \leqslant x) \rightarrow x = y$.

Informally: individuals which are part of one another are identical.

(A3) $x \leqslant y \leftrightarrow \bigwedge z(z \circ x \rightarrow z \circ y)$.

Informally: x is part of y just in case every individual which overlaps x overlaps y.

The notions of *derivability* (of a formula from a set of formulas) and of a *theorem* of **LGCI** are defined in the usual way.

The syntactical counterpart of the principle of individuation (2) can be expressed in **LGCI** as follows:

(7) $x = y \leftrightarrow \bigwedge z(Atz \rightarrow (z \leqslant x \leftrightarrow z \leqslant y))$.

Since Goodman regards such a principle of individuation as central to the notion of an individual, one would expect (7) to be a theorem of **LGCI.** But this is not the case. Given the axioms of **LGCI**, (7) turns out to be provably equivalent to:

(8) $\bigwedge x \bigvee y(Aty \wedge y \leqslant x)$.

An independence example of (8) relative to the axioms of **LGCI** is afforded by a complete atomless system of Boolean algebra upon removal of the zero element, and with the primitive '$\leqslant$' interpreted as the inclusion relation.[2]

In as much as (7) is not a theorem of **LGCI**, this system does not appear to be a system specifically of individuals in Goodman's sense. Since **LGCI** is a reconstruction of the Leonard-Goodman system, the intuitive adequacy of that system is questionable. The result of adding (8) to the axioms of **LGCI** can however be regarded as an adequate calculus of individuals which shall be considered in section 7.

5. *The calculus of individuals* **CI 1.** Since the syntactical counterpart (7) of the principle of individuation (2) fails to hold in the Leonard-Goodman system, we propose a number of alternative systems, in each of which (7) appears as a theorem and which have therefore some claim to being calculi of individuals. The first of these calculi, **CI 1**, shall have for its theorems just those formulas which, under a natural interpretation, are true in every universe of individuals for every part-whole relation.

The syntactical characteristics of **CI 1** are those given in section 3. In addition, the vocabulary of **CI 1** comprises the identity symbol '=', introduced here (unlike in **LGCI**) by the following definitional axiom:

(D4) $x = y \leftrightarrow (x \leqslant y \wedge y \leqslant x)$.

2. The author owes to Alfred Horn an example of such an algebra and verification of the independence of (8). The independence is also suggested by Tarski's discussion (in [5]) of atomless Boolean algebras and mereology.

The following *proper axioms* of **Cl 1** are read as asserting, respectively, that '$\leqslant$' expresses a reflexive, transitive, and atomistic relation:

(Ax1) $x \leqslant x$,
(Ax2) $(x \leqslant y \wedge y \leqslant x) \rightarrow x \leqslant z$,
(Ax3) $\bigwedge z((Atz \wedge z \leqslant x) \rightarrow z \leqslant y) \rightarrow x \leqslant y$.

Given the axioms of **Cl 1** and the rules of inference mentioned in section 3, the notions of *derivability* and of a *theorem* in **Cl 1** are characterized in the customary manner. We are not here interested in the theorems of **Cl 1**, except to the extent of observing that the principle of individuation (7) is a theorem of **Cl 1** (by virtue of D4 and Ax3).

Next, we characterize the *semantics* of **Cl 1**. A **Cl 1** *model* is to be an ordered couple (U, R), where U is any non-empty set, and R (the intended extension of '$\leqslant$') is an atomistic partial ordering defined on U. An *assignment* for a **Cl 1** model (U, R) is a function from the set of variables into U; and if a is such an assignment and α is a variable, then a^{α}_{x} is the assignment for (U, R) which differs from a at most by assigning x to α.

Given an assignment a for a **Cl 1** model (U, R), the notion that a *satisfies* ϕ *in* (U, R) is recursively characterized for all formulas ϕ of **Cl 1** in the usual manner. In particular, if ϕ is a primitive atomic formula of the form "$\alpha \leqslant \beta$", then a satisfies ϕ in (U, R) just in case $\mathrm{a}(\alpha) R \mathrm{a}(\beta)$. All further semantical notions appropriate to **Cl 1**, notably the notions of *truth, semantical soundness* and *completeness* receive customary meaning in terms of satisfaction in **Cl 1** models.

The defined symbol '=' has the meaning of true identity, for it is easily verified that an assignment a for a **Cl 1** model satisfies a formula of the form "$\alpha = \beta$" just in case $\mathrm{a}(\alpha) = \mathrm{a}(\beta)$.

Since (Ax1)-(Ax3), together with (D4), are the syntactical analogues, in terms of '$\leqslant$', of the conditions on atomistic partial orderings R which serve as interpretations of '$\leqslant$', the semantical soundness and completeness of **Cl 1**, relative to **Cl 1** models, is an immediate consequence of the soundness and completeness of the lower predicate calculus. By virtue of Theorem 1, the non-empty fields of any atomistic partial orderings are universes of individuals for some part-whole relation. Conversely, the restriction of a part-whole relation to any universe of individuals delimited within its field is clearly an atomistic partial ordering. Thus, the theorems of **Cl 1** are exactly those formulas of **Cl 1** which are true in every model (U, R) such that U is a universe of individuals for some part-whole relation, and R is the restriction of that part-whole relation to U. Thus, **Cl 1** may be regarded as a most comprehensive calculus specifically of individuals.

6. *Universes of individuals with various closure properties.* On intuitive grounds, mentioned earlier, we have refrained from imposing on all universes of individuals the condition that the sum of any individuals shall in turn be an individual. However, it may be of some interest to formalize calculi of individuals which are, like the Leonard-Goodman system, appropriate to just those universes of individuals which are closed under

sums or products. Preliminary to such a formalization, some additional notions are needed. The infimum, relative to a partial ordering R, of objects x and y (in symbols, $inf_R(x,y)$) is understood to be the unique object z satisfying the two conditions that (a) z bears R to both x and y, and (b) for all u, if u bears R to both x and y then u bears R to z. Such an infimum is said to *exist* just in case there is a unique z satisfying these two conditions. If R is a partial ordering, then $inf_R(x,y) = sup_R\{z : zRx \text{ and } zRy\}$. In case that R is a part-whole relation the infimum, relative to R, of two objects is intuitively regarded as their 'product' or as the sum of their common parts.

Relative to the system **CI 1** the notion of a specifiable subset of a universe of individuals is characterized as follows: Suppose that (U, R) is a **CI 1** model; then a set S is said to be *specifiable in U and R* just in case there exist an assignment a for (U, R), a variable α, and a formula ϕ (of **CI 1**) such that for all x, a^{α}_{x} satisfies ϕ in (U, R) if and only if $x \in S$.

Next, we characterize universes of individuals which are closed, roughly speaking, under 'non-empty' finite products, under finite sums, under both, or under 'non-empty' products and sums of specifiable subsets. More specifically, we define:

Definition 4. Suppose that R is a part-whole relation; then

(a) U is a **CI 2** - *universe for R* just in case U is a universe of individuals for R satisfying the following condition:

(C1) for all x and y in U, if there exists a z such that zRx and zRy then $inf_R(x,y)$ is in U.

(b) U is a **CI 3** - *universe for R* just in case U is a universe of individuals for R satisfying the following condition:

(C2) for all x and y in U, (a) $sup_R(x,y)$ is in U, and (b) for every R-least element z of U, if $zR\ sup_R\{x,y\}$ then zRx or zRy.

(c) U is a **CI 4** - *universe for R* just in case U is a universe of individuals for R satisfying both of the conditions (C1) and (C2).

(d) U is a **CI 5** - *universe for R* just in case U is a universe of individuals for R satisfying the condition (C1), and in addition the following condition:

(C3) If S is non-empty and specifiable in U and R_U, then $sup_R S$ is in U.

In Definition 4(c), reference to the condition (C2) may be equivalently replaced by reference to (C2)(a).

The following definitions serve to characterize certain closure conditions on atomistic partial orderings which have syntactical analogues in the language of **CI 1**. The correspondence of these conditions with the conditions (C1)-(C3) will become apparent in Theorem 2 below.

Definition 5. Suppose that R is an atomistic partial ordering; then

(a) R is *product-closed* just in case for all a and b in the field of R the following condition obtains: if for some x, xRa and xRb then there exists a y such that all and only those R-least elements bear R to y which bear R to both a and b.

(b) R is *sum-closed* just in case for all a and b in the field of R there exists a y such that all and only those R-least elements bear R to y which bear R to either a or b.
(c) R is *strongly sum-closed* just in case for every set S which is specifiable in the field of R the following condition obtains: if some R-least element is in S then there exists a y such that all and only those R-least elements bear R to y which are elements of S.

The following theorem is the analogue of Theorem 1 for the cases of **Cl 2 - 5** - universes:

Theorem 2. *Let R be any relation. Then there exists a part-whole relation R' such that the field of R is, respectively,* (a) *a* **Cl 2** *- universe,* (b) *a* **Cl 3** *- universe,* (c) *a* **Cl 4** *- universe, or* (d) *a* **Cl 5** *- universe for R' if and only if R satisfies the following conditions*:

(1) *the field of R is non-empty,*
(2) *R is an atomistic partial ordering, and*
(3) *corresponding, respectively, to* (a)-(d) *above, either*
 (a) *R is product-closed,*
 (b) *R is sum-closed,*
 (c) *R is both product-closed and sum-closed, or*
 (d) *R is strongly sum-closed.*

Proof: (A) If the field of a relation R is a universe of either of the sort (a)-(d) for some part-whole relation R', then the conditions (1) and (2) and the appropriate condition among (3)(a)-(d) follow easily from the definitions and from Lemma 2.
(B) Suppose that a relation R satisfies the conditions (1)-(3). Referring back to the proof of Theorem 1, let B, f, g, and S be characterized as in that proof. Thus, R is isomorphic, by g, to $\subseteq_{\Re g}$, and $\Re g$ is a universe of individuals for the part-whole relation $\subseteq_S$.
(a) We show that if R is product-closed then $\Re g$ satisfies the condition (C1) imposed on **Cl 2** - universes with respect to $\subseteq_S$. Assuming the hypothesis, suppose that for some $g(x)$, $g(y) \in \Re g$ and for some $z \in S$, $z \subseteq g(x)$ and $z \subseteq g(y)$. Since the empty set is not in S assume, without loss of generality, that for some u, $z = \{u\}$. By the specification of g, we can let $u = f(u_0)$ and $u_0 Rx$ and $u_0 Ry$. Hence, since R is product-closed, there exists a w such that for all v, if v is an R-least element then, vRw if and only if vRx and vRy. Since R-least elements are mapped, by g, onto singletons which are included in elements of the range of g and since g is an isomorphism, we have: all and only those singletons are included in $g(w)$ which are included in both $g(x)$ and $g(y)$. Thus, $g(w)$ is the infimum, relative to $\subseteq_{\bar{S}}$, of $g(x)$ and $g(y)$.
(b) We show that if R is sum-closed then $\Re g$ satisfies the condition (C2) imposed on **Cl 3** - universes with respect to $\subseteq_S$. Assuming the hypothesis, let $g(x)$, $g(y) \in \Re g$. Since R is sum-closed, there exists a w such that for all v, if v is an R-least element then, vRw if and only if vRx or vRy. Since R-least elements are mapped by the isomorphism g onto singletons, we have: all and only those singletons are included in $g(w)$ which are included

in either $g(x)$ or $g(y)$. Thus it is easy to see that $g(w)$ is the supremum, relative to $\subseteq_S$, of $g(x)$ and $g(y)$.
(c) Clearly, if R is both product-closed and sum-closed, then the range of g is **CI 4** - universe for $\subseteq_S$.
(d) We show that if R is strongly sum-closed then $\Re g$ satisfies the conditions (C1) and (C3) imposed on **CI 5** - universes. Assuming the hypothesis, we note first that for all a and b in the field of R, the set S' of all R-least elements bearing R to both a and b is specifiable in the field of R (in fact, by the formula of **CI 1** "$Atx \wedge x \leqslant a \wedge x \leqslant b$"). It is now easy to show that R is product-closed, the 'product' of two elements being the 'sum' of their common R-least parts. Thus, $\Re g$ satisfies the condition (C1) by the argument (a). Secondly, to show that condition (C3) obtains, assume that T is non-empty and specifiable in $\Re g$ and $\subseteq_S$, so that for some formula ϕ of **CI 1** and for all x, a^{α}_{x} satisfies ϕ in $(\Re g, \subseteq_S)$ if and only if $x \epsilon T$. Let U be the set of all $g(b)$, such that $g(b)$ is a least element, by $\subseteq$, of $\Re g$ and for some $x \epsilon T$, $g(b) \subseteq x$. U is specifiable in $\Re g$ and $\subseteq_S$ (in fact, by the formula of **CI 1** "$At\beta \wedge \bigvee\alpha(\phi \wedge \beta \leqslant \alpha)$", where α, β are variables). Let V be the set of all b, such that b is an R-least element and for some $x \epsilon T$, $g(b) \subseteq x$. Clearly, V is non-empty and specifiable in $\mathfrak{F}R$ and R'. Since R is strongly sum-closed, there exists a y such that all and only those R-least elements bear R to y which are in V. It is easy to show for all x in T that every least element, by $\subseteq$, of $\Re g$ which is included in x is also included in $g(y)$. Thus, by Lemma 2, every element of T is included in $g(y)$. Further, if z be some element of S such that all elements of T are included in z, then every least element, by $\subseteq$, of $\Re g$ which is included in $g(y)$ is also included in z, and thus $g(y)$ itself is included in z. Hence, $g(y)$ satisfies the conditions of the supremum, relative to $\subseteq$, of T. Thus, $\Re g$ satisfies the condition (C3) and is a **CI 5** - universe for $\subseteq_S$.

Letting h and R' be specified as in the proof of Theorem 1, it follows that the field of R is a **CI 2-5** - universe respectively for R' as R satisfies the respective conditions (3)(a)-(d).

7. *The calculi of individuals* **CI 2 - CI 5**. Corresponding to the universes of individuals **CI 2 - CI 5** characterized above, four calculi of individuals shall now be formalized whose theorems are regarded as necessary truths peculiar to individuals in one of these four universes.

The syntactical features of the calculi of individuals **CI 2 - CI 5** are those of **CI 1**. In particular, all definitional axioms (D1)-(D4) and all proper axioms (Ax1)-(Ax3) of **CI 1** are axioms of each of the systems **CI 2 - CI 5**. In addition, the following is a *proper axiom of* **CI 2**:

(Ax4) $\bigvee x(x \leqslant a \wedge x \leqslant b) \rightarrow \bigvee y \bigwedge x(Atx \rightarrow (x \leqslant y \leftrightarrow (x \leqslant a \wedge x \leqslant b)))$.

Informally: the part-whole relation is product-closed.

The following *proper axiom* is peculiar to **CI 3**:

(Ax5) $\bigvee y \bigwedge x(Atx \rightarrow (x \leqslant y \leftrightarrow (x \leqslant a \vee x \leqslant b)))$.

Informally: the part-whole relation is sum-closed.

The *proper axioms of* **Cl 4** are just those of **Cl 1**, together with (Ax4) and (Ax5). The *proper axioms of* **Cl 5** are just those of **Cl 1**, together with all formulas given by the following schema where 'y' does not occur in ϕ:

(AxS6) $\bigvee x(Atx \wedge \phi) \rightarrow \bigvee y \bigwedge x(Atx \rightarrow (x \leq y \leftrightarrow \phi))$.

Informally: the part-whole relation is strongly sum-closed.

The following remark may serve to clarify the relation between our calculus **Cl 5** and the reconstructed Leonard-Goodman system **LGCl** presented in section 4. Let **Cl 5**$_\circ$ be the result of adding to the vocabulary of **Cl 5** the defined two-place predicate '$\circ$', of expanding the set of formulas of **Cl 5** so as to include all formulas of **LGCl**, and of adopting the definition (D3) of '$\circ$'. Then it is easy to show that every axiom of **LGCl** (as stated in section 4) as well as the sentence (8) (whose independence of the axioms of **LGCl** was mentioned) are theorems of **Cl 5**$_\circ$. Conversely, in the system which results from **LGCl** by adding the independent sentence (8) as a further axiom, every axiom of **Cl 5**$_\circ$ is provable.

A **Cl 2** - *model* (a **Cl 3** -, **Cl 4** -, **Cl 5** - *model*) is understood to be a **Cl 1** - model $\langle U, R\rangle$ whose extension assignment R is in addition product-closed (sum-closed, both product-closed and sum-closed, strongly sum-closed, respectively). All other semantical notions are characterized as in **Cl 1**, except that reference to **Cl 1** - models is appropriately replaced by reference to **Cl 2** - **Cl 5** - models. Since the axioms (Ax4), (Ax5) and (AxS6) are the syntactical counterparts of the conditions of being product-closed, sum-closed, and strongly sum-closed imposed on the extensions in corresponding universes, the semantical soundness and completeness of each, **Cl 2** - **Cl 5**, relative to **Cl 2** - **Cl 5** - models respectively, are again an immediate consequence of the soundness and completeness of the lower predicate calculus. Thus, by virtue of Theorem 2, the theorems of **Cl 2** (**Cl 3, Cl 4, Cl 5**) are just those formulas which are true in every model whose universe is a **Cl 2** - (**Cl 3** -, **Cl 4** -, **Cl 5** -) universe for some part-whole relation. Analogues to the closure conditions imposed on these universes find expression in calculi of individuals which are presumed to be acceptable to a nominalist.

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