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NON-EMPTY COMPLEX TERMS

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The logic of traditional A,E,I and O propositions containing conjunctive and disjunctive terms was outlined by de Morgan and developed by Keynes. But neither writer worked out the logical effects, either of requiring complex terms to have an application, or of alternatively admitting empty terms. De Morgan briefly skirted this question by introducing "existence" and "non-existence" as ingredients of complex terms, manipulable in the same way as their ordinary ingredients.[1] Keynes noted that there was a problem because unqualified manipulation of complex terms sometimes results in a contradiction in terms or in a term with a non-existent subject, and suggested, but did not elaborate the consequences of his suggestion, that particular propositions be interpreted as implying, and universals not as implying the existence of their subjects.[2] In this paper I want to deal with the logic of referential or non-empty terms. After introducing the requirement that empty terms are to be excluded from the system of complex propositions,[3] I shall go on to discuss (a) the effect this requirement has on the validity of inferences involving complex terms, (b) the effect on relations of proposition, and (c) a special problem about relations between propositions that emerges in the system. The symbolism employed will be that of Łukasiewicz for symbolizing A,E,I and O forms of proposition, while "k" as in "kab", "v" as in "vab" and "n" as in "na", will be used to symbolize, respectively, conjunctive, disjunctive and negative terms (or negative ingredients of terms). For example, "$akabvncd$" is to be read as "All both a and b are either non-c or d". When connections between propositions need to be expressed, Russell's notation will be used, except that "$\vee$" will be substituted for his "v".

1. Augustus de Morgan, *Formal Logic*, 1847, p. 120.

2. J. N. Keynes, *Studies and Exercises in Formal Logic*, 4th ed., 1906, esp. p. 486 and p. 492.

3. I follow Keynes in using "complex proposition" to refer to A,E,I and O propositions one or both of whose terms are complex terms, While de Morgan used the phrase "complex term" in the sense followed by Keynes, he used "complex proposition" in a quite different way, viz., to refer to seven sets of *simple* A,E,I and O propositions that he distinguished (op. cit., pp. 65 ff.).

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I. *The Exclusion of Empty Terms*

Strawson has presented an interpretation of everyday universal and particular propositions according to which each proposition of this kind *presupposes* a true proposition to the effect that its subject-term is not empty.[4] Let us adopt his account of presupposing and apply it in the case of *all* terms, i.e., in the case of predicate as well as subject-terms, and in the case of complex as well as simple terms. Putative complex propositions like "All native inhabitants of Antarctica complain about the climate", "Some sheep-killers are carnivorous kangaroos", will then be excluded from our system on the ground that, since they have terms which apply to nothing, they are not propositions and the question of their truth or falsity does not arise. If a candidate *is* a proposition it presupposes true propositions in respect of each of its terms. At the same time, let us add the requirement that if a potential term, simple or complex, is to qualify as a term its contradictory must also be a term. Putative propositions containing terms that apply to everything, as well as those containing vacuous terms, will be excluded from our system. This requirement introduces negative terms and enables us to retain all the traditional forms of inference, including obversion and contraposition.

The extension of these requirements to complex propositions has been challenged by Prior on the ground that it would be restrictive to the point of eliminating universal propositions from our system. Prior maintains that the consequent rule for a system containing complex terms "would be that if '*X*' and '*Y*' are terms then so are 'at-once-*X*-and-*Y*' and 'either-*X*-or-*Y*'". But this, he goes on to point out, leads to the conclusion that no true universal propositions would be expressible in our system. For if "*X*" and "*Y*" are terms then (1) "*XY*" would have application, so that "No *X* is a *Y*" would be false, and (2) since "non-Y" would also be a term in the system, "*X* and non-*Y*" would have application, so that "Every *X* is a *Y*" would be false.[5]

The rule specified by Prior, however, is much more stringent than is necessary for excluding empty complex terms. That exclusion is achieved, and true universal propositions are retained in the usual way, if we merely insist on the weaker rule that *if* a conjunctive or disjunctive term occurs in the system it must apply to something and not apply to everything. Given "de Morgan's Laws", that *kab* and *vnanb,* and *vab* and *knanb*, are pairs of contradictory terms,[6] it follows that for every permissible conjunctive term and vice versa. With this weaker rule, given any two permissible simple terms, *a* and *b,* it follows that some complex terms formed from *a* and *b* and their contradictories will be permissible terms, though they will not necessarily be *kab* or *vab*. Suppose, as is the case with combinations of "Englishmen" and "poets," the four particular propositions, *Iab*, *Ianb*,

4. P. F. Strawson, *Introduction to Logical Theory.* Compare also, H. L. A. Hart, "A Logician's Fairy Tail," *Philosophical Review*, April, 1951.

5. A. N. Prior, *Formal Logic*, 2nd ed., p. 142.

6. De Morgan, op. cit., p. 116.

Ibna, *Inanb*, all happen to be true; the four conjunctive terms, *kab*, *kanb*, *kbna*, *knanb*, and the four corresponding disjunctive terms, *vnanb*, *vnab*, *vnba*, *vab*, will all both be non-empty and not apply to everything. If any of these propositions is false, as, e.g., is the case with "Some squares are circles," the corresponding conjunctive term will be empty and the corresponding disjunctive term will apply to everything. But it is a fact of logic that at most two of these four particular propositions can be false together subcontrariety). Consequently, given two simple terms, possibly four, possibly three, but at least two of the conjunctive (and corresponding disjunctive) terms derivable from them, are permissible complex terms.

II. *Complex Inferences*

To discuss the effects of the exclusion of empty terms it is necessary to summarize the main types of inference involving complex propositions. Keynes sets out these inference-forms in considerable detail, but a much briefer account will be given here, differing from that of Keynes in the following ways. (a) A simpler and more rigorous order of presentation will be employed, thus avoiding repetitions found in parts of Keynes' account. (b) There is a large number of equivalent or derivative inference-forms, many of which concern disjunctive terms, for in line with de Morgan's Laws, every form of conjunctive inference has a corresponding form of disjunctive inference. Most of these forms will be omitted here. (c) Laws of Distribution, Commutation etc, as applied to ingredients of complex terms, are not listed. (d) Also disregarded are complications of detail which arise if terms such as *kakbc*, *kavbc*, are considered.[7]

1. *The Addition and Omission of Conjuncts and Disjuncts*. These are immediate inferences the rules for which can be given by reference to the distribution of terms.

1.1. Conjuncts (or determinants) can be added to distributed terms to form or modify conjunctive terms.
1.2 Conjuncts can be omitted from undistributed terms.

The corresponding rules for disjunctive terms will be added in this case.

1.3 Disjuncts can be added to undistributed terms.
1.4 Disjuncts can be omitted from distributed terms.

Since the reverse processes are not permissible, any proposition derived by these rules is in subaltern relation to the original proposition.

The requirements of our system confine the substitution of propositions for propositional forms to cases of non-empty terms. This affects some types of complex inference in a special way. With some inference-forms, the substitution of permissible premises for premise-forms guarantees that

7. It is worth noting that de Morgan's own account is complicated because he concentrates on examples like *Akabvcd* in which both terms are complex.

the corresponding conclusions are also permissible. But in certain cases, the presence of permissible terms in the premises does not ensure the formation of permissible terms in the conclusions. These cases are therefore subject to a special "conclusion-restriction": the substitution of apparent conclusions for conclusion-forms can be made only if they truly presuppose non-empty terms. This conclusion-restriction applies to 1.1 and 1.3 above. For example, it applies to $Eac \supset Ekabc$. Thus, we cannot derive "No feathered lions are able to fly" from "No lions are able to fly"; such as apparent inference is neither valid nor invalid; it is a non-inference, a case which does not arise in our system.

2. *The Addition and Omission of Superfluous Conjuncts.* These are inferences in which a conjunct, already present in (or as) one term, is added to or omitted from the other term of the proposition. There are four basic forms.

2.1 $Akabc \equiv Akabkac$ 2.2. $Ekabc \equiv Ekabkac$

2.3 $Ikabc \equiv Ikabkac$ 2.4 $Okabc \equiv Okabkac$

Of these, 2.2 and 2.4 are subject to the conclusion-restriction.

What in some logic textbooks is called "Inference by Added Determinants" consists of the two steps, (i) $Aac \supset Akabc$ (by 1.1), (ii) $Akabc \supset Akabkbc$ (by 2.1).

3. *Conversion, Obversion and Contraposition.* The ordinary rules for immediate inference apply to complex propositions, but there are also a number of analogous special cases.

3.1 With E and I propositions, as well as ordinary conversion there is a process I shall call "partial conversion" in which a conjunct is transferred from the subject or predicate to the predicate or subject.

3.11 $Ekabc \equiv Ekacb \equiv Ekbca$ 3.12 $Ikabc \equiv Ikacb \equiv Ikbca$

3.11 is subject to the conclusion-restriction. Thus, from "No carnivorous lions are tigers" we can derive "No carnivorous tigers are lions," but not "No lion-tigers are carnivorous."

3.2 We can also obtain propositions that are "partial obverses" of each other.

3.21 $Akabc \equiv Eakbnc$ 3.22 $Okabc \equiv Iakbnc$

3.21 is subject to the conclusion-restriction.

3.3 With A and O propositions, there is a process analogous to contraposition by which there is an interchange of ingredients of complex terms in subjects or predicates with their contradictories in predicates or subjects.

3.31 $Akabc \equiv Aavnbc \equiv Ekbnca \equiv Akbncna$

3.32 $Okabc \equiv Oavnbc \equiv Ikbnca \equiv Okbncna$

3.31 is subject to the conclusion-restriction.

The derivation of 3.31 can be illustrated as follows.

(i) $Akabc$

(ii) $Ekabnc$ (by ordinary obversion)

(iii) *Eakbnc*	(by partial conversion)
(iv) *Aavnbc*	(by ordinary obversion)
(v) *Ekbnca*	(from (iii) by ordinary conversion)
(vi) *Akbncna*	(from (v) by ordinary obversion)

Here (ii) is the obverse and (iii) a partial obverse of (i), but with each of (iv), (v) and (vi), (i) has been partly contraposed. Since the term "partial contrapositive," already employed in traditional logic, is unavailable, let us make use of the term "transpositive" to refer to these special forms. It will be seen that in transposing a complex proposition there is a question of how many contradictories are involved in the interchange of ingredients. Thus, (iv) and (v) involve only one, (vi)[8] involves two, while the ordinary contrapositive, *Ancvnanb*, involves all three, of the contradictories of the ingredients of *Akabc*.

These transpositives, partial obverses and partial converses are derivative forms of complex inference. As illustrated in the above table, given partial conversion and ordinary immediate inference the other forms can be derived. Partial conversion itself can be derived from the rules about superfluous conjuncts. Thus, $Ekabc \equiv Ekabkac \equiv Ekackab \equiv Ekacb$. But it is convenient to specify these inference-forms separately.

4. *Complex and Compound Propositions.* In certain types of inference, complex propositions are associated with conjunctions and disjunctions of *propositions*. The basic types are as follows.

4.1 $(Aab \,.\, Aac) \supset Aakbc$

Since $(Aakbc \supset Aab)$. $(Aakbc \supset Aac)$ (by 1.2) we have: $(Aab \cdot Aac) \equiv Aakbc$

4.2 $(Aab \,.\, Iac) \supset Iakbc$	4.3 $(Aab \vee Aac) \supset Aavbc$
4.4 $(Oab \vee Oac) \equiv Oakbc$	4.5 $(Aavbc \,.\, Eab) \supset Aac$
4.6 $(Akabc \,.\, Akanbc) \supset Aac$	4.7 $(Aac \,.\, Abd) \supset Akabkcd$

4.7 is an example of a complicated, derivative form. Thus, (i) $(Aac \supset Akabc) \cdot (Abd \supset Akabd)$ (by 1.1), and (ii) $(Akabc \,.\, Akabd) \supset Akabkcd$ (by 4.1).

Of these forms, 4.3, 4.4 and 4.7 are subject to the conclusion-restriction.

III. *Logical Opposition*

Complex propositions provide a range of special cases of the five relations listed in the square of opposition, and of the relations of equivalence and indifference or independence. This topic, which neither de Morgan nor Keynes takes up, is also affected by the question of empty terms. The relations which hold in the present system are set out below. An incidental feature of this system, compared with systems admitting empty terms, is that the number of cases of indifference is minimized.

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8. John Anderson, of Sydney University, in his *Studies in Empirical Philosophy*, esp. p. 130, argued for the importance of this type of proposition in dealing with causal questions, and called it the "virtual contrapositive" of the original proposition.

1. *Simple and Complex Propositions.* There are two main cases to be considered, the relations between simple and complex, and between complex and complex, propositions. Let us consider the former first by taking the four forms of simple propositions which have the subject-predicate filling *a* - *c* in relation to forms of complex proposition which have (i) the filling *kab* - *c* and (ii) the filling *a* - *kbc*, i.e., *Aac* (*Eac* etc.) is to be taken with *Akabc*, with *Aakbc*, and so on. Of these, when the paired propositions are both *A*'s, both *E*'s, both *I*'s, or both *O*'s, the relations (superaltern or subaltern) have already been covered by the rules for adding and omitting conjuncts. With the remainder the relations can be summed up as follows.

(a) With paired *A* and *O* forms, and paired *E* and *I* forms, relations are in all cases either contrary or subcontrary. For example, $Aac \supset \sim Okabc$; $\sim Aac \supset Oakbc$. Since there are no cases of contradictory relation there is thus a contrast with the relations which hold in the square of opposition between *A* and *O*, and *E* and *I*, propositions.

(b) Paired *A* and *E* forms are, with one exception, in contrary relation.

(c) Paired *I* and *O* forms are in subcontrary relation.

(d) With the pairing of *A* and *I*, and *E* and *O*, forms, *A*'s stand to *I*'s, and *E*'s stand to *O*'s with one exception, in superaltern relation.

The two exceptions referred to in (b) and (d) are the relations of *Aac* to *Eakbc*, and *Eakbc* to *Oac*, respectively. In each case the relation is indifference. There is, however, a peculiarity about these exceptions connected with the question of non-empty terms. Thus, suppose, given propositions of these forms, the information happens to be available that *kab* is a non-empty term. In that event the exceptions cease to be exceptions, for in the two cases relations are inferable which are, respectively, contrary and superaltern. To illustrate, let us take the first case, *Aac* and *Eakbc*. With an example like "All swans are birds" and "No swans are green birds," there is no inference to truth or falsity, though both propositions happen to be true. But with an example like "All swans are birds" and "No swans are white birds," where it is true that "Some swans are white" and hence that "white swans" is non-empty, the relation is contrary. The distinction can be explained as follows. Given two propositions of the forms *Aac* and *Eakbc* the question is whether *Ekabc* is a proposition or not. If it is a proposition we can establish contrary relation between the given propositions, for $Ekabc \supset \sim Aac$ (contrary relation), and $Ekabc \equiv Eakbc$ (partial conversion). But *Ekabc* is a proposition, usable in this way, only if it is true that *Iab*.

2. *Complex and Complex Propositions.* When complex propositions have the same terms their relations are, of course, parallel to the relations of simple propositions with the same terms. But suppose we take forms of the pattern *kab* - *c* in relation to forms of the pattern *a* - *kbc*. Here (i) where the paired propositions are both affirmative, or both negative, they exemplify complex inferences and are either in subaltern relation or are equivalent. (ii) *A*'s and *O*'s are in contrary or else in subcontrary relation, *A*'s and *E*'s are in contrary relation, and *I*'s and *O*'s are in subcontrary relation. (iii) Relations between *E* and *I* propositions depart from

the correspondence with relations between simple and complex propositions. Thus, *Ekabc* and *Iakbc*, and *Eakbc* and *Ikabc*, are in contradictory relation.

IV. *Relations of Presupposed Propositions*

In the present system, complex propositions presuppose true propositions to the effect that their terms are non-empty. But this requirement creates problems within the system. Unusual difficulties emerge when we try to determine what relations hold between presupposed propositions and the propositions that presuppose them. In this concluding section I want to specify these difficulties, and to show how they can be overcome.

Complex propositions of the pattern *kab - c* each presuppose *Iab*, and those of the pattern *a - kbc* each presuppose *Ibc*. For example, *Akabc* presupposes *Iab*. But it also *entails* this proposition. This can be shown as follows.

(i)	*Akabc*	
(ii)	*Ikabc*	(by subalternation)
(iii)	*Iakbc*	(by partial conversion)
(iv)	*Iab*	(by omission of conjuncts)

Similarly, for example, *Eakbc* presupposes, but also entails, *Ibc*.

(i)	*Eakbc*	
(ii)	*Ekbca*	(by conversion)
(iii)	*Akbcna*	(by obversion)
(iv)	*Ikbcna*	(by subalternation)
(v)	*Ibkcna*	(by partial conversion)
(vi)	*Ibc*	(by omission of conjuncts)

In the remaining cases, except for *Oakbc*, it can be shown similarly that the relevant proposition is entailed as well as presupposed.

This consequence that the very same proposition can be both presupposed and entailed by another proposition may seem a peculiarity, but it is a harmless one. The basic distinction between presupposing and entailing is unaffected. Thus, suppose *Akabc* is a specific proposition; then a condition of its being a proposition, true or false, is that it truly presupposes *Iab*; but it has to qualify as a proposition, and in addition be *true*, in order to entail *Iab* as a true conclusion. Admittedly, if *Akabc* is false, *Okabc* is true and this also entails *Iab*, so one way or the other *Iab* is entailed as a true conclusion. Nevertheless, the difference remains that it is here truly presupposed *twice*, and only entailed *once* as a true conclusion.

As a connected peculiarity, however, there is the question of which of the seven relations of opposition etc., hold between these presupposed and presupposing propositions. Now, ordinarily we might treat presupposing as a relation among propositions that is quite different from other logical relations.[9] For example, given that a simple universal proposition of the

9. Compare Strawson, op. cit., p. 175.

form *Aab* presupposes *a exists* , we can say that these propositions, being a subject-predicate and an existential proposition respectively, and there being no formal rule of inference connecting them, are not commensurate; so that with respect to the seven ordinary relations they have no relation (unless, extending the scope of this relation, we said it was indifference). But in the case of a complex proposition, the presupposed proposition is itself plainly expressible as a subject-predicate proposition (*ab exists* as *Iab*), and, as we have seen, it already has a role in our system as an entailed proposition. Therefore, in these cases the presupposed proposition should have an ordinary logical relation to the presupposing proposition. But what is the relation? *Akabc* entails but is not entailed by *Iab*, so that it should be in superaltern relation to it; and likewise *Ekabc* etc., should be in superaltern relation to it. But a standard feature of subalternation is that if the subaltern proposition is false, the superaltern proposition must also be false. Here, however, there is the peculiarity that if *Iab* is false, *Akabc* etc., are not propositions and so are not false.

To deal with this peculiarity, let us look more carefully at the seven ordinary logical relations. The differences between these relations may be summed up briefly by considering the possibilities, in regard to truth and falsity, with which each relation is compatible and incompatible. Let us indicate the possibilities a relation leaves open, in respect to any two propositions in the relation, by using "11," "10," "01," and "00," to mean, respectively, that both propositions may be true, that the first proposition may be true and the second false, that the first may be false and the second true, and that both may be false. We can now set out, in the following table, the possibilities each relation leaves open.

equivalence:	11, 00
contradictory relation:	10, 01
subcontrary relation:	11, 10, 01
subaltern relation (of the first proposition to the second):	11, 10, 00
superaltern relation (of the first proposition to the second):	11, 01, 00
contrary relation:	10, 01, 00
indifference:	11, 10, 01, 00

Two relations thus leave open two possibilities and rule out two, four relations each leave open three possibilities and rule out one, while indifference leaves open all four possibilities. Now, this accounts for seven cases, but mathematically there are *fifteen* combinations of this kind available. Thus, given the type of grouping involved, we have what we can call, for brevity, one quadruple (as with indifference), four trebles, six doubles and four singles. The unused combinations consist of four additional cases in which two possibilities are left open, and of the four cases in each of which only one possibility is left open, as follows.

(a): 11, 10	(c): 10, 00
(b): 11, 01	(d): 01, 00

(e): 11 (g): 01
(f): 10 (h): 00

If we expand the set of logical relations to include these new cases as possible relations, we can cope with the peculiarity referred to above Thus, of the new cases, it will be seen that (a) and (b) each incorporate features of subcontrarity and of subalternation. For example, (b) rules out the possibility that the first of two propositions is true and the second false (as does superaltern relation) and also the possibility that both propositions are false (as does subcontrary relation). But the conditions for this "subcontrary-superaltern" relation are fulfilled by the troublesome examples cited above. For example, if *Akabc* is true, *Iab* can't be false (entailment), but they also can't both be false, for if *Iab* happens to be false then *Akabc* is not a proposition and so is not false. In other words, they have a relation only if they are both propositions, and the nature of their relation is accurately specified by (b). At the same time, with (c) and (d) we have a new type of contrariety. For example, (d) rules out the possibility of both propositions being true, but is also incompatible with the first proposition being true and the second false. Relevant examples of this relation are also available. Thus, consider the relation *Eab* has to *Akabc*. Both can't be ture, for the latter entails *Iab* which is the contradictory of *Eab*. But also *Eab* can't be true and *Akabc* false, for if *Eab* happens to be true, *Akabc* is not a proposition and so is not false.

It is worth noting that the four remaining cases, (e), (f), (g) and (h) can also be exemplified by reference to complex terms. Thus, suppose we admit such forms as *Iakab* as propositional forms, and apply to them the rule that terms be non-empty. The relation *Iab* has to *Iakab* is then a case in which the combination for (e), 11, alone is possible, for if either is true the other is true (superfluous conjuncts), but if *Iab* is false *Iakab* is not a proposition and so is not false; i.e., if both are propositions both are true. Again, (f), which has (g) as its converse, is exemplified by *Iakab* and *Eab*, for if both are propositions the first is true and the second false. Finally, (h) is exemplified by *Eab* and *Eakab*, for unless *Eab* is false *Eakab* is not a proposition, but if it is a proposition it must be false.

The special problem confronting the development of a system of non-empty complex terms, the fact that both presupposing and presupposed propositions belong to the one system of inferences and relations, can thus be overcome by introducing the concept of these additional logical relations. At the same time, the introduction of these relations in the present context has the incidental effect of showing that, what might have appeared to be only a theoretically possible expansion of the list of relations, can be exemplified by categorical propositions.

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