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ON PREDICATE LETTER FORMULAS WHICH HAVE NO SUBSTITUTION INSTANCES PROVABLE IN A FIRST ORDER LANGUAGE.

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We shall investigate the following question in this discussion. Does there exist an algorithm A which operates on a recursively enumerable formal system S couched in the first order predicate calculus P (say the formulas of *S* are constructed from logical symbols of P with predicate and individual symbols from given finite or infinite lists) such that if S is simple consistent, then A(S) is a satisfiable predicate letter formula which has no substitution instance provable in S? A partial solution is given in the theorem below. The notation used is from [1].

Theorem 1 (Kleene): For every recursively enumerable and simple consistent formal system S, *couched in the first order predicat calculus, there is a satisfiable formula* F *of* P *where* F *has no substitution instance provable in* S *and* F *can be effectively found, given* S.

The following proof is due to S. C. Kleene in [2]. We shall repeat the argument here, since [2] is not readily available.

Because S is recursively enumerable, we can enumerate recursively all the provable formulas of S. From each provable formula of S we can recover the finitely many formulas of P of which it is a substitution instance. Thus we can recursively enumerate the formulas of P which have substitution instances provable in S. Suppose the formulas of P in this enumeration are: $F_0, F_1, F_2, \ldots$ Then

1) F_i is satisfiable (i=0, 1,2, . . .) ,

for if F_i were not satisfiable, then $\neg F_i$ would be valid and hence provable in P by Gödels completeness theorem. So if $F_i{}^*$ is any one of the substitution instances of F_i, which is provable in S, we would have $\neg F_i{}^*$ also provable and thus S is not simple consistent.

Consider the predicate $T_1(x,x,y)$ in [1, p.281] and the formulas K_x in [1, p. 434, Remark 2] for $R(x,y) \equiv T_1(x,x,y)$.

2) $(y)\overline{T}_1(x,x,y) \equiv \overline{(Ey)}T_1(x,x,y) \equiv [K_x$ is unprovable in P$]$
$\equiv [K_x$ is not valid$] \equiv [\neg K_x$ is satisfiable$]$

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We can now go through the enumeration: F_0, F_1, F_2, . . . and examine each F_i to tell whether it is $\neg K_x$ for some x. (This can effectively be done since the number of symbols in $\neg K_x$ is larger than x.) Therefore we get a recursively enumerable class of numbers x, ($\hat{x}$(Ey)R(x,y) with R(x,y) a recursive predicate), consisting of those x's for which $\neg K_x$ is in the enumeration: F_0, F_1, F_2, ... We have shown that R(x,y) can be effectively found given S. For each such x, $\neg K_x$ is satisfiable by 1) and hence by 2) $(y)\overline{T}_1(x,x,y)$. Thus

3) $(Ey)R(x,y) \rightarrow (y)\overline{T}_1(x,x,y)$.

By [1, Thm. IV, p. 281] there is a number f (which can be effectively found from R using the method in the proof of Thm IV) such that

4) $(Ey)R(x,y) \equiv (Ex)T_1(f,x,y)$.

Hence

5) $(\overline{Ey})R(f,y) \equiv (\overline{Ey})T_1(f,f,y) \equiv (y)\overline{T}_1(f,f,y)$.

Suppose (Ey)R(f,y). Then by 3), $(y)\overline{T}_1(f,f,y)$ and hence by 4), $(\overline{Ey})R(f,y)$, contradicting the assumption. Thus

6) $(\overline{Ey})R(f,y)$,

and hence by 5)

7) $(y)\overline{T}_1(f,f,y)$.

Thus by 6), $\neg K_f$ is not in the enumeration: F_0, F_1, F_2, . . . (i.e. no *substitution instance of* $\neg K_f$ *is provable in* S). But by 7) with 2), $\neg K_f$ is satisfiable. Thus $\neg K_f$ is an F for the theorem. (i.e. there is an algorithm A such that if S is simple consistant then A(S) is $\neg K_f$ and $\neg K_f$ is an F for the theorem).

Now notice how A(S) acts if S is not simple consistent. First of all, the set $\hat{x}$(Ey)R(x,y) consists of all of the integers. Hence if f is a number such that $(Ey)R(x,y) \equiv (Ey)T_1(f,x,y)$ we have $(Ey)T_1(f,f,y)$, since $f \in \hat{x}(Ey)R(x,y)$. But his means by 2),

$$[K_f \text{ is provable in P}] \rightarrow K_f \text{ is valid} \rightarrow \neg K_f = A(S) \text{ is not satisfiable.}$$

Consequently if S is not simple consistent then A(S) is not satisfiable. The following theorem is a generalization of this.

Theorem 2. There is no algorithm A(S) *which operates on recursively enumerable formal systems* S *couched in* P, *such that* A(S) *always produces satisfiable predicate letter formulas and if* S *is simple consistent then* A(S) *has no substitution instance provable in* S.

To prove the theorem we construct a sequence of formal systems: S_1, S_2, S_3, . . . , each of which has the properties described in the theorem, but the existence of any algorithm defined on this system having the properties described in the theorem leads necessary to a contradiction.

If Q is a formal system, it is convenient to abbreviate the statements;

F is a formula of Q and F is a provable formula of Q, by F ε Q and $\vdash^{Q}$ F respectively. Should g be a formal object of P, let [g] designate its Gödel number.

Suppose that R represents Robinson's number theoretic formal system in [1, Lemma 18b, 49]. By [1, Thm. 43(b)] there is a number theoretic system R' couched in the same symbols as R except the function symbols for addition, multiplication and the successor function are replaced by predicate symbols (say the successor function is replaced by '(,), and we can find a correspondence θ between R and R' such that;

(i) $F \varepsilon R \rightarrow F^{\theta} \varepsilon R'$
(ii) F(x) ε R, where x occurs free → for all integers n we can find variables $x_1, \ldots, x_n$ such that $(F(n))^{\theta}$ is $\exists x_1 \exists x_2 \ldots \exists x_n('(0,x_1) \& '(x_2,x_3) \& \ldots \& '(x_{n-1},x_n) \& F^{\theta}(x_n))$ where n is the corresponding numeral for n
(iii) $\vdash^{R} F \equiv \vdash^{R'} F^{\theta}$

For $i = 0, 1, 2, \ldots$ we define a recursively enumerable formal system S_i by adding the following formalism to R'.

(a) Individual symbols (numerals): 0, 0', 0'', . . .
(b) Predicate symbol: G( ).
(c) Formation rule: If t is a term then G(t) is a formula
(d) Axioms:

Suppose that $F(x_1, \ldots, x_n) \varepsilon P$ contains only the variables $x_1, \ldots, x_n$ free. If $f = [F(x_1, \ldots, x_n)]$, let $F_f([x_1], \ldots, [x_n])$ designate the formula which results from $F(x_1, \ldots, x_n)$ by replacing every occurrence of x_i with $[x_i]$ $(i=1, \ldots, n)$. Then for each such f we have the axioms:

$I(f): \; G(f) \leftrightarrow F_f([x_1], \ldots, [x_n])$

where f is the numeral corresponding to f. (Notice, since it can be effectively decided whether an integer f is the Gödel number of a formula of P, axioms $I(f)$ can be recursively enumerated.)

Consider now the enumeration predicate $(Ey)T_2(z,x_1,x_2,y)$ in [1, p. 281]. From [1, ex. 2, p. 305] we can find a formula $T(z,x_1,x_2) \varepsilon R$ such that for all natural numbers n,m,p where n,m,p are the corresponding numerals respectively, we have

$$8) \quad (Ey)T_2(n,m,p,y) \equiv \vdash^{R} T(n,m,p) \equiv \vdash^{R'} \exists x_1 \ldots \exists x_n \exists y_1 \ldots \exists y_m \exists z_1 \ldots \exists z_p ('(0,x_1) \& \ldots \& '(x_{n-1},x_n) \& '(0,y_1) \& \ldots \& '(y_{m-1}, y_m) \& '(0,z_1) \& \ldots \& '(z_{p-1}, z_p) \& T^{\theta}(x_n, y_m, z_p))$$

for variables: $x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_p$ having no occurrence in $T^{\theta}(z,x_1,x_2)$.

Suppose that the variables: $x, y_1, \ldots, y_n$ have no occurrence in $T^{\theta}(z,x_1,x_2)$. Then for $n = 1,2,3, \ldots$ we have,

$$\Pi_1(n): \; \exists y_1 \ldots \exists y_n ('(0,y_1) \& \ldots \& '(y_{n-1},y_n) \& T^{\theta}(y_n,x_1,x_2)) \leftrightarrow T^{\theta}(n,x_1,x_2)$$

$\mathrm{II}_2(n): \exists y_1 \ldots \exists y_n ('(0,y_1) \& \ldots \& '(y_{n-1},y_n) \& T^\theta(z,y_n,x_2)) \backsim T^\theta(z,n,x_2)$
$\mathrm{II}_3(n): \exists y_1 \ldots \exists y_n ('(\bar{0},y_1) \& \ldots \& '(y_{n-1},y_n) \& T^\theta(z,x_1,y_n)) \backsim T^\theta(z,x_1,n)$
(n is the numeral corresponding to n),

$\mathrm{III}_i \quad \forall x(T^\theta(i,i,x) \supset G(x))$
(i is the numeral corresponding to i).

Thus for all natural numbers n,m,p where n,m,p are the corresponding numerals respectively, we have by $\mathrm{II}_1(n)$, $\mathrm{II}_2(m)$, $\mathrm{II}_3(p)$ and 8)

(9) $(EY)\, T_2(n,m,p,y) \equiv \overset{S_i}{\vdash} T^\theta(\mathrm{n,m,p})$.

We shall now return to the proof of Theorem 2.

Suppose their exists an algorithm A as described in the theorem. Then the correspondence between S_i and F_i, where $A(S_i) = F_i$, determines a general recursive function $f(i) = [F_i]$. Let g be the Gödel number of $f(i)$. In order to show that S_g is simple consistent, it is necessary to prove the following lemma.

Lemma 1. Suppose $F \in P$ *where* F *contains free only the variables:* $x_1, \ldots, x_n$ *and contains the predicate symbols* $A_1(\ell_1), \ldots, A_k(\ell_k)$. *(*$A_i(\ell_i)$*) is a predicate symbol where the number of attached variables is equal to the natural number* $\ell_i \geq 0$, $i = 1, \ldots, k$.*) Then if* F *is satisfiable we can find number theoretic predicates:* $A_1(\ell_1), \ldots, A_k(\ell_k)$, *for arbitrary natural numbers;* $y_1, \ldots, y_n$ *such that:* $y_1, \ldots, y_n, A_1(\ell_1), \ldots, A_k(\ell_k)$ *satisfy* F.

We may regard F as a logical functional $F(x_1,\ldots,x_n,A_1(\ell_1),\ldots,A_k(\ell_k))$ defined by the truth tables for: $\supset$, &, $\vee$, $\neg$, $\exists$ and $\forall$ with $\{t,f\}$ constituting the range, where $x_1, \ldots, x_n$ vary over the natural numbers and $A_1(\ell_1), \ldots, A_k(\ell_k)$ vary over number theoretic predicates. Thus since F is satisfiable we have

$$F(z_1, \ldots, z_n, A_1(\ell_1), \ldots, A_k(\ell_k)) = t$$

for some natural numbers: $z_1, \ldots, z_n$ and number theoretic predicates: $A_1(\ell_1), \ldots, A_k(\ell_k)$ whose domains are the natural numbers. Of course we make no restriction that $z_i \neq z_j$, $i \neq j$. Now define the following function

$$h_i(x) = \begin{cases} z_i & \text{of } x = y_i \\ y_i & \text{if } x = z_i \\ x & \text{otherwise} \end{cases}$$

Let $A_i^*(\ell_i)$ $(i = 1, \ldots, k)$ be the predicate which results from $A_i(\ell_i)$ by replacing every occurrence of the variables corresponding to: $x_1, \ldots, x_k$ with: $h_1(x_1), \ldots, h_k(x_k)$ Therefore

$$F(y_1, \ldots, y_n, A_1^*(\ell_1), \ldots, A_k^*(\ell_k)) = F(z_1, \ldots, z_n, A_1(\ell_1), \ldots, A_k(\ell_k)) = t$$

and the lemma is proved.

We can show that S_g is simple consistent by finding a model for it. This we do now.

First observe that for any assignment of number theoretic predicates to the predicate symbols of P the axioms $I(f)$, under the intuative interpretation of the logical symbols, allow to define a number theoretic predicate $G(x)$. If we assign only predicates whose domains consist of all the natural numbers to the predicate symbols of P we observe that the domain of G(x) are all Gödel numbers of formulas of P. Also under the intuative interpretation of the successor and enumeration predicate we obviously have a model for axioms: $II_1(n)$, $II_2(n)$, $II_3(n)$ $(n = 1,2,3, \ldots)$. Suppose $F(x_1, \ldots, x_n, A_1, \ldots, A_k) \in P$ where: $A_1, \ldots, A_k$ are all the predicate symbols and only the variables $x_1, \ldots, x_n$ occur free. Suppose also that $[F(x_1, \ldots, x_n, A_1, \ldots, A_k)] = f(g)$. Since by assumption $F(x_1, \ldots, x_n, A_1, \ldots, A_k)$ is satisfiable there are number theoretic predicates: $\mathcal{A}_1, \ldots, \mathcal{A}_k$, by Lemma 1, such that $F([x_1], \ldots, [x_n], \mathcal{A}_1, \ldots, \mathcal{A}_k) = t$. Now assign any number theoretic predicates to the predicate symbols of P except to the predicate symbols: $A_1, \ldots, A_k$ assign: $\mathcal{A}_1, \ldots, \mathcal{A}_k$. We shall interpret $T^{\theta}(z, x_1, x_2)$ of course as the predicate $(EY)T_2(z, x_1, x_2, y)$. Since g is the Gödel number of the function $f(i)$ we have

$$(x)\ ((EY)\ T_2(g,g,f(g),y) \,\&\, x \neq f(g) \rightarrow (EY)\ T_2(g,g,x,y))$$

But under the assignment to the predicate symbols of P we have that $G(f(g))$ is true. Thus

$$(x)\ ((EY)\ T_2(g,g,x,y) \rightarrow G(x))$$

and axiom III_g is satisfied.

Thus by 9) and modus ponens on axiom $III_{\bar{g}}$ we have,

$$\overset{S_g}{\vdash}\ G(f(g))$$

and by $I(f(g))$,

$$\overset{S_g}{\vdash}\ F_{f(g)}([x_1], \ldots, [x_n])\ .$$

where f(g) is the numeral for $f(g)$. But $F_{f(g)}([x_1], \ldots, [x_n])$ is a substitution instance of $F(x_1, \ldots, x_n)$ and we have a contradiction.

REFERENCES

[1] S. C. Kleene, Introduction to Metamathematics, D. Van Nostrand, Co., 1952.

[2] S. C. Kleene, Memorandum on non-satisfiable Formula, June 1955.

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