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ON INDUCTION

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Philosophers who worry about induction generally take for granted that induction is something like deduction, but carried out by means of certain peculiar inductive axioms or inference rules. Of course, deductive logic can be extended in a certain way into a sound and semantically complete logic of probabilities[1]. Nevertheless, such a logic of probabilities cannot justly be called an inductive logic since it is entirely deductive; that is, the so-called inductive inferences (from sample to population and so on) are not provable in it. Moreover, if the inductive inferences were provable in such a logic of probabilities, then, since most of them have innumerable counterinstances, there would be no good sense in which that logic would be sound. All of this suggests that the usual approach made to induction by philosophers is misguided.

A quite different approach was made by R. Carnap[2]; although he constructed a theory which he called inductive logic, his theory is in fact an extension of deductive metamathematics. Moreover, Carnap's inductive logic is an extension of that part of metamathematics which deals with concepts having to do with the interpretation of object language expressions. In other words, Carnap's inductive logic is a branch of semantics.

In this paper, we follow Carnap in dealing with induction semantically; however, the theory of induction which we construct is somewhat different from the one constructed by Carnap. Also, since our theory is both deductive and within semantics, we refrain from calling it either inductive or a logic.

1. SYMBOLS, TERMS, AND FORMULAS

Our object language contains the following symbols:

(1) the logical constants $\sim$('not'), $\rightarrow$('only if'), $\wedge$('and'), $\vee$('or'), $\leftrightarrow$('if and only if'), ι('the'), $\bigwedge$('for any'), $\bigvee$('for some'), and $\mathbf{I}$('is identical with'); we call the first five of these sentential connectives and the next three variable binders;

(2) a denumerable infinity of distinct

(a) individual variables,

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(b) individual constants, and

(c) predicates of any positive number of places.

By { } ('the empty set') and ϵ ('is a member of'), we mean the first individual constant and second 2-place predicate respectively. We understand **I** to be the first 2-place predicate.

We use the symbols '$<$', '$>$' and '{', '}' in the metalanguage to mark the boundaries of non-empty finite sequences and sets respectively. The letter '*m*' will be used as a metalinguistic variable ranging over positive integers. Terms and formulas will be understood as follows:

(1) all variables and individual constants are terms;

(2) for any m-place predicate p and m-term sequence of terms t, $<pt>$ is a formula; and

(3) for any variable v and formulas f and g.

(a) $<\text{ı} vf>$ is a term and

(b) $<\sim f>$, $<f \rightarrow g>$, $<f \wedge g>$, $<f \vee g>$, $<f \leftrightarrow g>$, $<\bigwedge vf>$, and $<\bigvee vf>$ are formulas.

In what follows, we omit sequence marks according to the usual conventions for the omission of parentheses and write two-place predicates between their arguments instead of in front of them. Given terms t and u and a term or formula f, we understand freedom and PStuf (the result of properly substituting t for u in f) as follows:

(1) if $u = f$, then u is free in f and PS$tuf = t$;

(2) if $u \neq f$, then

(a) if f is a variable or individual constant, then u is not free in f and PS$tuf = f$;

(b) for any m-place predicate p and m-term sequence of terms v, if $f = <pv>$, then u is free in f just in case u is free in some member of the range of v and PS$tuf = <p$ the m-term sequence w such that $w(i) =$ PS$tuv(i)$ for any i in the domain of $w>$;

(c) for any sentential connective c and formula g and h,

(1) if $f = <cg>$, then u is free in f just in case u is free in g and PS$tuf = <c$PS$tug>$ and

(2) if $f = <gch>$, then u is free in f just in case u is free in either g or h and PS$tuf = <$PS$tugc$PS$tuh>$; and

(d) for any variable binder b, variable v, and formula g, if $f = <bvg>$, then u is free in f just in case u is free in g and v is not free in u and

(1) if u is not free in f, then PS$tuf = f$;

(2) if u is free in f, then

(a) if v is not free in t, then PS$tuf = <bv$PS$tug>$ and

(b) if v is free in t and w is the first variable not occurring in either f or t, then PS$tuf = <bw$PStuPS$wvg>$.

2. INTERPRETATION AND TRUTH

By an interpreter, we mean a function i of the following kind:

(1) the domain of $i =$ the set of all individual constants, predicates, sentential connectives, and variable binders;

(2) there is a set s such that

(a) for any individual constant c, $i(c)$ is in s;

(b) for any m-place predicate p, $i(p)$ is included in the set of all m-term sequences whose ranges are included in s;

(c) $i(\{\ \})$ = the empty set, $i(\mathrm{I})$ = the set of all 2-term sequences r such that, for some x in s, $r = \langle xx \rangle$, and $i(\epsilon)$ = the set of all r such that, for some x and y in s, y is a set, x is a member of y, and $r = \langle xy \rangle$;

(d) $i(\text{ɿ})$ = the function d such that the domain of d = the set of all subsets of s and, for any r in the domain of d, either there is just one object q in r and $d(r) = q$ or there is not just one object in r and $d(r)$ = the empty set;

(e) $i(\wedge)$ = the function u such that the domain of u = the set of all subsets of s and, for any r in the domain of u, either $r = s$ and $u(r) = 1$ or $r \neq s$ and $u(r) = 0$;

(f) $i(\vee)$ = the function e such that the domain of e = the set of all subsets of s and, for any r in the domain of e, either r is not empty and $e(r) = 1$ or r is empty and $e(r) = 0$; and

(3) for any sentential connective c.

(a) if $c = \sim$, then $i(c)$ = the function whose domain is $\{01\}$ and which assigns $1-t$ to any t in its domain and

(b) if $c \neq \sim$, then $i(c)$ is the function f whose domain is the set of all 2-term sequences whose ranges are included in $\{01\}$ such that, for any t and u in $\{01\}$, $f(\langle tu \rangle)$ = the n such that either $c = \rightarrow$ and n = the smallest member of $\{1, (1-t) + u\}$ or $c = \wedge$ and n = the smallest member of $\{tu\}$ or $c = \vee$ and n = the greatest member of $\{tu\}$ or $c = \leftrightarrow$ and n = (1-the greatest member of $\{tu\}$) + the smallest member of $\{tu\}$.

Given an interpreter i, we understand **U**i (the universe of i) to be the set satisfying (2) above with respect to i. If x is included in **U**i and the empty set is in x, then **Red** ix (the reduction of i to x) is the interpreter j such that

(1) for any individual constant c, either $i(c)$ is in x and $j(c) = i(c)$ or not and $j(c)$ = the empty set and

(2) for any predicate p, $j(p)$ = the set of all s in $i(p)$ such that the range of s is included in x.

If t is a term or a formula, then a is an assigner for t by i just in case a is a function such that

(1) every variable which occurs in t is in the domain of a,

(2) the domain of a is included in the set of all variables, and

(3) the range of a is included in **U**i.

We say that a is a proper assigner for t by i just in case a is an assigner for t by i and the domain of a = the set of all variables which occur in t. Given an assigner for t by i a, variable v, and z in **U**i, $a(\frac{v}{z})$ is the assigner for t by i b such that b is a with the pair v, $a(v)$ removed and the pair v,z added in its place.

Given an assigner for t by i a, we understand **Int** $ia(t)$ (the interpretation with respect to i and a of t) as follows:

(1) if t is a variable, then **Int** ia $(t) = a(t)$;

(2) if t is an individual constant, then **Int** $ia\ (t) = i(t)$;

(3) for any m-place predicate p and m-term sequence of terms u, if $t = \langle pu \rangle$, then **Int** $ia\,(t)$ = the z such that either the m-term sequence s such that, for any j in the domain of s, $s(j)$ = **Int** $ia\ (u(j))$ is in $i(p)$ and $z = 1$ or not and $z = 0$; and

(4) for any formulas f and g, sentential connective c, variable v, and variable binder b,

(a) if $t = \langle cf \rangle$, then **Int** $ia\ (t) = (i(c))($**Int** $ia\ (f))$;

(b) if $t = \langle fcg \rangle$, then **Int** $ia\ (t) = (i(c))(\langle$**Int** $ia\ (f)$**Int** $ia\ (g)\rangle)$;and

(c) if $t = \langle bvf \rangle$, then **Int** $ia\ (t) = (i(b))$(the set of all z in **U**i such that **Int** $ia\binom{v}{z}\ (f) = 1$).

Given a formula f and interpreter i, we say that f is true by i just in case, for any proper assigner for f by i a, **Int** $ia\ (f) = 1$.

3. THE PRINCIPLE OF INDUCTION

Given a formula f, interpreter i, and finite subset of **U**i x such that the empty set is in x, we understand **T**$xi(f)$ (the degree of truth in x by i of f) to be the z such that, for some k and l,

(1) k = the number of members of the set of all proper assigners for f by **Red** ix a such that **Int (Red** ix) $a\ (f) = 1$,

(2) l = the number of members of the set of all proper assigners for f by **Red** ix, and

(3) z = the fraction k/l.

This definition[3] embodies the principle that, if a formula f holds among the members of a finite set x of objects, then, on the basis of what we know from x, f holds in general. In other words, our definition of degree of truth can be understood as an exact and strengthened formulation of the principle of induction. In what follows, we call the definition the principle of induction. A consequence of this identification is the disappearance of the problem of justifying induction; just like any other definition, the principle of induction is simply a convention which establishes how we shall use certain expressions. We are, however, left with the problem of how far we can go towards identifying maximal degree of truth with respect to a finite sample x with truth before this identification can lead us into error. The answer is, of course, that this identification almost always can lead us into error and so that it is usually advisable to refrain from making it. We can act on the basis of maximal degree of truth with respect to a finite sample without trying to delude ourselves into believing that it is always the same as truth.

4. THE PRINCIPLE OF INDUCTION AND MILL'S CANONS

In his *System of Logic*, J. S. Mill listed five statements which are to guide us in our search for laws among the facts of experience. These, the famous five canons of induction, were expressed by Mill as follows[4]:

(1) If two or more instances of the phenomenon under investigation have only one circumstance in common, the circumstance in which alone all the instances agree is the cause (or effect) of the given phenomenon.

(2) If an instance in which the phenomenon under investigation occurs and an instance in which it does not occur have every circumstance in common save one, that one occurring only in the former, the circumstance in which alone the two instances differ is the effect, or the cause, or an indispensable part of the cause, of the phenomenon.

(3) If two or more instances in which the phenomenon occurs have only one circumstance in common, while two or more instances in which it does not occur have nothing in common save the absence of that circumstance, the circumstance in which alone the two sets of instances differ is the effect, or the cause, or an indispensable part of the cause, of the phenomenon.

(4) Subduct from any phenomenon such part as is known by previous inductions to be the effect of certain antecedents, and the residue of the phenomenon is the effect of the remaining antecedents.

(5) Whatever phenomenon varies in any manner whenever another phenomenon varies in some particular manner is either a cause or an effect of that phenomenon, or is connected with it through some fact of causation.

Mill believed that the canons embodied five methods (those of agreement, difference, agreement and difference, residues, and concomitant variations respectively) of finding laws of nature.

If we understand Mill's phenomena, circumstances, and antecedents to be sets and his causes to be sets memberships in which are necessary conditions, then, with the aid of the principle of induction, we can prove a semantic analogue to the conjunction of the five canons. Suppose that i is an interpreter, s is a finite subset of $\mathsf{U}i$ of which the empty set is a member, the set of all members of sets in $\mathsf{U}i$ is included in $\mathsf{U}i$, a through e are distinct individual constants, and x and y are distinct variables. Obviously, $\mathsf{T}si(<\bigvee x\, x\in a \vee a\mathbf{I}\{\ \}> \wedge \bigwedge y<y\in e \rightarrow \bigvee x\, x\in y \vee y\mathbf{I}\{\ \}> \wedge b\in e \wedge \bigwedge x<x\in a \rightarrow x\in b>) = 1$ when one of the five following formulas is true by **Red** is:

(1) $<\bigvee x\, x\in a \vee a\mathbf{I}\{\ \}> \wedge \bigwedge y<y\in e \rightarrow \bigvee x\, x\in y \vee y\mathbf{I}\{\ \}> \wedge b\in e \wedge \bigwedge x<x\in a \rightarrow x\in b> \wedge \bigwedge y<y\in e \wedge \bigwedge x<x\in a \rightarrow x\in y> \rightarrow y\mathbf{I}b>$

(2) $<\bigvee x\, x\in a \vee a\mathbf{I}\{\ \}> \wedge \bigwedge y<y\in e \rightarrow \bigvee x\, x\in y \vee y\mathbf{I}\{\ \}> \wedge b\in e \wedge \bigwedge x<x\in a \rightarrow x\in b> \wedge \bigwedge x<\sim x\in a \rightarrow \sim x\in b> \wedge \bigwedge y<y\in e \wedge \sim y\mathbf{I}b \rightarrow <\bigwedge x<x\in a \rightarrow x\in y> \leftrightarrow \bigwedge x<\sim x\in a \rightarrow x\in y\gg$

(3) $<\bigvee x\, x\in a \vee a\mathbf{I}\{\ \}> \wedge \bigwedge y<y\in e \rightarrow \bigvee x\, x\in y \vee y\mathbf{I}\{\ \}> \wedge b\in e \wedge \bigwedge x<x\in a \rightarrow x\in b> \wedge \bigwedge y<y\in e \wedge \bigwedge x<x\in a \rightarrow x\in y> \rightarrow y\mathbf{I}b> \wedge \bigwedge y<y\in e \wedge \bigwedge x<\sim x\in a \rightarrow x\in y> \rightarrow \bigwedge x<x\in y \leftrightarrow \sim x\in b\gg$

(4) $<\bigvee x\, x\in a \vee aI\{\ \}> \wedge \bigwedge y<y\in e \rightarrow \bigvee x\, x\in y \vee yI\{\ \}> \wedge b\in e \wedge c\in e \wedge d\in e \wedge \bigwedge y<y\in e \rightarrow <\bigwedge x<x\in c \rightarrow x\in y> \leftrightarrow \bigwedge x<x\in y \leftrightarrow x\in d\gg \wedge \bigwedge y<y\in e \rightarrow <\bigwedge x<x\in c \wedge x\in a \rightarrow x\in y> \leftrightarrow \bigwedge x<x\in y \leftrightarrow x\in d \wedge x\in b\gg \wedge \bigwedge y<y\in e \rightarrow <\bigwedge x<x\in a \rightarrow x\in y> \leftrightarrow \bigwedge x<x\in y \leftrightarrow x\in c\gg$

(5) $<\bigvee x\, x\in a \vee aI\{\,\}> \wedge \bigwedge y<y\in e \rightarrow \bigvee x\, x\in y \vee yI\{\,\}> \wedge b\in e \wedge \bigwedge x<x\in a \leftrightarrow x\in b>$

The formulas (1), (2), (3), and (5) are restricted set-theoretical analogues to the conditions of the first, second, third, and fifth canons respectively while (4) is a restricted and corrected set-theoretical analogue to the condition of the fourth canon. $< \bigvee x\ x \epsilon a \vee a \mathbf{I} \{\ \} > \wedge \bigwedge y < y \epsilon e \rightarrow \bigvee x\ x \epsilon y \vee y \mathbf{I} \{\ \} > \wedge b \epsilon e \wedge \bigwedge x < x \epsilon a \rightarrow x \epsilon b >$ is a set-theoretical analogue to the assertion that b is a cause among the circumstances e of the phenomenon a.

Notice that, if one of (1) through (4) is true by **Red** *is*, then it also follows that $\mathsf{T}si(b \mathbf{I} \iota y \ll \bigvee x\ x \epsilon a \vee a \mathbf{I} \{\ \} > \wedge \bigwedge y < y \epsilon e \rightarrow \bigvee x\ x \epsilon y \vee y \mathbf{I} \{\ \} > \wedge y \epsilon e \wedge \bigwedge x < x \epsilon a \rightarrow x \epsilon y \gg) = 1$. Since the just mentioned formula is a set-theoretical analogue to the assertion that b is *the* cause among the circumstances e of the phenomenon a, this result fits Mill's words more closely than the previous one with respect to the first four canons.

5. THE PRINCIPLE OF INDUCTION AND INDUCTIVE INFERENCES

The principle of induction can also be used for the proof of semantic analogues to certain of the so-called inductive inferences. Let i, s, a, b, c, x, and y be as in the previous section, t be a term, and f be a formula whose only free variable is x. It can then be shown that

(1) if, for any finite subset of $\mathsf{U}i\ r$ such that the empty set is in r, f is true by **Red** *ir*, then $\mathsf{T}si(f) = 1$;

(2) if f is true by **Red** *is*, then $\mathsf{T}si(f) = 1$;

(3) if f is true by **Red** *is*, then $\mathsf{T}si(\bigwedge xf) = 1$;

(4) if f is true by **Red** *is*, then $\mathsf{T}si(\mathsf{PS}\,txf) = 1$; and

(5) if $\bigwedge x < x \epsilon c \rightarrow < \bigvee y\ y \epsilon x \vee x \mathbf{I} \{\ \} > \wedge < a \epsilon x \leftrightarrow b \epsilon x \gg$ is true by **Red** *is*, then $\mathsf{T}si(\bigwedge x < x \epsilon c \wedge < \bigvee y\ y \epsilon x \vee x \mathbf{I} \{\ \} > \wedge a \epsilon x \rightarrow b \epsilon x >) = 1$.

(1) through (5) represent the inferences from population to sample, from sample to population, from sample to universal generalization, from sample to instance, and by analogy respectively.

NOTES

1. One such extension was discussed by the author in 'On probability logics' (*Notre Dame Journal of Formal Logic*, vol. 6, 1965).

2. The curious reader is referred to Carnap's books *The Logical Foundations of Probability* (Chicago, 1950) and *The Continuum of Inductive Methods* (Chicago, 1952).

3. The definition is an adaption to standard semantics of the definition of degree of truth given by the author in 'Contributions to syntax, semantics, and the philosophy of science' (*Notre Dame Journal of Formal Logic*, vol. 5, 1964).

4. The canons are quoted from E. Nagel's edition *John Stuart Mill's Philosophy of Science* (New York, 1950).

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