THE SYNTAX OF PROJECTIVE GEOMETRY

WOODROW JAFFEE

In a recent paper[1], Prof. Menger describes the results of joining points and intersecting lines in a projective plane, by chains of symbols for points and lines *without any use of symbols for the operations*. For example, he writes P Q m R n for the point obtained by joining the points P and Q, intersecting the resulting line with the line m, joining the point of intersection to the point R, and intersecting the resulting line with the line n.

If a line w, called the line of infinity, is distinguished, then the line that passes through P (P not on w) and is coincident with or parallel to m, is described by the chain $m w$ P. If on w two points V and U are distinguished, and the lines through them are called *vertical* and *horizontal* respectively, then for any finite point Q (i.e. any point not on w), the vertical and horizontal lines through Q are described by Q V and Q U - symbols that can be also used in an affine plane. If furthermore, a non-vertical and non-horizontal identity line j is distinguished, the result of substituting a finite point Q into a non-vertical line m is the point[2]

$$m [Q] = (Q \ U \ j \ V \ m \ U) (Q \ V)$$

In particular, $j [Q]$ = Q for each point Q not on w.

Only points and lines are denoted by chains while the vacuum and the universe (i.e. the empty element and the whole plane) are excluded. Hence, the intersection of two distinct points, the join of two distinct lines, and the join and intersection of a point and a non-incident line are outside of the scope of the symbolism. So are, moreover, the join and the intersection of incident elements. No point or line can be combined (e.g. intersected) with itself; nor can a line be combined with a point on that line.

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1. Karl Menger, Frammenti piani autoduali e relative sostituzioni, *Accademia Nazionale dei Lincei*, Serie VIII vol. 30 fasc. 5 (1961).

2. This substitution is, of course, a projectivization of the general geometric substitution of a 1-place function into a 1-place function. Cf. Menger, *Calculus—A Modern Approach*, Boston 1953 and Axiomatic Theory of Functions of Fluents in *The Axiomatic Method*, ed. Henkin et al. Amsterdam, 1959.

Menger's symbolism is further based on the following two syntactic conventions:

I. *The symbols for points differ visibly (that is, typographically) from the symbols for lines.* Capital letters in Roman type (from P to Z) are used for points, and lower case letters in *italic* type, for lines.

II. *Chains are read from left to right.* Thus P Q m stands for (P Q)m, while P (Q m) and m P Q do not designate elements. But, of course, P Q m = m (P Q).

Which chains of letters do designate elements? Using Greek letters for both points and lines and writing $|\alpha| = 1$ or 2 according as the element designated by the letter α is a point or line, we can formulate, first of all, the following:

Rule Concerning Pairs. $\alpha\beta$ *designates an element if and only if the elements designated by α and β are distinct and*

(1) $|\alpha| + |\beta| \not\equiv 0$ (mod. 3)

in which case

$$|\alpha\beta| \equiv |\alpha| + |\beta| \quad \text{(mod. 3)}.$$

(1) rules out that $|\alpha| \neq |\beta|$. If α and β designate distinct points (or distinct lines), then $|\alpha| + |\beta| = 2$ (or 4) and $|\alpha\beta| = 2$ (or 1); that is to say, $\alpha\beta$ designates a line (or a point).

We furthermore can state the following:

Inductive Rule. *If $\alpha_1\alpha_2 \ldots \alpha_{n-1}\alpha_n$ (where $n > 1$) designates an element, then so does $\alpha_1\alpha_2 \ldots \alpha_{n-1}$.*

From these two rules one readily obtains the following:

Characterization of Well-Formed Chains in Plane Geometry. *In order that the chain $\alpha_1\alpha_2 \ldots \alpha_n$ designate an element it is necessary and sufficient that*

    a) *the element designated by α_k is distinct from the element designated by $\alpha_1\alpha_2 \ldots \alpha_{k-1}$ for k = 2, 3, . . . , n*

    b) $|\alpha_1| = |\alpha_2| \neq |\alpha_3| \neq |\alpha_4| \neq \ldots \neq |\alpha_n|$.

While in projective spaces of dimensions >3 a symbolism without operational symbols would lead to considerable complications, Menger's method will now be extended to the 3-dimensional projective space where only in one case are symbols for joining and intersecting needed.

In space, as they are in the plane, the empty element and the universe (i.e., the whole space) are excluded. Hence, the intersection of a point, as well as the join of a plane, with a distinct and non-incident element are outside of the scope of the symbolism. So are, moreover, the join and the intersection of two identical or incident elements. Hence, no element can be combined with itself, nor can a line be combined with a point or plane on that line, or a plane with a line or point on that plane.

Of the two syntactic conventions concerning the plane, the first will be supplemented by the stipulation that planes are to be designated by capital letters (from A to N) in *italic* type, and II will be retained without change.

For example, P h I V is the plane obtained by joining the point P to the line h, intersecting the resulting plane with the plane I, and joining the resulting line with the point V. Suppose certain elements are distinguished, viz., a plane N ("of infinity") and on N a line h ("of horizontality") and a point V ("of verticality") not on h.

All planes on h as well as all lines intersecting h may be called *horizontal*, all planes and lines on V, *vertical*. Then Ph is the horizontal plane through P while PV is the vertical line through P and m V is the vertical plane through the non-vertical line m. The plane (the line) through the point P that is parallel to the plane A (to the line m) is A N P (is m N P where m is not on N).

Suppose that also two non-vertical and non-horizontal planes, I and J, be distinguished. Then one can introduce an operation in space which is the analogue of the substitution of a point Q into a non-vertical line m in the plane resulting in a point $m[\mathrm{Q}]$. It is the substitution of an ordered pair of points, lying on a vertical line, into a non-vertical plane.[3] The result of substituting the pair P, Q into the plane A is a point $A[\mathrm{P},\mathrm{Q}]$ that can be described in terms of chains as follows:

A [P, Q] = (P h I V) (Q h J V) A h(P Q) if P $\neq$ Q

A [P, P] = (P h I V) (P h J V) A h(P V)

$A[\mathrm{P},\mathrm{Q}]$ as well as $A[\mathrm{P},\mathrm{P}]$ is a point on the vertical line through P (and Q). If, on I and J, one introduces the points of horizontality, X = h I, Y = h J then

P h I V = Y P I V (h I) and Q h J V = X Q J V (h J).

While the composition of lines that are either identical or skew, is outside the scope of this symbolism it is desirable to include the composition of lines that are neither identical nor skew. But such lines may be either intersected or joined, the result being a point and a plane, respectively. In order to distinguish these two conditions of two lines, symbols such as ∩ and ∪ for intersection and join seem to be indispensable. For example, the chain l ∩ m P ∪ n indicates the plane obtained by intersecting l with m, joining the point of intersection to P, and joining the resulting line with n. If $m = A\,B$ and $n = \mathrm{P\,Q}$ then the element l ∩ m P ∪ n may also be described by other chains such as l A P ∪ n, l B P ∪ n, l ∩ mP Q, and l A P Q.

Chains describing elements thus consist of designations of points, lines, and planes and of symbols ∩ and ∪. The latter, however, occur only

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3. This operation is the projectivization of Menger's general geometric construction of the substitution of an ordered pair of 2-place functions (traditionally called "functions of 2 variables") into a 2-place function. Cf. Menger, *Calculus—A Modern Approach.* Mimeo Edition, Chicago 1952, p. 224.

immediately before designations of lines. In order to characterize the well-formed chains, which describe elements. we shall use Greek letters not only for elements (and write $|\alpha| = 1$, 2 or 3 according as the element designated by α is a point, a line, or a plane) but also for *the designation of a line in conjunction with a preceding symbol $\cap$ or $\cup$*. A Greek letter will be referred to as *simple* if it designates an element, and as *non-simple* if it stands for a line symbol preceded by an operation symbol.

If γ stands for $\cap m$, we shall write $E(\gamma) = m$ and $|\gamma| = 3$.
If δ stands for $\cup n$, we shall write $E(\delta) = n$ and $|\delta| = 1$.

We then have the following:

Rule Concerning Pairs. *$\alpha\beta$ designates an element if and only if*
a) *α is simple.*
b) *if β is simple,* (1) *β designates an element that is neither identical with nor incident with the element designated by α, and* (2) $|\alpha| + |\beta| \not\equiv 0$ (mod. 4).
c) *if β is non-simple, then $|\alpha| = 2$ and $E(\beta)$ is neither identical with nor skew to the element designated by α.*
In this case, $|\alpha\beta| \equiv |\alpha| + |\beta|$ (mod. 4).

(2) excludes pairs P A and A P consisting of a point and a plane as well as pairs of lines m n. If α and β designate non-identical points (or non-identical planes) then $|\alpha| + |\beta| = 2$ (or 6), hence $|\alpha\beta| = 2$ and $\alpha\beta$ designates a line in either case. If, say, α designates a line and β designates a non-incident point (or non-incident plane), then $|\alpha| + |\beta| = 3$ (or 5) and $\alpha\beta$ designates a plane (or point).

If β stands for $\cap n$, then α stands for a line, say m, we have $|\alpha| + |\beta| = 2 + 3 \equiv 1$ (mod. 4); and indeed, $|m \cap n| = 1$. If β stands for $\cup n$, we have $|\alpha| + |\beta| = 2 + 1 = 3$; and indeed $|m \cup n| = 3$.

The Inductive Rule stipulated for chains in the plane will be retained verbatim for chains in space. Together with the Rule Concerning Pairs, the Inductive Rule implies, for example, that if $\alpha\beta\gamma$ designates an element, then

$$|\alpha\beta\gamma| \equiv |\alpha\beta| + |\gamma| = |\alpha| + |\beta| + |\gamma| \text{ (mod. 4)}.$$

If $\alpha_1\alpha_2 \ldots \alpha_n$ designates an element, then we have immediately that

$$|\alpha_1\alpha_2 \ldots \alpha_n| \equiv \sum_{i=1}^{n} |\alpha_i| \text{ (mod. 4)}.$$

These results lead to the following:

Characterization of Well-Formed Chains in 3-Dimensional Geometry. *In order that the chain $\alpha_1\alpha_2 \ldots \alpha_n$ designate an element it is necessary and sufficient that*
a) *α_1 is simple.*
b) *for any k = 2, 3, . . . , n if α_k is simple, then the element designated by α_k is neither identical to nor incident with the element designated by $\alpha_1\alpha_2 \ldots \alpha_{k-1}$.*

c) *for any* k = 2, 3, . . . , n *if* α_k *is non-simple, then* $|\alpha_1\alpha_2 \ldots \alpha_{k-1}| = 2$ *and* $E(\alpha_k)$ *is neither identical with nor skew to the element designated by* $\alpha_1\alpha_2 \ldots \alpha_{k-1}$.

d) $|\alpha_1| + |\alpha_2| + \ldots + |\alpha_k| \not\equiv 0$ (mod. 4) *for all* k = 2, 3, . . . , n.

Illinois Institute of Technology
Chicago, Illinois