SYSTEMS CLASSICALLY AXIOMATIZED AND PROPERLY CONTAINED IN LEWIS’S S3

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It is well known that Lewis’s modal systems S3, S4 and S5 can be classically axiomatized. That is, an axiomatic for those systems can be given with a finite number of axioms taking substitution for propositional variables and material detachment as the only primitive rules of inference. It will be shown in this paper that such an axiomatic is available for some systems properly contained in S3. Each section of the paper introduces new axiomatics for sub-systems of S3 and then gives new sub-systems which are classically axiomatized and in which all of Lewis’s primitive rules of inference are derivable. The symbolism throughout is that of [7] and “α is a thesis” is abbreviated as “\( \vdash \alpha \)”.

I. Lemmon in [4] gave new foundations for Lewis’s systems S1-S3 of [5] analogous to a systematic for T of Feys-von Wright [2, 8, 12] due to Gödel in [3]. In this section new foundations for Lemmon’s systems are described and two systems containing Lemmon’s S0.5 and properly contained in S3 are classically axiomatized.

The Lemmon systems are N-C-L calculi with K and E defined in the usual way by C and N, \( \& \) defined as LC and \( \& pq \) (strict equivalence) as \( K\& pq\& qp \). Propositional calculus (PC) is given by three rules:

(PCa) if \( \alpha \) is a tautology, then \( \vdash \alpha \);
(PCb) substitution for propositional variables;
(PCc) material detachment (that is, from \( \alpha \) and \( C\alpha\beta \) infer \( \beta \));

and Lewis’s systems are based on selections from the following rules and axioms:

(a) \( \vdash \alpha \) only if \( \vdash L\alpha \);
(a′) \( \alpha \) is a tautology or axiom only if \( \vdash L\alpha \);
(a′′) \( \alpha \) is a tautology only if \( \vdash L\alpha \);
(b) \( \vdash L\alpha\beta \) only if \( \vdash LCL\alpha L\beta \);
(b′) substitutability of strict equivalents;
(b′′) \( \vdash \alpha\beta \) only if \( \vdash \& L\alpha L\beta \);
(1) \( CLpqLCLpLq \);
(1′) \( CLpqCLpLq \);
(2) \( CLp\&p \);
(3) \( CKLPqLCqrLCPr \).

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Lemmon's foundations for the Lewis systems, S1-S3, are then given as:

\[ \begin{align*}
S_1 &= \{PC; (a'); (b'); (3); (2)\} \\
S_2 &= \{PC; (a'); (b); (1'); (2)\} \\
S_3 &= \{PC; (a'); (1); (2)\}
\end{align*} \]

the system of Feys-von Wright as:

\[ T = \{PC; (a); (1'); (2)\} \]

and a system introduced by Lemmon as:

\[ S_{0.5} = \{PC; (a''); (1); (2)\}. \]

Now from the following list of axioms:

\[ \begin{align*}
A_1. & \quad LCCNppp \\
A_2. & \quad LCCpqCCqCpr \\
A_3. & \quad LCPqNq \\
A_4. & \quad CLCpqCLpLq \\
A_5. & \quad CLp\phi
\end{align*} \]

the axiomatics for the above systems are taken as:

\[ \begin{align*}
M_0 &= \{PCb; PCc; A1; A2; A3; A4; A5\} \\
M_1 &= \{M_0; (b_f); AV; A2\} \\
M_2 &= \{M_0; (b_t); A1''; A3\} \\
M_3 &= \{M_0; A1'; A4'\} \\
M &= \{M_0; (a)\}.
\end{align*} \]

The adequacy of the revisions can be seen by observing first that all of Lemmon's systems contain \( M_0 \); (i) \( A1-A3 \) follow from \( PC \) and either (a); (a'), or (a''), and (ii) \( A4 \) either follows from (1) and (2) by \( PC \), or in \( S_1 \) is provable from (3):

\[ \begin{align*}
1. & \quad CKLCpqLCqLCpr & [3] \\
2. & \quad CLCpqCLCqLCpr & [1, PC] \\
3. & \quad CLCNqNpCLCNppLCNq & [2, PC] \\
4. & \quad \& CpqCNpNq & [PC, (a')] \\
5. & \quad \& pCNpp & [PC, (a')] \\
6. & \quad \& CNpqCNq & [PC, (a')] \\
7. & \quad CLCpqCLPQCNpq & [3, 4, 5, 6, (b')] \\
8. & \quad CLCNqNpCLCNqPQCNq & [2, PC] \\
9. & \quad CLCpqCLCpCNpq & [8, 4, 5, (b')] \\
10. & \quad CLCpqCLpLq & [7, 9, PC]
\end{align*} \]

Thus, \( S_{0.5} \) contains \( M_0 \) and \( T \) contains \( M \).

Secondly, (i) each of the remaining systems, \( S_1-S_3 \), contain \( (b') \)(cf. p. 178 of [4]), and thus contain \( A1' \) by (2) and (a'); while (ii) \( S_1 \) contains \( A2' \) by (3) and (a'), \( S_2 \) contains \( A3' \) by \( A3 \) and (b), and \( S_3 \) contains \( A4' \) by (1) and (a'). Thus \( S_1-S_3 \) contain \( M_1-M_3 \) respectively.
Conversely, it can be shown that $M_0$-$M_3$, and $M$ contain $S0.5$, $S1$-$S3$, and $T$. To this end the following theorems of $M_0$ are established.

Theorem 1. If $\vdash \alpha$ in $PC$, then $\vdash L\alpha$.

Proof. If $\alpha$ is an axiom of $PC$, then $L\alpha$ is given by $A1$-$A3$ since $CCNppp$, $CCpqCCqrCpr$, and $CpCNpq$ form a set of axioms for $PC$ (cf. Appendix of [7]), and if $\alpha$ is derived from the axioms of $PC$, then $L\alpha$ is given by:

1. $L\beta$ \hspace{1cm} [Induction hypothesis]
2. $LC\beta\alpha$ \hspace{1cm} [Induction hypothesis]
3. $CL\beta La$ \hspace{1cm} [2, A4]
4. $La$ \hspace{1cm} [1, 3]

Theorem 2. If $\vdash LCA\beta$ and $\vdash LC\alpha\gamma$, then $\vdash LCAK\beta\gamma$.

Proof. From hypothesis by Theorem 1 and $A4$.

Theorem 3. If $\vdash LCA\beta$ and $\vdash LC\beta\gamma$, then $\vdash LC\alpha\gamma$.

Proof. From hypotheses by Theorem 1 and $A4$.

Theorem 4. If $\vdash \alpha$ and $\vdash \beta$, then $\vdash K\alpha\beta$.

Proof. From hypotheses by Theorem 1 and $A5$.

Theorem 5. If $\vdash \alpha$ and $\vdash LCA\beta$, then $\vdash \beta$.

Proof. From hypotheses by $A5$.

Hence, $PC$ is contained in each of $M_0$-$M_3$, and $M$ by Theorem 1 and $A5$, and thus $M_0$ contains $S0.5$ and $M$ contains $T$.

Further, $M_3$ contains (b'). The proof is given by showing that (i) $\vdash \$a\beta$ only if $\vdash LLaL\beta$, (ii) $\vdash \$a\beta$ only if $\vdash NaNa\beta$, (iii) $\vdash \$a\beta$ only if $\vdash CCa\gamma C\beta\gamma$, and (iv) $\vdash \$a\beta$ only if $\vdash CC\alpha\gamma C\beta\gamma$. Now (i) follows by $A4'$ and (ii-iv) follow from $CLCpqLCNqNp$, $CLCpqLCCqrCpr$, and $CLCpqLCCrpCrq$, each of which are obtained by Theorem 1 and $A4$.

Moreover $M_2$ contains (b):

1. $LCA\beta$ \hspace{1cm} [Hypothesis]
2. $LCa\alpha$ \hspace{1cm} [Theorem 1]
3. $LCAKa\beta$ \hspace{1cm} [1, 2, Theorem 2]
4. $LCAKa\beta$ \hspace{1cm} [Theorem 1]
5. $LCLaLa$ \hspace{1cm} [Theorem 1]
6. $LCLaLKA\beta$ \hspace{1cm} [3, 4, 5, (b')]
7. $LCLaLKa\beta LCNa\beta KNa\beta$ \hspace{1cm} [A3']
8. $LCLaLKa\beta L\beta$ \hspace{1cm} [7, Theorem 1, (b')]
9. $LCLaL\beta$ \hspace{1cm} [6, 8, Theorem 3]

And by Theorem 1, in order to show that $M_1$-$M_3$ contain (a') it is sufficient to remark that (i) each system contains $LCL\rho$ by $A1'$ and (b'), (ii) $M_1$ contains $LCLCpqLCLqLCPq$ by $A2'$ and (b'), (iii) $M_2$ contains $LCLCpqLCpLq$ by $A2'$, (b') and Theorem 3, and (iv) $M_3$ contains $LCLCpqLCLpLq$ as $A4'$. 

Thus, to complete the proof that M1-M3 contain S1-S3 it need only be shown that each of the axioms of S1-S3 which is not an axiom of M1-M3 are provable in M1-M3. In M1, (3) follows from $LCKLC\neg pLC\neg qrLC\neg pr$ by A5. And in M3, (1) follows from $A4'$ by A5.

On the basis of the new foundations for Lewis's systems it is now possible to classically axiomatize some systems containing S0.5 and properly contained in S3. The systems to be considered are:

\[
R_1 = \{M_0; A6; A'1; A2'\}
\]

\[
R_2 = \{M_0; A6; A'1; A3'\}
\]

\[
R_3 = \{M3\}.
\]

Bull in [1] uses an equivalence relation employing schemata analogous to A6 as Lemma 1 part III (p. 212) while for R1-R3, since A6 yields (b'"), the proof that each of the systems contains (b') can be established as was the proof above that M3 contains (b'). And thus:

\[
R_1 = \{S1; A6\}
\]

\[
R_2 = \{S2; A6\}
\]

\[
R_3 = \{S3\}.
\]

Hence R1-R3 obviously contain S0.5. But R1 and R2 are properly contained in S3 as is shown by a variation of Parry's matrix of [6]: a regular expansion of the C-N matrix (as are all matrices considered in this paper) to eight values:

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with $L(*1*2*3*45678) = (26888888)$, where *n indicates a designated value. For with this matrix all the axioms and rules of S2 are designated together with A6 while $A4'(p/1,q/2) = LCLC12LCL1L2 = LCL2LC26 = LC6L5 = LC68 = L3 = 8$.

It may be noted that the independence of A6 in R1 and R2 is shown by Parry's original matrix where only 1 and 2 are designated values. For in this case all the axioms and rules of S2 are still designated while $A6(p/1,q/2) = CLC12LC21LCL1L2 = CL2CL1LC26 = C6C2L5 = C6C28 = C67 = 3$.

Moreover, the independence of A1' in R1-R3 is shown by a variation of Group IV of Lewis [5] in which $L(*1*234) = (1333)$, since $A1'(p/2) = LCLCN222 = LCLC322 = LCL2 = LC32 = L2 = 3$.

The independence of A3' in R2 is shown by Group V of Lewis [5] in
which $L(*1*234) = (2434)$, since $A3'(p/3,q/2) = LCL3LCN32 = LC3LC22 = LC3L1 = LC32 = L2 = 3$.

And though Lemmon has shown that $A2'$ is derivable in $S2$ (cf. p. 178 of [4]), and hence $R2$, it remains an open question as to whether $A2'$ is derivable or independent from the remaining axioms of $R1$. It should be noted that Lemmon describes a proof of $A2'$ in $S3$ (p. 179 of [4]), but in so doing applies $(a')$ to $(1')$ which is not an axiom of $S3$. However $A2'$ is derivable in $S3$, and hence $R3$, as follows:

1. $LCCpqCCqrCpr$ [PC, $(a')$]
2. $LCLCpqLCCqrCpr$ [1, (b)]
3. $LCLCpqLCLpLq$ [(1), $(a')$]
4. $LCLp$ [(2), $(a')$]
5. $LCLCpqCLpLq$ [3, 4, Theorem 3]
6. $LCLCpqLCqrLCpr$ [2, 5, Theorem 3]

Indeed, this derivation points up a significant difference between Lemmon's foundations for $S1$-$S3$ and those given here. The derivation requires line 5 whereas that thesis is independent of $S0.5$ (given Thomas's matrix of [11] in which $L(*1*234) = (2334)$, since $LCLC34CL3L4 = LCL2C34 = LCL2C34 = LC32 = L2 = 3$). And it is the absence of this thesis and $A1'$ which gives rise to the systems to be discussed in the next section.

Finally, although $A6$ is useful in classically axiomatizing systems properly contained in $S3$, its addition to system $T$ yields $S4$, and thus it is useless in attempting to axiomatize $T$. That $T$ with $A6$ yields $S4$ clearly follows from the fact that the proper axiom of $S4$, $LCLpLLp$, follows from $A6$ in $T^0 (= PC; (a); A4)$:

1. $CLCqpCLCpqLCLpLq$ [A6, PC]
2. $CLCqpCLCpqCLCCpplqLCLcppLq$ [1, PC]
3. $LCqCp$ [PC, $(a)$]
4. $CLCCpplqLCLcppLq$ [2, 3, PC]
5. $LCqCCppq$ [PC, $(a)$]
6. $CLqLCCpplq$ [5, A4 PC]
7. $CLqLCLcppLq$ [4, 6, PC]
8. $CLcppCLCCppgLqqLq$ [PC]
9. $LCq$ [PC, $(a)$]
10. $CCLcppgLqqLq$ [8, 9, PC]
11. $LCCLcppgLqqLq$ [10, $(a)$]
12. $CLCcppgLqLLq$ [11, A4, PC]
13. $CLqLLq$ [7, 12, PC]
14. $LCLqLLq$ [13, $(a)$]

Thus this section gives a classical axiomatization for two systems $R1$ and $R2$ which contain $S0.5$ and are properly contained in $S3$.

II. Sobociński in [9] describes a system, $S3^*$, which is properly contained in $S3$ and is classically axiomatized. In this section, $S3^*$ will be given a new basis and other systems properly contained in $S3^*$ will be classically axiomatized.
System S3* is an N-K-M calculus with C and E given their usual definitions, $\mathcal{E}pq$ defined as $\text{NMK}pNq$, $\mathcal{E}pq$ as $K\mathcal{E}pq \mathcal{E}qp$, and L as $\text{NMN}$. The rules of inference are $\text{PCb}$ adjusted to N-K-M and $\text{PCc}$ for N-K (that is from $\alpha$ and $\text{NKaN}\beta$ infer $\beta$). The following are the axioms of S3*:

$$
\begin{align*}
Z1. & \quad \text{NMK}p\text{NK}p\beta \\
Z2. & \quad \text{NMKKp}qNq \\
Z3. & \quad \text{NMKKKp}r\text{NK}q\text{rNK}pNq \\
Z4. & \quad \text{NMKNMK}p\text{n}q\text{NNMK}N\text{MqNMP}p \\
Z5. & \quad \text{KNM}p\text{pN}p \\
\end{align*}
$$

While the new basis for S3* is the N-C-L calculus:

$$
\text{R3}^* = \{\text{MO}; \text{A4}^1\}.
$$

To see that R3* contains S3* first observe that the definitions of E, C, $\mathcal{E}$, and L are provable in the form of strict equivalences, when K is given its usual N-C definition and $M = \text{NLN}$.

Now the proof of section I that M3 contains (b') in no way relies on A1'. Thus R3* contains (b'). And hence the axioms of S3*, are obtainable from the following theses of R3*: $\text{LCpKpp}$, $\text{LCKpqq}$, $\text{LCKKR}p\text{NKq}r\text{KpN}q$, $\text{LCLpqLCLpL}q$, and $\text{CLpp}$, while the rules of S3* are obtainable from the rules of R3*.

Conversely, to show that S3* contains R3* it must be remarked that the definitions of K, E, $\mathcal{E}$, and M are provable in the form of strict equivalences when C is given its usual N-K definition and $L = \text{NMN}$. Thus, once the substitutability of strict equivalents is shown for S3*, it is clear that $A4'$ follows from Z4, $A5$ from Z5, and $A4$ from $A4'$ and $A5$ by $\text{PC}$.

Hence, besides showing (b') in S3* it is sufficient to show that S3* contains PC and that if $\vdash \alpha$ in PC then $\vdash \text{NMN}\alpha$ in S3*, in order to complete the proof that S3* contains R3*. For in such a case, $A1-A3$ will be theses of S3*.

To this end, the following meta-rules and theses of S3* given passim in [9] are required.

$$
\begin{align*}
\text{RI.} & \quad \vdash \alpha \text{ and } \vdash \text{NMKaN}\beta \text{ only if } \vdash \beta. \\
\text{RIII.} & \quad \vdash \text{NMKaN}\beta \text{ and } \vdash \text{NMK}bN\gamma \text{ only if } \vdash \text{NMKaN}\gamma. \\
\text{RIV.} & \quad \vdash \text{NMKaN}\beta \text{ only if } \vdash \text{NMKMaN}\beta. \\
\text{RV.} & \quad \vdash \text{NMKaN}\beta \text{ and } \vdash \text{NMKaN}\gamma \text{ only if } \vdash \text{NMKaN}\beta\gamma. \\
Z7. & \quad \text{NMKN}p\beta \quad Z12. & \quad \text{NMKNKNp}r\text{NNK}r\text{p} \\
Z8. & \quad \text{NMKNMKp}q\text{NNMNKNqNNp}p \quad Z16. & \quad \text{NMKKpqNK}p\beta \\
Z9. & \quad \text{NMKpNNNp} \quad Z21. & \quad \text{NMKKpqN}p
\end{align*}
$$

It is now possible to derive the following meta-rules.

$$
\begin{align*}
\text{RVI.} & \quad \vdash \text{NMKaN}\beta \text{ only if } \vdash \text{NMKN}\beta\alpha. \\
\text{Proof:} & \quad \text{From hypothesis by Z8 and RI.} \\
\text{RVIII.} & \quad \vdash \text{NMKaN}\beta \text{ only if } \vdash \text{NMKKa}\gamma\text{NK}\beta\gamma.
\end{align*}
$$
Proof:
1. \( \text{NMKaN} \beta \) \[\text{Hypothesis}\]
2. \( \text{NMMKaN} \alpha \gamma \) \[Z2\]
3. \( \text{NMKN} \alpha \gamma \) \[1, 2, \text{RVII}\]
4. \( \text{NMKN} \alpha \gamma N \gamma \) \[Z2\]
5. \( \text{NMKN} \alpha \gamma N \gamma \) \[3, 4, \text{RV}\]

**RIX:** \( \vdash \alpha \beta \text{ and } \vdash \gamma \text{ only if } \vdash \delta \text{ where } \delta \text{ results from } \gamma \text{ by replacing } \alpha \text{ by } \beta \) \((\beta \text{ by } \alpha)\) in one or more places.

Proof: The meta-rule follows immediately from (i) \( \vdash \alpha \beta \text{ only if } \vdash \delta \text{MaM} \beta \), (ii) \( \vdash \alpha \beta \text{ only if } \vdash \delta \text{NaN} \beta \), (iii) \( \vdash \alpha \beta \text{ only if } \vdash \delta \text{K} \alpha \gamma K \beta \gamma \), and (iv) \( \vdash \alpha \beta \text{ only if } \vdash \delta \text{K} \gamma \alpha K \gamma \beta \), which are obtained from RIV, RVIII, RVIII, and RVIII with Z16, respectively.

**RX:** If \( \vdash \alpha \text{ in PC}, \) then \( \vdash \alpha \).

Proof: The meta-rule is established by deriving a sufficient set of axioms for PC, B1-B4, below, is such a set, given by Sobociński in [10].

\[
\begin{align*}
B1. & \quad \text{NKPNNP} \\
B2. & \quad \text{NKpNqNq} \\
B3. & \quad \text{NKNNKpNqNqNqNq} \\
B4. & \quad \text{KNKNpNqNKNqNqNq} \\
\end{align*}
\]

**RXI:** If \( \vdash \alpha \) then \( \vdash \text{NMNa} \).

Proof: In case \( \alpha \) is an axiom of PC, \( \text{NMNa} \) is given by:

\[
\begin{align*}
\text{NMNNKpNpNp} & \quad \text{[Z1, Z7, Z9, RIX]} \\
\text{NMNNKpNqNq} & \quad \text{[Z2, Z7, Z9, RIX]} \\
\text{NMNNKKpNqNqNqNq} & \quad \text{[Z3, Z7, Z9, RIX]} \\
\text{NMNNKNNpNpNKNqNqNp} & \quad \text{[Z12, Z7, Z9, RIX]} \\
\end{align*}
\]

and in case \( \alpha \) is derived from the axioms of PC, the proof is completed by the following derivation.

1. \( \text{NMN} \beta \) \[\text{Induction hypothesis}\]
2. \( \text{NMNNK} \beta \) \[\text{Induction hypothesis}\]
3. \( \text{NK} \beta \) \[2, Z7, Z9, \text{RIX}\]
4. \( \text{NMN} \alpha \beta \) \[3, \text{RVII}\]
5. \( \text{NMKN} \alpha \beta \beta \) \[4, Z4, \text{RI}\]
6. \( \text{N} \alpha \beta \) \[1, 5, \text{RI}\]

Thus RIX-RXI complete the proof that S3* contains R3*.

With S3* given a basis analogous to those of section I, the following systems are now defined:

S2* = \{M0; \{b'; A3'\}\}; \quad \text{R2* = \{M0; A6, A3'\};}
S1* = \{M0; \{b'\}\}; \quad \text{R1* = \{M0; A6 \}}.

The independence of A2' from all the systems under consideration in this section is given by the Thomas matrix, for A2' (p/1, q/2, r/4) = CCLC12CLC24LC14 = LCL2CL3L4 = LC3C34 = LC32 = L2 = 3. Thus
S3 properly contains R3*, while proofs of the previous section establish that

\[ R2^* = \{S2^*; A6\} \]
\[ R1^* = \{S1^*; A6\} \]

as well as showing that each of the following systems properly contains its predecessor: S0.5, R1*, R2*, and R3*.

Thus this section gives a classical axiomatization for three systems, R1*, R2*, and R3*, which properly contain S0.5 and are properly contained in S3.

III. In [9] Sobociński has also described a system, S3°, analogous to S1° and S2° of Feys [2], which is properly contained in S3. In this section, analogues, R1°-R3°, of these systems will be classically axiomatized.

Systems S1°-S3° have the same primitive basis as S3*, while their rules of inference are the four given by Lewis [5]: substitution for propositional variables adjusted to N-K-M; substitutability of strict equivalents; adjunction (that is, \( \vdash \alpha \) and \( \vdash \beta \) only if \( \vdash K\alpha \beta \)); and strict detachment (that is, \( \vdash \alpha \) and \( \vdash NMK\alpha N\beta \) only if \( \vdash \beta \)). Their axioms are drawn from:

\[
\begin{align*}
F1. \ & NMKKpqpNp \\
F2. \ & NMKKpqNKKp \\
F3. \ & NMKKKpqrNKpKqr \\
F4. \ & NMKnKpKp \\
F5. \ & NMKNMKpNqNMQKqNNMKp
\end{align*}
\]

so that, with the above rules of procedure:

\[
\begin{align*}
S1° &= \{F1; F2; F3; F4; F5\}; \\
S2° &= \{S1°; K1\}; \\
S3° &= \{S1°; L1\},
\end{align*}
\]

The R-systems to be considered are:

\[
\begin{align*}
R1° &= \{M0; A6; A2'; A3'\}; \\
R2° &= \{M0; A6; A2'; A3'\}; \\
R3° &= \{M0; A2'; A4'\};
\end{align*}
\]

and it will be shown that

\[
\begin{align*}
R1° &= \{S1°; A5; A6\}; \\
R2° &= \{S2°; A5; A6\}; \\
R3° &= \{S3°; A5\}.
\end{align*}
\]

As in section II the required definitions are provable in both the R- and S-systems as strict equivalences, as is the substitutability of strict equivalences. Thus R1°-R3° contain: F1-F4 by Theorem 1 and A5; F5 by A2'; and the remaining rules of inference by Theorems 4 and 5. Moreover, R2° contains K1:

\[
\begin{align*}
1. \ & LCNMNPqNMNCNPq \quad [A3', (b'), M0] \\
2. \ & LCMNCNPqMNPq \quad [1, (b'), M0]
\end{align*}
\]
and finally, $L1$ follows in $R3^o$ by $A4'$.

Conversely, the required containments are obvious, since $S1^o$ together with $A5$ yields $M0$ as is clear from Feys [2], and thus the adequacy of $R1^o-R3^o$ is established.

It should be noted that the presence of $A5$ with $S3^o$ ($S1^o$, or $S2^o$) defines a system which properly contains $S3^o$ ($S1^o$, or $S2^o$) and is properly contained in $S3$ ($S1$, or $S2$). $A5$ is shown to be independent from $S1^o-S3^o$ by interpreting $L$ as $verum$, and the independence of $A1'$ from $R1^o-R3^o$ was given in section I. (But whether or not it is possible to classically axiomatize $S3^o$ remains an open question.)

Finally, the proofs of independence given in the two previous sections show that (i) $R1-R3$ properly contain $R1^o-R3^o$ respectively, just as these systems properly contain $R1^*-R3^*$ respectively, while (ii) each of the following systems properly contains its predecessor: $S0.5, R1^o, R2^o, R3^o$ and $S3$.

Thus this section gives a classical axiomatization for three systems $R1^o-R3^o$ which contain $S0.5$ and are properly contained in $S3$.

IV. In conclusion the appended table shows the containment relations between the systems discussed in this paper.

\[
\begin{align*}
R1 & \leftarrow R2 \leftarrow R3 \text{ (S3)} \\
R1^o & \leftarrow R2^o \leftarrow R3^o \text{ (S3$^o$; CL$pp$)} \\
S0.5 & \leftarrow R1^* \leftarrow R2^* \leftarrow R3^* \text{ (S3$^*$)}
\end{align*}
\]

REFERENCES


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