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SOME RESULTS ON FINITE AXIOMATIZABILITY IN MODAL LOGIC

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A system of propositional calculus S may be said to be *finitely axiomatizable* if there is a finite set of schemata $\{A_1, \ldots, A_n\}$ such that $\vdash_S A_i$ $(1 \leq i \leq n)$ and any theorem of S can be derived from $A_1, \ldots, A_n$ using detachment (*modus ponens*) alone. In [5], McKinsey and Tarski show, among other things, that the Lewis systems S4 and S5 are finitely axiomatizable, and in [6] the result is extended to S3[1]. The purpose of the present note is to prove a quite general theorem, due to Tarski, giving sufficient and necessary conditions under which a system is finitely axiomatizable, and to use this result to establish that the members of a certain class of modal systems, including T, S2, and E2, are not finitely axiomatizable[2].

1. In what follows, it will frequently be convenient, and never seriously ambiguous, to use the names of propositional logics as names of the corresponding classes of theorems. This is the case with the following fundamental theorem[3]. Any system is, of course, understood to be closed with respect to substitution and detachment.

Theorem 1. Let S *be any system. Then a sufficient and necessary condition for* S *to be not finitely axiomatizable is that there be an infinity of systems* $S_0, S_1, \ldots, S_n, \ldots$ *such that* $S_n \subseteq S_{n+1}$ *and* $S_n \neq S$ *for all n and* $S = \bigcup S_n$.

Proof. Suppose there are systems S_n such that $S_n \subseteq S_{n+1}$ and $S_n \neq S$ for all n, and $S = \bigcup S_n$, and consider any finite set $\{A_1, \ldots, A_m\}$ of theorem-schemata of S. Then for A_i there is a system S_{a_i} such that $\vdash_{S_{a_i}} A_i$. Let $p = \max \{a_1, \ldots, a_m\}$. Then $\vdash_{S_p} A_i$ for all i $(1 \leq i \leq m)$, so that any consequence of $\{A_1, \ldots, A_m\}$ is in S_p. But S_p is properly included in S, so that $\{A_1, \ldots, A_m\}$ cannot provide an axiomatization for S. Conversely, suppose S is not finitely axiomatizable, and consider an enumeration $A_1, \ldots, A_m, \ldots$ of the theorem-schemata of S. Then the systems S_n whose axiom-schemata are $\{A_1, \ldots, A_{n+1}\}$ and sole rule of inference detachment are evidently such that $S_n \subseteq S_{n+1}$ and $S = \bigcup S_n$. That $S_n \neq S$ follows from the assumption that S is not finitely axiomatizable.

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It will be useful to begin by giving formulations of the systems E2 and T. We consider the following schemata and rules:

A1: $A \rightarrow (B \rightarrow A)$;
A2: $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$;
A3: $(-A \rightarrow -B) \rightarrow (B \rightarrow A)$;
A4: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$;
A5: $\Box A \rightarrow A$.

R1: $\dfrac{A, A \rightarrow B}{B}$; **R2**: $\dfrac{A \rightarrow B}{\Box A \rightarrow \Box B}$; **R3**: $\dfrac{A}{\Box A}$

Then E2 has as axiom-schemata *A1-A5* and rules **R1** and **R2**, whilst T has as axiom-schemata *A1-A5* and rules **R1** and **R3**[4]. The usual definitions of other connectives, including $\Diamond$, are presupposed. We use T for $p \rightarrow p$.

We next construct an infinity of systems $E2^n$, for each natural number n, as follows:

$E2^0$ is just E2;
$E2^n$ $(n \geqslant 1)$ has as axioms any theorem of E2, together with $\Box^n T$ $(= \underbrace{\Box \ldots \Box}_{n} T)$, and sole rule of inference **R1** (detachment).

Lemma 1. $E2^n \subseteq E2^{n+1}$ for all n.

Proof. Since $\vdash_{E2^{n+1}} \Box^{n+1} T$ by *A5* $\vdash_{E2^{n+1}} \Box^n T$.

Lemma 2. *If* $\vdash_T A$, *then* $\vdash_{E2} \Box^n T \rightarrow A$ *for some* n.

Proof, by induction on the number of occasions **R3** is used in the given proof of A. If B is an axiom, then $\vdash_{E2} T \rightarrow B$. An application of **R3** in a T-proof yielding $\Box B$ from B can be paralleled by an application of **R2** in a corresponding E2-proof yielding $\Box^{n+1} T \rightarrow \Box B$ from $\Box^n T \rightarrow B$.

Lemma 3. $T = \bigcup E2^n$.

Proof. Clearly $\vdash_T \Box^n T$ for any n, so that $E2^n \subseteq T$, whence $\bigcup E2^n \subseteq T$. Conversely, if $\vdash_T A$, then $\vdash_{E2^n} A$ for some n, by Lemma 2.

Lemma 4. $T \nsubseteq E2^n$ *for all* n.

Proof is delayed until the second part of the paper, where matrices $\mathfrak{M}^n$ are defined which distinguish $E2^n$ from T.

Theorem 2. T *is not finitely axiomatizable.*

Proof immediate from Theorem 1 and Lemmas 1, 3, and 4, if we take $E2^n$ as the systems S_n.

Let us designate as ET the system resulting from E2 by the addition of the axiom:

A6: $\Box T \rightarrow \Box\Box T$.

Lemma 5. $\vdash_T A$ *iff* $\vdash_{ET} \Box T \rightarrow A$.

Proof. $\vdash_T$ *A6*, so that ET $\subseteq$ T. Hence if $\vdash_{ET} \Box T \rightarrow A$ then obviously $\vdash_T A$.

The converse is proved by induction on the length of A's T-proof. If B is an axiom of T, then $\vdash_{\mathrm{ET}} \Box T \rightarrow B$. Suppose B results by **R1** from C and $C \rightarrow B$, and suppose $\vdash_{\mathrm{ET}} \Box T \rightarrow C$ and $\vdash_{\mathrm{ET}} \Box T \rightarrow (C \rightarrow B)$. Then by *A2* $\vdash_{\mathrm{ET}} \Box T \rightarrow B$. Suppose $\Box B$ results by **R3** from B, and suppose $\vdash_{\mathrm{ET}} \Box T \rightarrow B$. Then $\vdash_{\mathrm{ET}} \Box\Box T \rightarrow \Box B$ by **R2**, whence $\vdash_{\mathrm{ET}} \Box T \rightarrow B$ by *A6*.

Theorem 3. ET *is not finitely axiomatizable.*

Proof. If $\{A_1, \ldots, A_n\}$ provided a finite axiomatization of ET, then by Lemma 5 $\{A_1, \ldots, A_n, \Box T\}$ would provide one for T, counter to Theorem 2.

We now turn to the rather more difficult task of showing that E2 and S2 are not finitely axiomatizable. In fact, as is shown in [4], S2 = $\mathrm{E2}^1$, so that the result for S2 will follow from the fact that none of the systems $\mathrm{E2}^n$ can be finitely axiomatized.

As before, we define a hierarchy of systems included in E2. The starting point of this hierarchy is a system e2 whose axiom-schemata are *A1-A5*, as for E2, but whose sole rule of inference is **R1** (thus e2 *is* finitely axiomatizable). Putting $\mathrm{e2}_0 = \mathrm{e2}$, we define systems $\mathrm{e2}_m$ inductively as follows: the axioms of $\mathrm{e2}_{m+1}$ are all the axioms of $\mathrm{e2}_m$, together with all sentences $\Box A \rightarrow \Box B$ where $\vdash_{\mathrm{e2}_m} A \rightarrow B$; sole rule of inference for all systems is **R1**.

Lemma 6. $\mathrm{e2}_m \subseteq \mathrm{e2}_{m+1}$ *for all m.*

Proof immediate from the definition of $\mathrm{e2}_m$, since the axioms of $\mathrm{e2}_{m+1}$ include those of $\mathrm{e2}_m$.

Next, we use the systems $\mathrm{e2}_m$ to define a further series of hierarchies of systems. The systems $\mathrm{e2}_m^n$ for $n \leq m$ shall have as axioms all axioms of $\mathrm{e2}_m$ together with $\Box^n T$, and sole rule of inference **R1**. All these systems are closed with respect to substitution, since $\vdash_{\mathrm{e2}_m} \Box^n T \rightarrow \Box^n(A \rightarrow A)$ for $n \leq m$.

Lemma 7. $\mathrm{e2}_m^n \subseteq \mathrm{e2}_{m+1}^n$ *for all* $m \geq n$.

Proof again immediate, since any axiom of $\mathrm{e2}_m^n$ is one of $\mathrm{e2}_{m+1}^n$.

Lemma 8. $\mathrm{e2}_m^n \subseteq \mathrm{e2}_m^{n+1}$ *for all* $n < m$.

Proof by *A5*, whence any axiom of $\mathrm{e2}_m^n$ is either an axiom or a theorem of $\mathrm{e2}_m^{n+1}$.

Lemma 9. $\vdash_{\mathrm{E2}^n} A$ *iff* $\vdash_{\mathrm{E2}} \Box^n T \rightarrow A$.

Proof. If $\vdash_{\mathrm{E2}} \Box^n T \rightarrow A$, then $\vdash_{\mathrm{E2}^n} \Box^n T \rightarrow A$ whence of course $\vdash_{\mathrm{E2}^n} A$. Conversely, suppose $\vdash_{\mathrm{E2}^n} A$. By induction on the length of proof, it is easy to show that $\vdash_{\mathrm{E2}} \Box^n T \rightarrow A$.

Lemma 10. $\vdash_{\mathrm{e2}_m^n} A$ *iff* $\vdash_{\mathrm{e2}_m} \Box^n T \rightarrow A$, *for* $n \leq m$.

Proof similar to that of Lemma 9.

Lemma 11. *If* $\vdash_{\mathrm{E2}} A$, *then* $\vdash_{\mathrm{e2}_m} A$ *for some m.*

Proof by induction on the number of occasions **R2** is used in the E2-proof of A. If B is an axiom of E2, then $\vdash_{e2} B$. Suppose $\Box B \rightarrow \Box C$ results from $B \rightarrow C$ by **R2**, and suppose $\vdash_{e2_m} B \rightarrow C$. Then by the definition of $e2_{m+1}$ $\Box B \rightarrow \Box C$ is an axiom of $e2_{m+1}$.

Lemma 12. $E2 = \bigcup e2_m$.

Proof immediate from Lemma 11, if we note that $e2_m \subseteq E2$ for all m.

Lemma 13. $E2^n = \bigcup e2^n_m$ $(m \geqslant n)$ *for all* n.

Proof. If $\vdash_{e2^n_m} A$ $(m \geqslant n)$ then $\vdash_{e2_m} \Box^n T \rightarrow A$ by Lemma 10, whence $\vdash_{E2} \Box^n T \rightarrow A$ and so $\vdash_{E2^n} A$ by Lemma 9, so that $e2^n_m \subseteq E2^n$ for all $m \geqslant n$. Conversely, suppose $\vdash_{E2^n} A$. Then $\vdash_{E2} \Box^n T \rightarrow A$ by Lemma 9, whence by Lemma 11 $\vdash_{e2_m} \Box^n T \rightarrow A$ for some m, whence $\vdash_{e2_{m+n}} \Box^n T \rightarrow A$, and so $\vdash_{e2^n_{m+n}} A$ by Lemma 10.

Lemma 14. $E2 \neq e2_m$ *for all* m; $E2^m \neq e2^m_m$ *for all* m.

Again, we delay a proof by means of matrices until the second part of the paper.

Lemma 15. $e2^n_m \neq E2^n$, *for all* $m, n, m \geqslant n$.

Proof. Suppose $e2^n_m = E2^n$, $m \geqslant n$. The case that $m = n$ is ruled out by Lemma 14, so that $m > n$. Then $e2^n_m \subseteq e2^m_m$ (by Lemma 8) $\subseteq E2^m$ (by Lemma 13). Now if $\vdash_{E2^m} A$ then $\vdash_{E2} \Box^m T \rightarrow A$ (by Lemma 9), whence $\vdash_{E2^n}$ $\Box^m T \rightarrow A$, whence $\vdash_{e2^n_m} \Box^m T \rightarrow A$ (by the hypothesis), whence $\vdash_{e2_m}$ $\Box^m T \rightarrow A$, whence $\vdash_{e2^m_m} A$; so that $E2^m \subseteq e2^m_m$, and $E2^m = e2^m_m$, contrary to Lemma 14. This proves the lemma.

Theorem 4. E2 *is not finitely axiomatizable; none of the systems* $E2^n$ (*including* $E2^1 = S2$) *is finitely axiomatizable.*

Proof. That E2 is not finitely axiomatizable follows from Theorem 1, together with Lemmas 6, 12, and 14. The cases of $E2^n$ are covered by the same theorem, together with Lemmas 7, 13, and 15.

II. Our proof of Theorems 2, 3, and 4 is incomplete until we have established Lemmas 4 and 14. To this end, we develop sets of matrices of the form $\mathfrak{M} = \langle M, D, \cup, \cap, -, \mathsf{P}\rangle$, where $\langle M, \cup, \cap, -\rangle$ is a Boolean algebra and D (the set of designated elements of M) is a filter (additive ideal) of M. We intend that $A \rightarrow B$ shall be interpreted as $-x \cup y$, $-A$ as $-x$, and $\Box A$ as $-\mathsf{P}-x$ for $x, y \varepsilon M$. It follows that *A1-A5* will be satisfied by any matrix, and that **R1** is also satisfied (if $x \varepsilon D$ and $-x \cup y \varepsilon D$ then $y \varepsilon D$)[5]. We use the symbols U and 0 for the unit element and null element of the Boolean algebra; it will always be the case that $U \varepsilon D$, $0 \not\varepsilon D$, of course.

Let P^n be the set of the first n positive integers $\{1, \ldots, n\}$, and put $M^n = \mathfrak{P} P^n$, the set of all subsets of P^n. For $x, y \varepsilon P^n$, put $R = \{\langle x,y\rangle : x = y + 1 \vee (x \neq 1 \wedge x = y)\}$. For $A \subseteq P^n$, put $\mathsf{P}A = \{x : (\exists y)(Rxy \wedge y \varepsilon A)\} \cup \{1\}$. For $A \subseteq P^n$, put $D^n = \{A : n \varepsilon A\}$ so that D^n is a filter of M^n. Finally, put $\mathfrak{M}^n = \langle M^n, D^n, \cup, \cap, -, \mathsf{P}\rangle$[6].

For a sentence A, we write $f^{n(A)}$ for the matrix-function corresponding to A in $\mathfrak{M}^n$. Then it is easy to verify that the axioms and rules of E2 are satisfied by $\mathfrak{M}^n$. Indeed, we have that if $\vdash_{E2} A$ then $f^{n(A)} \equiv U$.[7] Further, putting $\mathbf{N}A (A \subseteq P^n) = -\mathbf{P}-A$, we have:

$$\mathbf{N}U = -\mathbf{P}0 = U - \{1\};$$
$$\mathbf{NN}U = \mathbf{N}(U - \{1\}) = -\mathbf{P}\{1\} = U - \{1,2\};$$
$$\mathbf{N}^i U (\underbrace{\mathbf{N} \ldots \mathbf{N}}_{i} U) = U - \{1, \ldots, i\} \quad (i \leqslant n).$$

Theorem 5. $\mathfrak{M}^{n+1}$ *satisfies* E2^n.

Proof. As already observed, if $\vdash_{E2} A$ then $f^{n+1(A)} \equiv U$; since $\vdash_{E2} T$, $f^{n+1(\Box^n T)} \equiv U - \{1, \ldots, n\} = \{n+1\}$. But $\{n+1\} \varepsilon D^{n+1}$, so that all axioms of E2^n are satisfied by $\mathfrak{M}^{n+1}$. Since D^{n+1} is a filter of M^{n+1}, **R1** is also satisfied, which proves the theorem.

Theorem 6. $\mathfrak{M}^{n+1}$ *falsifies* $\Box^{n+1}T$.

Proof. Since $f^{n+1(T)} \equiv U$, $f^{n+1(\Box^{n+1}T)} \equiv 0$ and $0 \notin D^{n+1}$.

Lemma 4 is an immediate consequence of Theorems 5 and 6, since $\vdash_T \Box^{n+1} T$ for any n.

For Lemma 14, we define a different series of matrices. M^n, as before, is the set of all subsets of P^n. Also, the relation R is as defined earlier, but we employ a new possibility operator $\mathbf{P}'$:

$$\mathbf{P}'0 = \{1\};$$
$$\mathbf{P}'U = U;$$
$$\text{for } A \neq 0, A \neq U (A \subseteq P^n),\ \mathbf{P}'A = \{x : (\exists y)(Rxy \wedge y \varepsilon A)\}.$$

We have correspondingly, for $\mathbf{N}' = -\mathbf{P}'-$, that

$$\mathbf{N}'U = U - \{1\};$$
$$\mathbf{N}'0 = 0;$$
$$\text{for } A \neq 0, A \neq U (A \subseteq P^n),\ \mathbf{N}'A = \{x : (\forall y)(Rxy \rightarrow y \varepsilon A)\}.$$

D^n is still $\{A : n \varepsilon A\}$, but we shall also be interested in subsets $D^n_i (i \leqslant n)$, where $D^n_i = \{A : (\forall x)(x \geqslant i \rightarrow x \varepsilon A)\}$. Then $D^n_n = D^n$, and, for each $i \varepsilon P^n$, $D^n_i \subseteq D^n$. Further, D^n_i is always a filter of M^n. We put $\mathfrak{M}'^n = \langle M^n, D^n, \cup, \cap, -, \mathbf{P}' \rangle$, and $\mathfrak{M}'^n_i = \langle M^n, D^n_i, \cup, \cap, -, \mathbf{P}' \rangle$.

We note the following $\mathbf{N}'$-values in $\mathfrak{M}'^n (\mathfrak{M}'^n_i)$:

$$\mathbf{N}'U = U - \{1\};$$
$$\mathbf{N}'\mathbf{N}'U = -\mathbf{P}'\{1\} = U - \{2\};$$
$$\mathbf{N}'\mathbf{N}'\mathbf{N}'U = -\mathbf{P}'\{2\} = U - \{2,3\};$$
$$\mathbf{N}'^i U = U - \{2, \ldots, i\} \quad (2 \leqslant i \leqslant n).$$

As before, T is $p \rightarrow p$; we also put T' as $\Box p \rightarrow p$.

Lemma 16. $\mathfrak{M}'^n_2$ *satisfies* e2.[8]

Proof. It is obvious that if A is a tautology then $f^{n(A)} \equiv U$, so that $A1$-$A3$ are satisfied. (In particular, $f^{n(T)} \equiv U$.) For $A5$, it suffices to test the matrix-function $-A \cup \mathbf{P}'A (A \subseteq P^n)$. If either $A = 0$ or $A = U$, $-A \cup \mathbf{P}'A = U$.

If $A \neq 0$, $A \neq U$, then, if $x \neq 1$ and $x \varepsilon A$, since Rxx, we have $x \varepsilon \mathbf{P}'A$. Hence $-A \cup \mathbf{P}'A \cup \{1\} = U$. Thus either $-A \cup \mathbf{P}'A = U$ or $-A \cup \mathbf{P}'A = U - \{1\}$; in any case, $-A \cup \mathbf{P}'A \,\varepsilon\, D_2^n$, so that $A5$ is satisfied by $\mathfrak{M}'^n_2$. (In particular, either $f^{n(T')} = U$ or $f^{n(T')} = U - \{1\}$ and there is an assignment from $\mathfrak{M}'^n_2$ such that $f^{n(T')} = U - \{1\}$, namely that assigning to p the value $U - \{1\}$.) For $A4$, it suffices to test the matrix-function

$$g = -(\mathbf{N}'A \cap \mathbf{P}'B) \cup \mathbf{P}'(A \cap B).$$

If $A = 0$, $\mathbf{N}'A = 0$, and the left-hand side of g is U, so that $g = U$. If $A = U$, the right-hand side is $\mathbf{P}'B$, so that by Boolean algebra $g = U$. If $B = 0$, the right-hand side is $\mathbf{P}'0$ and the left-hand side $-(\mathbf{N}'A \cap \mathbf{P}'0)$, so that by Boolean algebra $g = U$. If $B = U$, g becomes $-\mathbf{N}'A \cup \mathbf{P}'A$, so that $g = U$ or $g = U - \{1\}$ by reasoning identical with that in the case of $A5$ above. Finally, suppose $A \neq 0$, $A \neq U$, $B \neq 0$, $B \neq U$, and suppose $x \,\varepsilon\, \mathbf{N}'A \cap \mathbf{P}'B$. Then $(\forall y)(Rxy \rightarrow y \,\varepsilon\, A)$, $(\exists y)(Rxy \wedge y \,\varepsilon\, B)$. It follows that $(\exists y)(Rxy \wedge y \,\varepsilon\, A \cap B)$; since $A \cap B \neq 0$, $A \cap B \neq U$, we have $x \,\varepsilon\, \mathbf{P}'(A \cap B)$ and $g = U$. Hence for any assignment either $g = U$ or $g = U - \{1\}$, so that $A4$ is satisfied by $\mathfrak{M}'^n_2$. That **R1** is also satisfied follows from the fact that D_2^n is a filter.

Lemma 17. $\mathfrak{M}'^{n+2}_{m+2}$ *satisfies* $\mathrm{e}2^m_m$, *for* $m \leq n$.

Proof by induction on m up to n. As basis, by Lemma 16 we have $\mathfrak{M}'^{n+2}_2$ satisfies $\mathrm{e}2^0_0 = \mathrm{e}2$. Suppose then that $\mathfrak{M}'^{n+2}_{m+2}$ satisfies $\mathrm{e}2^m_m$ for $m < n$, and consider $\mathfrak{M}'^{n+2}_{m+3}$, $\mathrm{e}2^{m+1}_{m+1}$. The axioms of $\mathrm{e}2^{m+1}_{m+1}$ contain all axioms of $\mathrm{e}2_m$, together with all sentences $\Box A \rightarrow \Box B$ where $\vdash_{\mathrm{e}2_m} A \rightarrow B$ and also $\Box^{m+1} T$. If A is an axiom of $\mathrm{e}2_m$, then $f^{n+2(A)} \varepsilon D^{n+2}_{m+2}$ (by Lemma 7 and the inductive hypothesis), whence $f^{n+2(A)} \varepsilon D^{n+2}_{m+3}$ (by definition of D^n_i), so that A is satisfied by $\mathfrak{M}'^{n+2}_{m+3}$. Suppose $\vdash_{\mathrm{e}2_m} A \rightarrow B$. Then $f^{n+2(A \rightarrow B)} \varepsilon D^{n+2}_{m+2}$. It follows that $f^{n+2(\Box(A \rightarrow B))} \varepsilon D^{n+2}_{m+3}$ (compare the analysis of $\mathbf{N}'$ preceding Lemma 16). But $f^{n+2(\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B))} \varepsilon D_2^{n+2}$, as is shown in the proof of Lemma 16, and so $\varepsilon D^{n+2}_{m+3}$. Since D^{n+2}_{m+3} is closed under detachment, we have $f^{n+2(\Box A \rightarrow \Box B)} \varepsilon D^{n+2}_{m+3}$, and $\Box A \rightarrow \Box B$ is satisfied by $\mathfrak{M}'^{n+2}_{m+3}$. Finally, since $f^{n+2(T)} \equiv U$, either $f^{n+2(\Box^{m+1} T)} \equiv U - \{1\}$ (when $m = 0$) or $f^{n+2(\Box^{m+1} T)} \equiv U - \{2, \ldots, m+1\}$ (when $m \geq 1$). But all values $U - \{1\}$, $U - \{2, \ldots, m+1\}$ $\varepsilon D^{n+2}_{m+3}$, so that $\Box^{m+1} T$ is satisfied by $\mathfrak{M}'^{n+2}_{m+3}$. Since **R1** is also satisfied, $\mathfrak{M}'^{n+2}_{m+3}$ satisfies $\mathrm{e}2^{m+1}_{m+1}$, and the induction is complete.

Theorem 7. $\mathfrak{M}'^{n+2}$ *satisfies* $\mathrm{e}2^n_n$.

Proof immediate from Lemma 17, if we put $m = n$ and note that $\mathfrak{M}'^{n+2}_{n+2} = \mathfrak{M}'^{n+2}$.

Theorem 8. $\mathfrak{M}'^{n+2}$ *falsifies* $\Box^{n+1} T \rightarrow \Box^{n+1} T'$.

Proof. We noted in the proof of Lemma 16 that $f^{n+2(T)} \equiv U$ in $\mathfrak{M}'^{n+2}$, whilst there is an assignment such that $f^{n+2(T')} = U - \{1\}$. For this assignment, $f^{n+2(\Box^{n+1} T)} \equiv U - \{1\}$ (when $n = 0$) or $U - \{2, \ldots n + 1\}$ (when $n \geq 1$), so that $f^{n+2(\Box^{n+1} T)} \varepsilon D^{n+2}$. For the same assignment, however, $f^{n+2(\Box^{n+1} T')} = U - \{2, \ldots, n + 2\} = \{1\}$, and $\{1\} \not\varepsilon D^{n+2}$. This assignment thus falsifies $\Box^{n+1} T \rightarrow \Box^{n+1} T'$.

Lemma 14 follows from Theorems 7 and 8, if we note that $\vdash_{E2}\Box^{n+1}T \to \Box^{n+1}T'$ by $n + 1$ applications of **R2** to $T \to T'$. Thus our proofs of Theorems 2-4 are now complete.

III. *Final Remarks.* It is shown in [3] that E2 is decidable. It therefore follows from Lemma 9 that all the systems $E2^n$ (including S2) are decidable. E2 admits as a derived rule the substitutability of material equivalents, but none of the systems $E2^n$ $(n > 0)$ does; thus for $n \geqslant 1$ $\vdash_{E2^n} \Box^{n-1}T \longleftrightarrow \Box^n T$, but it is not the case that $\vdash_{E2^n} \Box^{n+1}T$ (compare Theorem 6). Also, none of the systems $e2^{\frac{n}{m}}$ admit this rule, since $\vdash_{e2} T \longleftrightarrow T'$ but it is not the case that $\vdash_{e2^{\frac{n}{m}}} \Box^{p+1}T \longleftrightarrow \Box^{p+1}T'$, where $p = \max(m,n)$ (compare Theorem 8). If we *add* the rule (or equivalently **R2**) to the systems $E2^n$ $(n \geqslant 1)$ then the systems all collapse into T. This shows the importance, in defining extensions of systems, of specifying the rules of inference. In a recent paper [1], Åqvist (p. 81, compare (d)) in effect assumes that any extension of S2 will preserve Becker's rule; that this is not so is shown by the system $E2^2$, an extension of S2 containing $\Box^2 T$ but not equal to T.[9]

The system T(D) of [3], which is T modified by replacing *A5* by $\Box A \to -\Box -A$, can be shown not finitely axiomatizable by very simple modifications of the above arguments, and the same holds for T(C) of [3], which is T modified by dropping *A5* altogether. The systems E3, E4, E5 (see [2]) turn out to be finitely axiomatizable (I owe this result to an idea of Dana Scott's). It seems reasonable, with this evidence, to conjecture that, if a modal system has finitely many distinct and irreducible modalities, then it is finitely axiomatizable. The converse does not hold, if only because of e2. More interesting counterexamples are the systems T^n, which result from T (with **R3**) by adding $\Box^n A \to \Box^{n+1}A$. These can readily all be shown to be finitely axiomatizable, although for $n \geqslant 2$ T^n contains infinitely many distinct and irreducible modalities, as is shown in Sugihara [10].

NOTES

1. See also Sobociński [8] for further results.
2. For definitions of these systems, see [2]. Thus an open question of Sobociński [7] is settled in this paper, and the partial results of Åqvist [1], 6 and 6.1, are extended.
3. This theorem is merely a special case of a theorem due to Tarski in 1928; see [11], p. 36, Theorem 20. In fact, the condition that $S_n \subseteq S_{n+1}$ is redundant, as Tarski's theorem shows. It is useful for our purposes, however, since without it the matrix proof that $S_n \neq S$ in particular applications would not go through. Our present formulation is in [11], p. 362, Theorem 25 (Tarski, 1935-36).
4. All modal systems considered are thought of as having $\to$, $-$, and $\Box$ as primitive connectives. Equivalence of these formulations to those in [2] is trivial.
5. See, for example, Stoll [9], Chapter 6 (esp. p. 280).

6. The elements of M^n may be thought of, in the manner perhaps more familiar to some modal logicians, as n-sequences of *1*'s and *0*'s, where a *1* in the i'th position represents that i is a member of the corresponding element. In this light, the operation P can be described in a quite intuitive way.

7. This follows from [3]; indeed, $\langle M^n, \cup, \cap, -, \mathrm{P}\rangle$ are epistemic algebras in the sense of that paper; notice that R is reflexive in $U - \{1\}$.

8. The matrix $\mathfrak{M}'^2_2$ is used in [2] to distinguish S0.5 from S0.9.

9. Thus the question whether *all* systems between S2 and T are Halldén-unreasonable seems still open (compare Åqvist [1], 4.1), though I conjecture that all systems $E2^n$ are in fact Halldén-unreasonable.

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