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The Structure of Pleasant Ideals

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Abstract Continuing the work begun in Pleasant Ideals (*Notre Dame Journal of Formal Logic* vol. 32 (1991) pp. 612–617), we investigate the relationships among selective, normal and pleasant ideals. Our major result is that any selective ideal extending NS_κ is normal.

1 Introduction and Preliminaries The investigation of normal ideals, ideals that are closed under diagonal unions, has been ongoing for many years. In our [2] we introduced the concept of a pleasant ideal, an ideal that is closed under diagonal unions indexed by members of the ideal. It seems that to ask an ideal to be pleasant is very close to asking it to be normal, in the sense that pleasantness combines with several other ideal properties to imply normality.

Our set theoretic notation is standard. The axiom of choice is assumed throughout so a cardinal is identified with the set of its ordinal predecessors. The letters κ and λ will be reserved for cardinals, while α, β, etc. will represent ordinals.

An ideal on a regular uncountable cardinal κ is a collection of subsets of κ that is closed under subset and finite union. Our ideals will contain all singletons and be $< \kappa$ complete, and thus will extend $I_\kappa \equiv \{X \subseteq \kappa \mid |X| < \kappa\}$. If I is an ideal on κ, then I^* will denote the the dual filter and I^+ will be the co-ideal $\{X \subseteq \kappa \mid X \notin I\}$. If I is an ideal and $A \in I^+$, then $I \upharpoonright A$ is the ideal $\{X \subseteq \kappa \mid X \cap A \in I\}$.

If $A \subseteq \kappa$ and $f : A \to \kappa$, f will be called regressive if $f(\alpha) < \alpha$ for $\alpha \in A - \{0\}$, and weakly regressive if $f(\alpha) \leq \alpha$. If I is an ideal, then f is I-small if $f^{-1}(\{\xi\}) \in I$ for every $\xi < \kappa$.

The nonstationary ideal on κ, NS_κ, is defined by $A \in NS_\kappa \iff$ there is a club $C \subseteq \kappa$ such that $A \cap C = \emptyset$. It is well known that NS_κ is the smallest normal ideal, i.e., the smallest ideal that is closed under diagonal unions, so if $X_\alpha \in NS_\kappa$ for all $\alpha < \kappa$, then $\underset{\alpha<\kappa}{\nabla} X_\alpha \equiv \{\xi < \kappa \mid (\exists \alpha < \xi)(\xi \in X_\alpha)\} \in NS_\kappa$. If I is any normal ideal and if $Q \in I^+$, then there is no I-small regressive function on Q.

An ideal I on κ is called a p-point if for any I-small $f : \kappa \to \kappa$ there exists a set $X \in I^*$ such that $f \upharpoonright X$ is I_κ-small. I is a q-point if for any I_κ-small $f : \kappa \to \kappa$ there exists a set $X \in I^*$ such that $f \upharpoonright X$ is one-to-one. I is selective if I is both a

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p-point and a q-point. I is called quasinormal if for any sequence $\{X_\alpha\}_{\alpha<\kappa}$ of sets in I there is a set $Q \in I^*$ such that $\mathop{\nabla}_{\alpha\in Q} X_\alpha \equiv \{\xi\kappa \mid (\exists\alpha < \xi)(\alpha \in Q \text{ and } \xi \in X_\alpha)\} \in I$. It is not hard to show that an ideal is quasinormal if and only if it is selective, see for example Węglorz [3].

In [2], the basic facts about pleasant ideals are proven. An ideal I is said to be pleasant if it is closed under diagonal unions indexed by sets in I. In other words, if $X_\alpha \in I$ for all $\alpha < \kappa$, and $A \in I$ then $\mathop{\nabla}_{\alpha\in A} X_\alpha \in I$. I is pleasant if and only if for every $A \in I^+$ and every regressive I-small $f : A \to \kappa$, $f[A] \in I^+$. For any ideal on κ, the following are equivalent: I is normal; I is pleasant and extends NS_κ; I is pleasant and selective. We will extend this list in Section 3.

2 *Pleasant and Subpleasant Ideals*

In Baumgartner, Taylor, and Wagon [1] normal and subnormal ideals are examined and the implications of normality for the behavior of regressive functions are exposed. In this section of this paper we will begin to carry out a similar analysis of pleasant and subpleasant ideals. An ideal I is subpleasant if it is a subset of a pleasant ideal. In [2] there are exhibits of ideals that are pleasant and not normal, and it is shown that every pleasant ideal is subnormal. In this section we will give another proof of this result, which implies (Corollary 2.3) that an ideal is subnormal if and only if it is subpleasant.

Lemma 2.1 *Suppose I is a pleasant ideal on κ and C is a club. If $C \in I$, then I is improper.*

Proof: Without loss of generality, C consists entirely of limit ordinals. For $\alpha \in C$, let $X_\alpha = \sim C \cap \{\xi \mid \alpha < \xi < \hat{\alpha}\}$, where $\hat{\alpha} = \min(C - (\alpha \cup \{\alpha\}))$. Each X_α is bounded, and therefore in I. It is easy to check that $\mathop{\nabla}_{\alpha\in C} X_\alpha \cup [\sim C \cap \min C] = \sim C$.

Theorem 2.2 *Every proper pleasant ideal is subnormal.*

Proof: Suppose that I is a pleasant ideal. If I is I_κ, then $I \subseteq NS_\kappa$, and I is subnormal. For any other pleasant ideal I, we will show that $\langle I \cup NS_\kappa\rangle$ is pleasant and proper. Since any pleasant ideal extending NS_κ is normal, we will have a normal extension of I.

$\langle I \cup NS_\kappa\rangle$ must be proper, since otherwise there is some club in I, contradicting the properness of I. So all that is left is to show that $\langle I \cup NS_\kappa\rangle$ is closed under diagonal unions indexed by elements of $\langle I \cup NS_\kappa\rangle$.

It suffices to consider $\mathop{\nabla}_{\alpha\in X} U_\alpha$, where X and each U_α are elements of $I \cup NS_\kappa$.

Split each $U_\alpha = V_\alpha \cup W_\alpha$, where $V_\alpha \in NS_\kappa$ and $W_\alpha \in I - NS_\kappa$. Then $\mathop{\nabla}_{\alpha\in X} V_\alpha \in NS_\kappa \subseteq \langle I \cup NS_\kappa\rangle$, as NS_κ is normal. So it suffices to consider $\mathop{\nabla}_{\alpha\in X} W_\alpha$; i.e., without loss of generality each $U_\alpha \in I - NS_\kappa$. Now split $X = M \cup N$, with $M \in NS_\kappa$ and $N \in I$. $\mathop{\nabla}_{\alpha\in N} U_\alpha \in I$, as I is pleasant, so it suffices to consider $\mathop{\nabla}_{\alpha\in M} U_\alpha$. In other words, we can assume without loss of generality that $X \in NS_\kappa$.

So, without loss of generality, we are looking at $Z = \mathop{\nabla}_{\alpha\in X} U_\alpha$, where each U_α is stationary and in I, and X is nonstationary. Pick some $W \in I$ such that W is unbounded in κ. For $\alpha \in X$, let $f(\alpha) =$ least $\eta \in W$ such that $\eta \geq \alpha$. Let $V_\eta = \bigcup_{\alpha\in f^{-1}(\{\eta\})} U_\alpha$. Notice that each $V_\alpha \in I$, as I is $< \kappa$ complete, and $U_\alpha \subseteq V_{f(\alpha)}$. But then $\mathop{\nabla}_{\alpha\in X} U_\alpha \subseteq \mathop{\nabla}_{\alpha\in X} [U_\alpha \cap f(\alpha) + 1] \cup \mathop{\nabla}_{\eta\in W} V_\eta$. The first of these is in NS_κ, as it is a diagonal union of bounded sets, while the second term is in I, as I is pleasant.

So $\langle I \cup NS_\kappa \rangle$ is closed under diagonal unions indexed by sets in $I \cup NS_\kappa$, and so $\langle I \cup NS_\kappa \rangle$ is pleasant, as needed.

Corollary 2.3 *I is subpleasant if and only if I is subnormal.*

Proof: This follows immediately from the theorem above and from the fact that every normal ideal is pleasant.

In [1], one of the topics addressed is whether certain ideal properties are "local" or "global". A global property is one that is preseved under restriction: If I has the property, then so does $I \restriction A$ for every $A \in I^+$. Otherwise the property is local. For example, it is easy to see that normality is a global property, and that normality is the globalization of subnormality: I is normal if and only if $I \restriction A$ is subnormal for every $A \in I^+$. We will show that a similar situation holds for the property of pleasantness.

Theorem 2.4 *Suppose $I \restriction A$ is pleasant for each $A \in I^+$. Then I is normal (and trivially pleasant).*

Proof: Since each $I \restriction A$ is pleasant, each $I \restriction A$ is subnormal. Therefore I is normal.

Proposition 2.5 *There exist a pleasant ideal I and a set $A \in I^+$ such that $I \restriction A$ is not pleasant.*

Proof: We construct the needed ideal. Let I be the ideal generated by $I_\kappa \cup \{\lambda + 2 \mid \lambda \text{ is a limit ordinal}\}$. In [2] it is shown that $L = \{\lambda + 1 \mid \lambda \text{ is a limit ordinal}\} \notin P(I)$, the pleasant closure of I. We will show that $P(I) \restriction L$ is not pleasant. Notice that if $M = \{\lambda \mid \lambda \text{ is a limit ordinal}\}$, then $M \in P(I) \restriction L$, since $M \cap L = \varnothing$. But then by Lemma 2.1, since M is a club, if $P(I) \restriction L$ was pleasant, then it would be improper. But $L \notin P(I) \restriction L$, so $P(I) \restriction L$ is proper. Thus $P(I) \restriction L$ is not pleasant, as needed.

An ideal property is said to hold densely in I if, for every $A \in I^+$ there is some $B \in (I \restriction A)^+$ such that $I \restriction B$ has the property.

Proposition 2.6 *I is pleasant if and only if I is densely pleasant.*

Proof: The forward direction is obvious here. For the reverse direction we shall consider $A = \nabla_{\alpha \in Q} X_\alpha$, with Q and $X_\alpha \in I$. If $A \notin I$, by denseness there is some $B \in (I \restriction A)^+$ such that $I \restriction B$ is pleasant. But then $B \cap A \in I^+$, while $A = \nabla_{\alpha \in Q} X_\alpha \in I \restriction B$. This says $A \cap B \in I$, contradicting our choice of B.

3 *Pleasantness, Quasinormality, and Normality* In this section we will introduce a slight weakening of pleasantness and then we show that any extension of NS_κ that is quasinormal must also be normal. We also examine how one can force a pleasant ideal to become normal by adding sets to the ideal.

Definition 3.1 I is *prepleasant* if for every sequence B_α of bounded sets and for every $Q \in I$, $\nabla_{\alpha \in Q} B_\alpha \in I$.

There are nontrivial ideals that are not prepleasant. For example, consider the ideal $J_\kappa = \{X \subseteq \kappa \mid (\exists f : X \to \kappa)(\exists \theta < \kappa)$ such that f is regressive and $\leq \theta$ to 1 $\}$, introduced by Węglorz. $J_\kappa \subseteq NS_\kappa$, and $J_\kappa = NS_\kappa$ if and only if κ is a successor cardinal. That $J_\kappa \neq NS_\kappa$ for limit cardinals κ depends on the fact that $Z = \bigcup \{(\lambda, \lambda^+) \mid \lambda < \kappa$ and λ is a cardinal $\} \in NS_\kappa - J_\kappa$. But this same set can be used to show that J_κ is not prepleasant. Let $L = \{\lambda + 1 \mid \lambda$ is a cardinal$\}$, and let $A_{\lambda+1} = (\lambda + 1, \lambda^+)$. Then it is easy to check that $L \cup \nabla_{\lambda+1 \in L} A_{\lambda+1} = Z$ is not in J_κ, and thus J_κ is not prepleasant when κ is a limit cardinal.

Notice also that not every prepleasant ideal is pleasant. Suppose we have disjoint stationary sets Q and R and a collection $\langle Q_\alpha \rangle_{\alpha<\kappa}$ of pairwise disjoint stationary sets such that $Q = \nabla_{\alpha \in R} Q_\alpha$. Then we claim that the prepleasant closure of $I_\kappa \cup \{R\} \cup \{Q_\alpha\}_{\alpha<\kappa}$ is not pleasant, and in particular that it does not include Q. The only way that Q could be generated as a member of the prepleasant closure of $I_\kappa \cup \{R\} \cup \{Q_\alpha\}_{\alpha<\kappa}$ would be as a diagonal union, and if $Q = \nabla_{\alpha \in X} B_\alpha$ then one of the B_α's would have to be stationary.

Also notice that the definition of prepleasantness can be cast in terms of functions: I is prepleasant if and only if there is no regressive function $f : Q \to A$ such that $Q \in I^+$, $A \in I$, and f is I_κ-small.

Theorem 3.2 *If I is a p-point, then I is prepleasant if and only if I is pleasant.*

Proof: Clearly if I is not prepleasant then I is not pleasant, so let us assume that I is a p-point and I is not pleasant. Then we know there is a function $f : Q \to A$ such that f is I-small, $Q \in I^+$ and $A \in I$. If we extend f so that f is the identity function on $\kappa - Q$ and use the p-pointedness of I, we find a set $B \in I^*$ such that f is I_κ-small on B. But then $Q \cap B \in I^+$ and $f \restriction Q \cap B$ shows that I is not prepleasant.

Corollary 3.3 *If I is quasinormal and prepleasant, then I is pleasant, and therefore normal.*

Proof: This follows immediately from the preceding theorem and the fact that any quasinormal pleasant ideal is normal.

Theorem 3.4 *If $I \subseteq NS_\kappa$ and I is quasinormal, then I is normal. So for ideals contained in NS_κ, I is selective if and only if I is normal. (See [1]; Proposition IV.3.2).*

Theorem 3.5 *Suppose I is quasinormal and $NS_\kappa \subseteq I$. Then I is pleasant, and therefore I is normal.*

Proof: Suppose that I is not pleasant. Then there exist sets A and B_α (for $\alpha \in A$) such that each of them is in I and $\nabla_{\alpha \in A} B_\alpha \notin I$. Without loss of generality, if $\alpha < \beta$, then $B_\alpha \subseteq B_\beta$.

For each $\xi < \kappa$, define $D_\xi = \bigcup_{\alpha<\xi} B_\alpha$. Since each $D_\xi \in I$, and since I is quasinormal, we know there is a $Q \in I^*$ such that $\nabla_{\xi \in Q} D_\xi \in I$.

We claim that $Y = \nabla_{\alpha \in A} B_\alpha - \nabla_{\xi \in Q} D_\xi$ is not stationary. For if Y is stationary, notice that for each $\eta \in Y$ there is an $\alpha \in A$ such that $\alpha < \eta$ and $\eta \in B_\alpha$. Since $\eta \notin \nabla_{\xi \in Q} D_\xi$, for no $\xi \in Q$ can we have $\alpha < \xi < \eta$, since otherwise $\eta \in D_\xi$. So, for each $\eta \in Y$, Q is bounded below η. But then we can define a regressive function

$f : Y \to \kappa$ by $f(\eta) = sup(Q \cap \eta)$. By Fodor's Theorem this function is constant on an unbounded set, and so Q is bounded in κ. But $Q \in I^*$, and so Q is unbounded. Therefore Y is not stationary.

But this implies that $\nabla_{\alpha\in A} B_\alpha \in I$, contrary to assumption: $\nabla_{\alpha\in A} B_\alpha \subseteq [\nabla_{\alpha\in A} B_\alpha - \nabla_{\xi\in Q} D_\xi\,] \cup \nabla_{\xi\in Q} D_\xi$, and the first term in the union is in I, as I extends NS_κ, while the second term is in I by choice of Q.

Therefore I is pleasant, and since any pleasant extension of NS_κ is normal, I is normal.

In [2], various combinations of properties were proven to be equivalent to the normality of an ideal I. In particular, it is shown there that I is normal if and only if I is pleasant and extends NS_κ if and only if I is pleasant and selective. We can now add another equivalent set of conditions to this list, again emphasizing how close pleasantness is to normality.

Definition 3.6 Let $L = \{\lambda+1 \mid \lambda \text{ is a limit ordinal}\}$. If $Y \subseteq L$, let $B(Y) = \{\alpha - 1 \mid \alpha \in Y\}$.

Theorem 3.7 *If $L \in I$, then the pleasant closure of I extends NS_κ, and thus the pleasant closure of I is normal.*

Proof: We will show that if A is any nonstationary set, then $A \in P(I)$, where $P(I)$ is the pleasant closure of I, the ideal that is formed by iterating diagonal unions indexed by sets in I. For details on $P(I)$, see [2].

So, assume $A \in NS_\kappa$. As $L \in I$, we may assume without loss of generality that $A \cap L = \emptyset$. (Otherwise, as $L \in I$, $A \cap L \in I$. Then continue the proof with $A - L$).

As $A \in NS_\kappa$, we know $A = \nabla_{\alpha<\kappa} Y_\alpha$ with $Y_\alpha \in I_\kappa$. Define, for $\lambda + 1 \in L$, $X_{\lambda+1} = \bigcup_{\lambda\le\alpha<\lambda+\omega} Y_\alpha$. As κ is regular, we know $X_{\lambda+1} \in I_\kappa$. But then $A = \nabla_{\lambda+1\in L} X_{\lambda+1}$: If $\xi \in A$, then there is some $\beta < \xi$ such that $\xi \in Y_\beta$. Now $\beta = \theta + n$ for some limit ordinal θ and some $n \in \omega$. Since $Y_\beta \subseteq X_{\theta+1}$, we know that $\xi \in X_{\theta+1}$. The only question is whether $\theta + 1 < \xi$. But $\theta \le \beta < \xi$, so $\xi \neq \theta$. And since $\xi \notin L$, $\xi \neq \theta + 1$. Therefore $\theta + 1 < \xi$, and so $A = \nabla_{\lambda+1\in L} X_{\lambda+1}$, and $A \in P(I)$, as needed.

Corollary 3.8 *I is normal if and only if I is pleasant and $L \in I$.*

Proof: If I is pleasant and includes L, then I extends the nonstationary ideal, and thus I is normal.

Corollary 3.9 *I is normal if and only if I is pleasant and the set of limit ordinals is in I^*.*

Proof: In the non-obvious direction, we must only show that $L \in I$. But this is clear since the set of successor ordinals is in I, and L is a subset of the collection of successor ordinals.

This last corollary shows that the "canonical" club of limit ordinals is all that is needed to force a pleasant ideal to be normal. So if, for example, we start with I_κ, toss in L and then examine the pleasant closure of the resulting ideal, we will get NS_κ. In fact there is nothing all that special about the club of limit ordinals.

Definition 3.10 If S is a set of ordinals, S is *thin* if $\alpha \in S \to \alpha + 1 \notin S$.

Theorem 3.11 *If I is a pleasant ideal on κ and there is some thin $S \in I^*$, then I is quasinormal, and hence normal.*

Proof: Assume I is not quasinormal, and let $\langle X_\alpha \rangle_{\alpha<\kappa}$ be a counterexample to the quasinormality of I. So each $X_\alpha \in I$ and for any $R \in I^*$, $\nabla_{\alpha \in R} X_\alpha \in I^+$. In particular, $Q = \nabla_{\alpha \in S} X_\alpha \in I^+$. Define $f : Q \to \kappa$ by $f(\xi)$ = the least $\alpha \in S$ such that $\xi \in X_\alpha$. Now f is I-small and regressive, with range a subset of S. So the function $\hat{f} : Q \to \kappa$ defined by $\hat{f}(\xi) = f(\xi) + 1$ is weakly regressive, I-small, and (since S is thin) has range in I. Thus I is not pleasant.

A reasonable question would be to ask how many sets of what kind must be added to I_κ in order to force the pleasant closure of the the ideal to be normal. Although we do not have a complete answer to this question, we do have some interesting results. The next theorem shows that if you want to add nonstationary sets to I_κ in such a way as to force the pleasant closure of I to include the set L (and therefore be normal), you must add lots of subsets of L—in some sense a dense collection of subsets of L.

Theorem 3.12 *Suppose that $I \subseteq NS_\kappa$ and $L \in P(I)$. Then if Q is a stationary set of limit ordinals, there is some $Y \subseteq L$ such that $Y \in I$, $B(Y) \subseteq Q$ and $B(Y)$ is stationary.*

Proof: Assume that the theorem fails. We will examine a counterexample Q, chosen as follows: We know that $Q + 1 \equiv \{\gamma + 1 \mid \gamma \in Q\} \subseteq L$, so $Q + 1 \in P(I)$. If $Q + 1 \in I$, then $Y = Q + 1$, $B(Y) = Q$ shows that Q is not a counterexample. Thus $Q + 1 \in P(I) - I$. So there is some ordinal β such that $Q + 1 \in P_{\beta+1}(I) - P_\beta(I)$, where the subscript indexes the iterations in building the pleasant closure of I. (See [2].) We choose our counterexample Q so that $\beta + 1$ is minimal.

Now, since $Q + 1 \in P_{\beta+1}(I)$, we know $Q + 1 = \nabla_{\alpha \in T} X_\alpha$ for some $T \in I$ and $X_\alpha \in P_\beta(I)$. This gives us a natural weakly regressive function $f : Q \to T$. As Q is stationary, there is some $\xi \in T$ such that $\hat{Q} = f^{-1}(\{\xi\})$ is stationary. Since $\hat{Q} + 1 \subseteq X_\xi \in P_\beta(I)$, we know $\hat{Q}$ is not a counterexample to the theorem. So, find $\hat{Y} \subseteq L$ such that $\hat{Y} \in I$, $B(\hat{Y})$ is stationary and $B(\hat{Y}) \subseteq \hat{Q}$. But now, since $\hat{Q} \subseteq Q$, $\hat{Y}$ shows that Q is also not a counterexample to the theorem, a contradiction. Thus there is no counterexample, and the theorem holds.

An alternative approach to creating normal ideals via pleasant closures would be to begin with I_κ and add some stationary sets before closing under I-indexed diagonal unions. We show by example that the simplest attempts to do this do not work. For the following, let $W = \{\lambda < \kappa \mid \operatorname{cof} \lambda = \omega\}$

Theorem 3.13 *Suppose $A \subseteq L - \{\lambda + 1 \mid \lambda \in W\}$ and $B(A)$ is stationary. Then $A \notin P(I_\kappa \cup \{W\})$.*

Proof: Suppose we look at a counterexample A such that $A = \nabla_{\alpha \in Q} X_\alpha$ with $Q \in I$, $X_\alpha \cap (\alpha + 1) = \varnothing$, $X_\alpha \in P_\eta(I)$, with η minimal among all such A's and all such diagonal unions. This gives a regressive function $f : B(A) \to Q$ defined by $f(\gamma) =$ least α such that $\gamma + 1 \in X_\alpha$. The function f is regressive, not merely weakly regressive, since for each $\gamma + 1 \in A$, $\operatorname{cof}(\gamma) > \omega$. Since $B(A)$ is stationary, f is constant on a stationary set. But if $f^{-1}(\{\xi\})$ is stationary, then X_ξ is a subset of L, $B(X_\xi)$ is stationary, and $X_\xi \in P(I)$, contradicting the minimality of η. So there can be no such set A.

The above theorem can be generalized to include more general sorts of sets A, but suffices as stated as an illustration.

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