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THE CONSTRUCTION OF A STEINER TRIPLE SYSTEM ON SETS OF THE POWER OF THE CONTINUUM WITHOUT THE AXIOM OF CHOICE

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§1. *Introduction** It is most convenient to begin with a

Definition 1. *A set M is said to possess a Steiner triple system if there exists a family F of subsets of M such that a) every member of F is a set of exactly three elements of M, and b) every two distinct elements of M are found together in exactly one member of F.*[1]

Prior to 1945, investigations concerning Steiner triple systems dealt with problems of their existence and structure on finite sets. With regard to the existence question, a complete solution was achieved by M. Reiss, in [4], shortly after J. Steiner, in [8], initially posed the problem. Reiss [4] shows that a finite set M possesses a Steiner triple system if, and only if, the cardinality of M, denoted $\overline{\overline{M}}$, is congruent to 1 or 3 modulo 6. Since Reiss' result the existence and structure of Steiner triple systems on finite sets, and natural generalizations of such systems, have occupied a major part of research in combinatorial analysis.

A new slant on the Steiner problem was given, in 1945, by W. Sierpiński, who considers, in [5], the notion of a Steiner triple system on a non-finite set. In that note Sierpiński proves, with the aid of the axiom of choice, the following

Theorem SP. *Every non-finite set possesses a Steiner triple system.*

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1. cf. Frascella [2], p. 163.

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The role played by the axiom of choice in the proof of this theorem seems essential. Sierpiński even remarks in [5] that he was unable, without resort to the axiom of choice, to construct a Steiner triple system on a set of the power of the continuum. (The realization of just such a construction is the main concern of the present note). Sierpiński's result led B. Sobociński to conjecture and prove in [7] the following

Theorem SB. *Theorem* **SP** *is equivalent to the axiom of choice.*

Compatible with Sobociński's theorem, however, is the fact that there may be non-finite sets, of certain cardinalities, which can be shown to possess a Steiner triple system without employing the axiom of choice. That this, indeed, is the case was observed by Sierpiński in [6] where he mentions (without proof) that one may construct a Steiner triple system on a countably infinite set without resort to the axiom of choice. Such a construction is given in Vučković [9]. With this in mind we formulate the following

Definition 2. *A non-finite cardinal number* $\mathfrak{n}$ *is said to be a Steiner cardinal number if* $\mathfrak{n}$ *is the power of a set which can be shown to possess a Steiner triple system without the axiom of choice.*

From what has been said it is clear that $\aleph_0$ is a Steiner cardinal number. In fact, one can easily demonstrate, using the proof of Theorem **SP** in Sierpiński [5], that if a non-finite cardinal number $\mathfrak{m}$ can be shown to be an aleph[2] without the axiom of choice, then $\mathfrak{m}$ is a Steiner cardinal number. This remark follows from the fact that Sierpiński's construction of a Steiner triple system for a non-finite set M proceeds by elementary methods (i.e. methods independent of the axiom of choice) once the set M is well-ordered.

The question, however, remains open as what are the non-finite cardinal numbers which are not alephs, and yet, at the same time, are Steiner numbers.[3] The present note will demonstrate that the cardinal number $\mathfrak{c}$, representing the power of the continuum, is a Steiner cardinal number. That is, we shall prove, without the aid of the axiom of choice, the following

Theorem 3. *Every set of the power of the continuum possesses a Steiner triple system.*

We leave to future publications more general results concerning the extent of the class of Steiner cardinal number.[4]

2. By an aleph we mean a non-finite cardinal number which is the power of a well-ordered set.

3. In view of Theorem **SP**, such a question is meaningful only in a set theory without the axiom of choice.

4. A specific result to be included is the fact that all non-finite cardinal numbers of the form 2^n-1 are Steiner numbers.

§2. *A proof of Theorem* 3. In order to demonstrate that every set of the power of the continuum possesses a Steiner triple system, it is sufficient to show that a particular set C, of cardinality $\mathfrak{c}$, possesses such a Steiner system. This is so, since the elements of any other set C', of power $\mathfrak{c}$, are in one-one correspondence with the elements of C. Thus, the Steiner triple system of C will induce, in the obvious way, a Steiner triple system for C'. Consequently, to prove Theorem 3, we are justified in restricting our attention to a particular subset of the Cartesian plane.

Definition 4. *Let R represent the set of all real numbers. Then, let $L_i = \{<x, i>: x \varepsilon R\}$ for $i = 1, 2$ and 3.*

Definition 5. *Let $L = L_1 \cup L_2 \cup L_3$*

The set L is nothing other than the collection of all points on three distinct horizontal lines in the Cartesian plane. It is immediate that $\overline{\overline{L_i}} = \mathfrak{c}$ for $i = 1, 2$ and 3. Also, it is clear that $L_i \cap L_j = \phi$ whenever $i \neq j$. Hence, $\overline{\overline{L}} = 3\mathfrak{c}$. But it may be established, without the axiom of choice, that $3\mathfrak{c} = \mathfrak{c}$, from which we may conclude

$$\overline{\overline{L}} = \mathfrak{c}. \tag{1}$$

In virtue of (1) and the remarks made at the opening of §2., it will be sufficient, for a proof of Theorem 3, to construct, without the aid of the axiom of choice, a Steiner triple system for the set L. The realization and demonstration of just such a construction will be divided into four stages.

I. *An orientation of the lines L_1, L_2, and L_3.* To effect our construction of a Steiner triple system for the set L, it will be essential to order the lines L_1, L_2 and L_3 in a cyclic arrangement. Thus, we define an orientation in which we say, L_1 immediately precedes L_2, which immediately precedes L_3, which, in turn, immediately precedes L_1. Also note, that if L_i immediately precedes L_j, we may say, equivalently, L_j immediately succeeds L_i. To view this orientation it may be convenient to conceive of the lines L_1, L_2, and L_3 as edges of a triangular prism (see Figure 1).

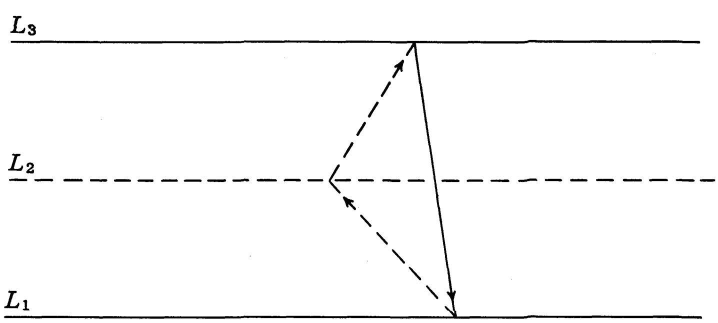

Figure 1.

With reference to the above orientation it is important to observe that if two of three lines, say L_i and L_j, are given, then it must be that L_i immediately precedes L_j or L_j immediately precedes L_i.

II. *The construction of Type A triples of L.* Before beginning the actual construction of the triples which will constitute a Steiner system for L, it is well to agree on some terminology. A point p of L has the form, $p = \langle x, i \rangle$, of an ordered pair where the first component x is a real number and the second component i is one of the integers *1*, *2* or *3*. By the *abscissa value* of a point of L we shall mean point's first component; likewise, by its *ordinate value*, the point's second component.

The triples which will be constructed from the points of L naturally divide into two kinds, which we label, for convenience, Type A and Type B. A *Type A triple* of L will be any unordered set of three distinct points of L, say p_1, p_2 and p_3, such that their abscissa values are equal. Thus a Type A triple will simply be a collection of three points of L, one from each of the lines L_1, L_2 and L_3, such that they lie directly above one another in the Cartesian plane.

III. *The construction of Type B triples of L.* A *Type B triple* of L will be an unordered set of three distinct points of L, say p_1, p_2 and p_3, subject to the following three conditions:

(B.1) two of the three points, say p_1 and p_2, must have the same ordinate value (i.e. they lie on the same line, say L_i, where i is their common ordinate value);

(B.2) the line L_i on which these two points, p_1 and p_2, lie must immediately precede (according to the orientation given in I) the line on which the remaining point p_3 lies;

(B.3) the abscissa value of this third point p_3 must equal the arithmetic mean of the abscissa values of the other two points, p_1 and p_2.

Geometrically, this says that three distinct points of L, p_1, p_2, and p_3, constitute a Type B triple if, and only if, the three points determine, in the plane, the vertices of an isosceles triangle with base on one of the lines L_1, L_2 or L_3 and vertex on the line which immediately succeeds the line on which the base lies. In Figure 2, three such isosceles triangles are drawn with bases on lines L_1, L_2 and L_3, respectively. One will observe that the two triangles, with bases on lines L_1 and L_2, point upward, while the third triangle with base lying on L_3 points downward. See Figure 2 on the following page. We now have all the triples needed in order to construct a Steiner system for the set L. Hence, we make the following

Definition 6. *Let S represent the collection of all Type* A *and Type* B *of the set L.*

IV. *Demonstration that S is a Steiner triple system for L.* To show that S is a Steiner triple system for the set L it is only necessary to prove that every two distinct points of L lie together in exactly one triple of S. To this end, let p_1 and p_2 be any two distinct points of L. We divide the demonstration into three cases.

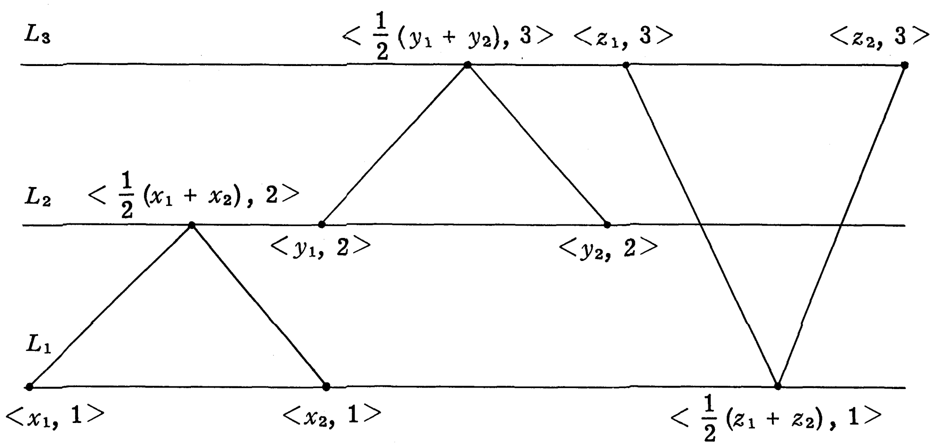

Figure 2

Case α. Suppose p_1 and p_2 have the same abscissa value but different ordinate value (i.e. the points lie on different lines but directly above one another). In this case, it is clear that there exists exactly one Type A triple which will contain p_1 and p_2; moreover, it is also evident that no Type B triple will contain p_1 and p_2 since the abscissa values of all points in a Type B triple are distinct, one from another.[5] But, in this case, the abscissa value of p_1 equals the abscissa value of p_2. Thus, it is clear that in Case α there exists exactly one member of S which contains both the points p_1 and p_2.

Case β. Suppose p_1 and p_2 have the same ordinate value but different abscissa values (i.e. p_1 and p_2 are distinct points on the same line, say L_i). Here it is impossible for any Type A triple to contain p_1 and p_2, since any three points constituting a Type A triple must all have the same abscissa value. To determine the unique Type B triple which contains p_1 and p_2 we let x_i = abscissa value of p_i for $i = 1$ and 2. Suppose L_j is that line which immediately succeeds L_i, the line which contains the points p_1 and p_2. Then, it is clear that the points $p_1 = <x_1, i>$ $p_2 = <x_2, i>$ and $p_3 = <\frac{1}{2}(x_1 + x_2), j>$ constitute a Type B triple which certainly contains the points p_1 and p_2. However, it is also clear that any Type B triple which contains the points p_1 and p_2 (since these points lie on the same line) must, in fact, contain that point of L which lies on the line immediately succeeding the line on which p_1 and p_2 lie, and whose abscissa value is the arithmetic mean of the abscissa values of p_1 and p_2. Hence, this point must be the p_3 given above. This proves that, under the assumption of Case β, there is exactly one Type B triple (and no Type A triple) of S which contains the points p_1 and p_2.

5. To see this it is only necessary to observe that if x_1, x_2 are real number such that $x_1 \neq x_2$, then their arithmetic mean $x = \frac{1}{2}(x_1 + x_2)$ has the property that $x \neq x_1$ and $x \neq x_2$.

Case γ. Now suppose the points p_1 and p_2 have different ordinate and different abscissa values (i.e. the points p_1 and p_2 lie on distinct lines and, moreover, are not directly above one another). Since their abscissa values are distinct, the points p_1 and p_2 are not contained in any Type A triple. To see that there is exactly one Type B triple of S which contains p_1 and p_2 we set $p_1 = \langle x_1, i \rangle$ and $p_2 = \langle x_2, j \rangle$. Under the assumptions of Case γ we may conclude that $x_1 \neq x_2$ and $i \neq j$ (i.e. p_1 lies on line L_i and has abscissa value x_1 and p_2 lies on line $L_j (\neq L_i)$ and has abscissa value $x_2 (\neq x_1)$). From what has been said concerning the orientation given in I, it must be that either line L_i immediately precedes line L_j or L_j immediately precedes L_i. Without loss of generality we may assume the latter possibility.

Now consider the three points $p_1 = \langle x, i \rangle$, $p_2 = \langle x_2, j \rangle$ and $p_3 = \langle 2x_1 - x_2, j \rangle$. It is clear that points p_2 and p_3 lie on the same line L_j and that the remaining point p_1 lies on the immediately succeeding line. Moreover, the abscissa value of p_1 is, in fact, the arithmetic mean of the abscissa values of the points p_2 and p_3 since $x_1 = \frac{1}{2}(x_2 + (2x_1 - x_2))$. It is clear then, that the points p_1, p_2 and p_3 constitute a Type B triple which contains p_1 and p_2. Moreover, since there is only one way to form an isosceles triangle with vertex p_1 and base lying on the line L_j, and having one of its end points p_2, we are forced to conclude that there is exactly one Type B triple which contains the points p_1 and p_2. Hence, under the assumption of Case γ, there is exactly one triple of S which contains the points p_1 and p_2.

Since the above three cases have exhausted all possibilities, it has been established that any two distinct points of L are contained in exactly one triple of S. This shows S to be a Steiner triple system for L and, in so doing proves Theorem 3.

§3. *Final remarks*. Without the axiom of choice it is not possible for us to conclude that every Steiner cardinal number is an aleph. If one examines the proof, given in Sobociński [7], that Theorem **SP** implies the axiom of choice, he will observe the following argument. Let an arbitrary non-finite cardinal number $\mathfrak{m}$ be given. Let $\aleph(\mathfrak{m})$ represents Hartogs' aleph for $\mathfrak{m}$ (i.e. $\aleph(\mathfrak{m})$ is the smallest well-ordered cardinal number having the property that $\aleph(\mathfrak{m}) \not\leq \mathfrak{m}$).[6] Since $\mathfrak{m} + \aleph(\mathfrak{m})$ is also a non-finite cardinal number, on the strength of Theorem **SP**, this cardinal number must be the power of a set which possesses a Steiner triple system. Employing this particular Steiner system, Sobociński goes on to show that the non-finite cardinal $\mathfrak{m}$ must, in fact, be an aleph.

Consequently, Sobociński's proof can be used to demonstrate, without the aid of the axiom of choice, that $\mathfrak{c}$ is an aleph if one can assume the cardinal number $\mathfrak{c} + \aleph(\mathfrak{c})$, not $\mathfrak{c}$, is the power of a set which possesses a

6. The existence of $\aleph(\mathfrak{m})$ for each non-finite cardinal number $\mathfrak{m}$, can be established without the axiom of choice.

Steiner triple system. Thus, on the basis of the well-known result of P. Cohen [1], it is possible to deduce the following

Theorem 4. *The theorem, that the non-finite cardinal number* $\mathfrak{x} + \aleph(\mathfrak{x})$ *is the power of a set which possesses a Steiner triple system, is independent of general set theory without the axiom of choice.*

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