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A NOTE ON NATURAL DEDUCTION

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The following extensions and modifications of the terminology and rules for natural deduction found in [1] furnish a complete system for quantification theory. Moreover, unlike the system of [1], *each deduction of this system is sound in all its lines.*

Extensions of the Terminology and Rules of [1]

1. *A restricted star* is any star in a column of stars whose initial star is prefixed by a free variable. This variable is called *the restricted variable of the column.*

2. Subjunction: a) To *subjoin a line* (n), to a *line and a restricted star*, $v *$ (m), write (n) as a later line than (m) and alongside all but the innermost column of stars which pass by (m);

   b) Suppose that the line (i) is earlier than the line (m), that the innermost column of stars passing by (m) is a column of restricted stars not passing by (i), and that all columns of stars passing by (i) also pass by (m). Then to *subjoin the line* (n) to the line (i) and the line and restricted star, $v *$ (m), write (n) later than (m) and alongside all but the innermost column of stars passing by (m).

3. The rule of premisses: a) At any stage in a deduction we may set down the symbol '∆' as a line provided that at this point we initiate a new column of restricted stars whose restricted variable is not yet a free or a restricted variable of the deduction.[1]

   b) At any stage in a deduction we may set down any schema provided that at that point we initiate a new column of stars. If the schema in question contains a free variable v which is not yet a free or a restricted variable of the deduction, then instead we may (but need not) initiate a column of restricted stars whose restricted variable is v.[2]

1. '∆' is supposed to be like a blank; it is not a schema and nothing may be subjoined to it.

2. We need not require that the restricted variables, mentioned in (a) and (b), be new to the deduction. Then, however, the following restriction must be placed on the operation of subjoining a line (n) to one or more lines (i), (j), . . . , (m): if any of (i), (j), . . . , (m) contain a variable v free, then we may not pass any columns of stars by (n) which are restricted to v. This restriction corresponds to the one in [2] placed on reiteration into restricted subordinate proofs.

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Modifications in the Rules and Terminology of [1].

1. **U.G.** is now: we may subjoin a universal quantification, *S*, to a line and restricted star $v *$ (m), provided that (m) is a conservative instance of *S* with respect to the instantial variable *v* and that the last premiss of (m) is 'Δ'.

2. **E.I.** is now: to the existential quantification (i) and to the line and restricted star $v *$ (m), we may subjoin (m) itself provided that *v* is not free in (m) and that the last premiss of (m) is a conservative instance of (i) with respect to the instantial variable *v*.[3]

3. Soundness in a line: we now say that a deduction is *sound in a given line* if the line is 'Δ', valid, or implied by the conjunction of all its premisses (if any) except 'Δ'.

At this point it is useful to consider some examples.

Example 1.

$*\ \ (1)\ (Ey)(x)Fxy$	
$*\ x * (2)\ \Delta$	
$*\ \ \ *\ y * (3)\ (x)Fxy$	
$*\ \ \ *\ \ \ *(4)\ Fxy$	(3)
$*\ \ \ *\ \ \ *(5)\ (Ey)Fxy$	(4)
$*\ \ \ *(6)\ (Ey)Fxy$	(1), $y*$(5) [**E.I.**]
$*(7)\ (x)(Ey)Fxy$	$x*$(6) [**U.G.**]
$*(8)\ (Ey)(x)Fxy \supset (x)(Ey)Fxy$	$*$(7)

Example 2.

$*(1)\ (x)(Fx \vee p)$	
$*\ \ *(2)\ -p$	
$*\ \ *\ x * (3)\ \Delta$	
$*\ \ *\ \ \ *(4)\ Fx \vee p$	
$*\ \ *\ \ \ *(5)\ Fx$	(2) (4) [**T.F.**]
$*\ \ *(6)\ (x)Fx$	$x*$(5) [**U.G.**]
$*(7) -p \supset (x)Fx$	$*$(6)
$*(8)\ (x)Fx \vee p$	(7) [**T.F.**]

Example 3.

$*\ \ (1)\ (x)(Fx.(Ey)Gy)$	
$*\ \ (2)\ Fx.(Ey)Gy$	(1)
$*\ z * (3)\ \Delta$	
$*\ \ *(4)\ (Ey)Gy$	(2)
$*\ \ *\ y * (5)\ Gy$	
$*\ \ *\ \ \ *(6)\ Gy \vee Hz$	(5)
$*\ \ *\ \ \ *(7)\ (Ey)(Gy \vee Hz)$	(6)
$*\ \ *(8)\ (Ey)(Gy \vee Hz)$	(4), $y*$(7) [**E.I.**]
$*\ \ (9)\ (x)(Ey)(Gy \vee Hx)$	$z*$(8) [**U.G.**]
$*\ \ (10)\ Fx.(x)(Ey)(Gy \vee Hx)$	(2) (9)

3. The system just sketched is not very novel, for **E.I.** and **U.G.** correspond to the rules given in [2].

The style of these deductions follows that of [1], but we can no longer pass from '*Fy*' to '*(x)Fx*' by means of **U.G.** That is why the deduction presented in example 2. is more complicated than the corresponding deduction in [1]. Nor can we pass directly from '*(Ex)Fx*' to '*Fy*' by means of **E.I.**, and this is why new columns of stars are initiated with line (3) in example 1. and with line (5) in example 3. Since '*x*' occurs free in step 2. of example 3., we cannot restrict the column of stars beginning with step (3) to '*x*', but must choose some other variable, in this case '*z*'.

Now for the proof of soundness. As in [1] we assume that an arbitrary deduction D is sound in lines (1) to (n-1) and show that it is sound in line (n). The proof for case P must be extended slightly since the rule of premisses has been extended in this paper. This is left to the reader. However, here we should consider the two new cases arising from the new **E.I.** and **U.G.**

Case **U.G.**: Here (n) is subjoined to $v *$ (m) and the last premiss of (m) is 'Δ'. Since D is sound in (m), (m) is valid or implied by the conjunction of all its premisses (if any) except 'Δ'. Let P be this conjunction. Let S be (m). P is also the conjunction of the premisses (if any) of (n). If (n) has no premisses, S must be valid. Hence so is (n), since it is a universal quantification of S. If (n) has premisses, then $P \supset S$ is valid. Thus $(v)(P \supset S)$ is valid, and since v is not free in P so is $P \supset (v)S$. But then P implies (n).

Case **E.I.**: Here (n) is subjoined to the line (i) and the line and restricted star, $v *$ (m). Let (n) be S_1 and (i) be $(\mathrm{E}v')S'_2$. Then (m) is also S_1 and the last premiss of (m) is S_2, a conservative instance of $(\mathrm{E}v')S'_2$ with respect to the instantial variable v. If (n) has no premisses (besides 'Δ') neither does (i). Thus $(\mathrm{E}v')S'_2$ and $S_2 \supset S_1$ are valid. Hence so is $(v)(S_2 \supset S_1)$, and since v is not free in S_1, so is $(\mathrm{E}v)S_2 \supset S_1$. But then S_1, i.e., (n) is valid. If (n) has premisses (besides 'Δ') let their conjunction be P. Then some of these premisses are also premisses of (i) or else (i) is valid. In either case P implies $(\mathrm{E}v')S'_2$. Also $P.S_2$ implies S_1. Thus $(v)(P \supset (S_2 \supset S_1))$ is valid. But since v is free in neither S_1 nor P, $P \supset ((\mathrm{E}v)S_2 \supset S_1)$ is valid. But then P implies S_1.

The proof of completeness presented in [1] may be easily adapted to show that the system outlined in this note is also complete.

REFERENCES

[1] W. V. Quine, *Methods of Logic*, New York, 1959.

[2] F. B. Fitch, *Symbolic Logic*, New York, 1952.

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