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STRONG REDUCIBILITY ON HYPERSIMPLE SETS[1]

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1. *Introduction.* In [5], Yates noted that all those hypersimple sets constructed by Dekker in [2] are nonhyperhypersimple sets with retraceable complements. The main purpose of this note is to draw attention to an easy construction, modifying very slightly the usual "cylinder" mapping, which (in view of the second theorem of [4]) leads to a proof that there are hypersimple, nonhyperhypersimple sets whose complements are not regressive in the sense of [3]. Actually, the mere existence of sets meeting this description is very easily shown on the basis of propositions in [3], [4], and [5]; what we wish to emphasize is that our construction provides an effective procedure for passing from a given hyperhypersimple set β to a hypersimple, nonhyperhypersimple, noncoregressive set α such that $\alpha \equiv_m \beta$.

For convenience, we use the following abbreviations: '**HS**', for the class of hypersimple sets; '**HHS**', for the class of hyperhypersimple sets; '$\bar{\mathbf{R}}$', for the class of sets with regressive complement. We denote the set of all natural numbers by '**N**'.

2. *Immune Cylindrification.* Let $\tau(x,y)$ be the usual recursive pairing function:

$$\tau(x,y) = x + \frac{1}{2}(x+y)(x+y+1);$$

let $\tau_1(x)$, $\tau_2(x)$ be its associated "unpairing" functions... thus, $x = \tau(\tau_1(x), \tau_2(x))$, for all x. We use the words "isolated set" as is customary: the number set β is isolated iff β is either finite or immune.

Lemma 1. Let $\{\omega_{\xi(n)}\}$ be a recursive sequence of r.e. sets (thus, the indexing function ξ is assumed recursive). Let α be an immune set. Let $\beta =_{df} \tau(\{0\} \otimes \alpha) \cup \bigcup_{n>0} \tau(\{n\} \otimes (\alpha - \omega_{\xi(n-1)}))$. Then β is immune $\Leftrightarrow$ $(\forall k \epsilon \alpha)$ $(\{y | k \notin \omega_{\xi(y)}\}$ is isolated).

(We use '$\otimes$' to denote the Cartesian product operation.)

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Proof. (i) Suppose β is immune. To argue by *reductio ad absurdum*, assume that k_0 is in α and such that $\{y|k_0 \notin \omega_{\xi(y)}\}$ is nonisolated. Let γ be an infinite r.e. subset of $\{y|k_0 \notin \omega_{\xi(y)}\}$. Then, clearly, $\{\tau(y,k_0)|\, y>0 \,\&\, y-1\epsilon\gamma\}$ is an infinite r.e. subset of β: contradiction. Hence, $(\forall k \in \alpha)(\{y|k \notin \omega_{\xi(y)}\}$ is isolated).

(ii) Suppose that $k\epsilon\alpha \Longrightarrow \{y|k\notin\omega_{\xi(y)}\}$ is isolated. By way of obtaining a contradiction, assume that β is not immune. Since α is infinite, β is infinite; hence, β has an infinite r.e. subset, say, γ. Consider the set $\tau_2(\gamma)$. Clearly $\tau_2(\gamma) \subseteq \alpha$; hence, since α is immune, $\tau_2(\gamma)$, being r.e., is finite. Let $\tau_2(\gamma) = \{k_0,\ldots,k_r\}$. Thus $\gamma = \bigcup_{i=0}^{r}(\gamma \cap \tau(N \otimes \{k_i\}))$; each set $\gamma \cap \tau(\mathsf{N} \otimes \{k_i\})$ is, obviously, r.e. If all $\gamma \cap \tau(\mathsf{N} \otimes \{k_i\})$ were finite, γ would be finite; hence, there is an i_0, $0 \leqq i_0 \leqq r$, such that $\gamma \cap \tau(\mathsf{N} \otimes \{k_{i_0}\})$ is infinite. But this clearly implies that $\{y|k_{i_0}\notin\omega_{\xi(y)}\}$ has an infinite r.e. subset: contradiction. It follows that β is immune. This finishes the proof of Lemma 1.

Definition A. Let α be a set of natural numbers; by the immune cylindrification of α (denoted by 'IC(α)') *we mean the set*

$$\tau(\{0\} \otimes \alpha) \cup \bigcup_{n>0} \tau(\{n\} \otimes (\alpha - \{0,\ldots,n-1\})).$$

Lemma 2 ([5, Theorem 5]). *An r.e. set β is hyperhypersimple $\Longleftrightarrow \overline{\beta}$ is infinite and there is no recursive sequence $\{\omega_{\xi(n)}\}$ of pairwisedisjoint r.e. sets (finite or infinite) such that $(\forall n)(\omega_{\xi(n)} \cap \overline{\beta} \neq \phi)$.*

Lemma 3 ([6]). *If α,β are simple sets, $\alpha \leqq_m \beta$, and β is hypersimple, then α is also hypersimple.*

Theorem 1. There exists an effective operation Φ: $F \to F$ (F the class of all r.e. sets) such that

(*i*) *Φ is 1-1 and distributes through $\cup$ and $\cap$;*
(*ii*) *$\alpha \in F \Longrightarrow \Phi(\alpha) \equiv_m \alpha$; and*
(*iii*) *Φ(HS) $\subseteq$ HS and Φ(HHS) $\subseteq$ HS - HHS.*

Proof. Let φ be a recursive function such that

$$\omega_{\varphi(n)} = \begin{Bmatrix} \bigcup_{k} \tau(\{k\} \otimes (\omega_n \cup \{j|j<k\})), & if\ \omega_n \neq \phi \\ \phi, & if\ \omega_n = \phi \end{Bmatrix}.$$

Define Φ by: $\Phi(\omega_n) = \omega_{\varphi(n)}$. Thus Φ is an effective operation on F; and assertion (i) is clear from the definition of Φ. As for (ii): It is plain that $\omega_n \leqq_m \Phi(\omega_n)$; indeed, the reducibility in that direction is one-one. If $\omega_n = \phi$, then $\Phi(\omega_n) = \phi$ also, whence $\Phi(\omega_n) \leqq_m \omega_n$. So suppose $\omega_n \neq \phi$; and let b be some element of ω_n. Then a many-one reduction ψ of $\Phi(\omega_n)$ to ω_n is obtained by setting $\psi = \tau_2$ on $\tau(\{0\} \otimes \mathsf{N})$, and, on $\tau(\{n+1) \otimes \mathsf{N}\}$, letting

$$\psi(x) = \begin{Bmatrix} \tau_2(x) & if\ x \notin \{\tau(n+1,0),\ldots,\tau(n+1,n)\}, \\ b & otherwise \end{Bmatrix}.$$

To prove (iii), we first note that if ω_n is simple, then $\Phi(\omega_n)$ is also simple; this follows from Lemma 1, since $\overline{\Phi(\omega_n)} = \mathrm{IC}(\overline{\omega_n})$ and the array $\omega_{\xi(0)} =$

$\{0\}, \ldots, \omega_{\xi(n+1)} = \{0,1,\ldots,n+1\}, \ldots$ satisfies the hypothesis of Lemma 1. Hence, by Lemma 3, if ω_n is hypersimple then also $\Phi(\omega_n)$ is hypersimple; and, finally, by Lemma 2, if ω_n is hyperhypersimple then $\Phi(\omega_n)$ is not hyperhypersimple (for note that $\tau(\{n\} \otimes \mathbf{N}) \cap \overline{\Phi(\omega_n)} \neq \phi$ holds for each n). The proof of Theorem 1 is complete.

Throughout the remainder of the paper, 'Φ' denotes the effective operation so denoted in the proof of Theorem 1.

Lemma 4 (1, *Corollary 4.1*]). *Suppose* $\omega_e \in \mathbf{HHS}$ *and* $\omega_f \in \overline{\mathbf{H}}\mathbf{S} \cap \overline{\mathbf{R}}$. *Then* ω_e *and* ω_f *are many-one incomparable.*

From Theorem 1 and Lemma 4 we at once conclude the following result:

Theorem 2. $\Phi(\mathbf{HHS}) \cap (\mathbf{HS} \cap \overline{\mathbf{R}}) = \phi$.

3. *Point-Decomposable Sets.* In this section, we introduce one more-or-less-natural generalization of the notion of regressive set; and we present a theorem showing that Lemma 4 fails by a wide margin with respect to this particular generalized regression concept.

Definition B. An infinite set α *of natural numbers is point-decomposable iff there is a recursive sequence* $\{\omega_{\xi(n)}\}$ *of pairwise-disjoint r.e. sets such that* (*i*) $\alpha \subseteq \bigcup_n \omega_{\xi(n)}$, *and* (*ii*) $(\forall n)(|\omega_{\xi(n)} \cap \alpha| = 1)$.

Remark. Suppose that we modify the definition of *regressive set* given in [3] by allowing a binary r.e. relation in place of a partial recursive function, thus:

A set α of natural numbers shall be said to be *generalized regressive* $\Longleftrightarrow_{df}$ α is finite or else α is infinite and there exist a nonrepetitive listing $a_0, a_1, \ldots$ of the membership of α and a binary r.e. relation R such that $(a_0, a_0) \in R$ and $(\forall n)((a_{n+1}, a_n) \in R)$ and, finally, $(\forall k > 0)$((If C is any R-chain from a_k to a_0, then C has length k).

It is a straightforward matter to verify that the infinite generalized regressive sets coincide with the point-decomposable sets.

We shall denote by '$\mathbf{PD}$' the class of sets of natural numbers having point-decomposable complements.

The next two lemmas are quite straightforward, and the reader should experience little difficulty in proving them for himself. The first is a very slight generalization of [7, Note II], while the second is really nothing more than a special case of [5, Theorem 5].

Lemma 5. Let β *be a maximal set, and suppose* $\alpha \leqq_m \beta$. *Then* α *nonrecursive* $\Longrightarrow \alpha \equiv_m \beta$.

Lemma 6. Let φ *be the recursive function defined in the proof of Theorem 1, so that* $\omega_{\varphi(n)} = \Phi(\omega_n)$ *for all n. There exists a recursive function* $\psi(x,y)$ *such that* (*i*) ω_n *noncofinite* $\Longrightarrow (\forall y)(\omega_{\psi(n,y)}$ *is a finite subset of* $\tau(\{y\} \otimes N)$ *such that* $|\omega_{\psi(n,y)} \cap \overline{\omega_{\varphi(n)}}| = 1$), *while* (*ii*) ω_n *cofinite* $\Longrightarrow \omega_{\psi(n,y)} = \tau(\{y\} \otimes \mathbf{N})$ *for all but finitely many y and, for the remaining finitely many y,* $\omega_{\psi(n,y)}$ *is a finite subset of* $\tau(\{y\} \otimes \mathbf{N})$ *such that* $|\omega_{\psi(n,y)} \cap \overline{\omega_{\varphi(n)}}| = 1$.

Theorem 3. There exists a recursive function ζ, *taking indices of sets in* **HS** *to indices of sets in* **HS** $\cap$ $\overline{\mathbf{PD}}$, *such that if* ω_n *is maximal then* $\omega_{\zeta(n)} \equiv_m \omega_n$.

Proof. We know from Theorem 1 that $\alpha \equiv_m \Phi(\alpha)$, for all α, and from Lemma 3 that $\Phi(\alpha)$ is in **HS** if $\alpha \in$ **HS**. Letting $\psi(x,y)$ be a function as described in Lemma 6, we can effectively find, given an index e of α, a one-one recursive function f which generates $\Phi(\alpha) \cup \bigcup_n \omega_{\psi(e,n)}$. Clearly, $f^{-1}(\Phi(\alpha))$ is one-one reducible to $\Phi(\alpha)$ (and hence $f^{-1}(\Phi(\alpha)) \leqq_m \alpha$), and is a member of $\overline{\mathbf{PD}}$ provided $\overline{\Phi(\alpha)}$ is infinite. By Lemma 3, $f^{-1}(\Phi(\alpha))$ is in **HS** if $\alpha \in$ **HS**. If α is maximal, then, by Lemma 5, $f^{-1}(\Phi(\alpha)) \leqq_m \Phi(\alpha) \equiv_m \alpha$ implies $f^{-1}(\Phi(\alpha)) \equiv_m \alpha$. Therefore, letting ζ be a recursive function such that

$$\omega_{\zeta(n)} = f_n^{-1}(\Phi(\omega_n))$$

(with f_n determined effectively from n, as above, for each n), we have the theorem.[2]

4. *Two Theorems Concerning the Operator* Φ. In this section of the note we exhibit a couple of results concerning properties preserved under application of Φ.[3] The second of the two theorems is due, both in statement and in proof, to K. I. Appel.

Theorem 4. $\Phi(\mathbf{HS} \cap \overline{\mathbf{PD}}) \subseteq \mathbf{HS} \cap \overline{\mathbf{PD}}$.

Proof. Suppose $\omega_e \in \overline{\mathbf{PD}}$; let $\{\omega_{\xi(n)}\}$ witness the fact. Since ω_e is r.e., we may suppose that $\bigcup \omega_{\xi(n)} = \mathbf{N}$; hence, there is a recursive function π such that, for every number k, $\pi(k)$ = the uniquely determined number $\xi(m)$ such that $k \in \omega_{\xi(m)}$. We begin by defining a certain recursive sequence of pairwise-disjoint r.e. sets, as follows:

$$\omega_{\zeta(0)} = \tau(\{0\} \otimes \omega_{\xi(0)});$$

$$\omega_{\zeta(1)} = \begin{Bmatrix} \tau(\{1\} \otimes \omega_{\xi(0)}), & \text{if } \xi(0) \neq \pi(0), \\ \tau(\{1\} \otimes \omega_{\xi(1)}), & \text{otherwise} \end{Bmatrix};$$

$$\omega_{\zeta(2)} = \tau(\{0\} \otimes \omega_{\xi(1)});$$

$$\omega_{\zeta(3)} = \begin{Bmatrix} \tau(\{1\} \otimes \omega_{\xi(1)}), & \text{if } \xi(0) \neq \pi(0) \;\&\; \xi(1) \neq \pi(0), \\ \tau(\{1\} \otimes \omega_{\xi(2)}), & \text{otherwise} \end{Bmatrix};$$

2. It follows, of course, from the foregoing argument that any element of **HS** is *comparable with* an element of **HS** $\cap$ $\overline{\mathbf{PD}}$.

3. A third property of Φ, which we will not bother to prove here, is this: If β is a simple set, then $\Phi(\beta)$ is a simple set, many-one equivalent to β, whose nonhyperhypersimplicity is witnessed by a recursive sequence $\{\omega_{\xi(n)}\}$ such that for every positive integer k there are infinitely many n for which $\omega_{\xi(n)}$ is a k-element subset of $\overline{\Phi(\beta)}$. Whether such a sequence must exist relative to β itself, if β is not in **HHS**, is something we do not at present know, although it is easy to establish the affirmative answer for the case in which only $k = 1$ is considered.

$$\omega_{\zeta(4)} = \begin{cases} \tau(\{2\} \otimes \omega_{\xi(0)}), & \text{if } \xi(0) \neq \pi(0) \;\&\; \xi(0) \neq \pi(1), \\ \tau(\{2\} \otimes \omega_{\xi(1)}), & \text{if } (\xi(0) = \pi(0) \vee \xi(0) \\ & \quad = \pi(1)) \;\&\; \xi(1) \neq \pi(0) \;\&\; \xi(1) \\ & \quad \neq \pi(1) \\ \tau(\{2\} \otimes \omega_{\xi(2)}), & \text{otherwise} \end{cases} ;$$

and so on.
In general, a set $\tau(\{n\} \otimes \omega_{\xi(m)})$ fails to be listed as $\omega_{\zeta(k)}$ for some k if and only if some of the numbers less than n belong to $\omega_{\xi(m)}$, say, $r_0, \ldots, r_t$; we say in such a case that $\tau(\{n\} \otimes \omega_{\xi(m)})$ *is placed on the waiting list.* The criterion for removing a set $\tau(\{n\} \otimes \omega_{\xi(m)})$ from the waiting list is that all of $r_0, \ldots, r_t$ are eventually listed in ω_e (which we enumerate bit by bit as we go along). We can form a recursive sequence consisting of the terms of $\{\omega_{\zeta(k)}\}$ together with all sets $\tau(\{n\} \otimes \omega_{\xi(m)})$ which *are placed on the waiting list but eventually are removed from it*; it is easy to see that such a sequence witnesses the point-decomposability of $\overline{\Phi(\omega_e)}$, which completes the proof of Theorem 4.

Theorem 5 (Appel). *If α is regressive, so is $IC(\alpha)$.*

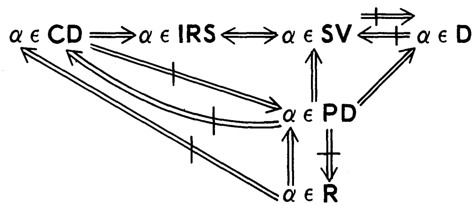
Proof. Let g be a partial recursive function which regresses α in the sense of [3]. Define a second partial recursive function p, as follows:

$$p(\tau(x,y)) = \begin{cases} \tau(x-1,y), & \text{if } x \neq 0; \\ \tau(g(y),g(y)), & \text{if } x = 0, g \text{ is defined} \\ & \quad \text{on } y, \text{ and } g(y) \neq y; \\ \tau(0,y), & \text{if } x = 0 \text{ and } g(y) = y \end{cases}$$

($p(\tau(x,y)$ is undefined in all remaining cases.) Then it is not difficult to see that p regresses $\mathrm{IC}(\alpha)$ relative to the listing $\tau(0,a_0), \ldots, \tau(a_0,a_0)$, $\tau(0,a_1), \ldots, \tau(a_1,a_1), \ldots$, where $a_0, a_1, a_2, \ldots$ is the listing of α relative to which α is regressed by g. This completes the proof of Theorem 5.

Corollary. $\Phi(\mathbf{HS} \cap \overline{\mathbf{R}}) \subset \mathbf{HS} \cap \overline{\mathbf{R}}$.

5. *Concluding Remarks.* Let '**R**' denote the class of infinite regressive sets, '**PD**' the class of point-decomposable sets, '**D**' the class of decomposable sets (i.e., sets α such that, for some i and j, we have $\omega_i \cap \omega_j = \phi$, $\alpha \subseteq \omega_i \cup \omega_j$, and $\alpha \cap \omega_i$, $\alpha \cap \omega_j$ are both infinite), '**IRS**' the class of sets α such that α has a subset $\beta \in \mathbf{R}$, '**SV**' the class of ("sequentially vulnerable") sets α such that there is a recursive sequence $\{\omega_{\xi(n)}\}$ of pairwise-disjoint r.e. sets satisfying $\omega_{\xi(n)} \cap \alpha \neq \phi$ for all n, and, finally, '**CD**' the class of **SV** sets α for which the sequence $\{\omega_{\xi(n)}\}$ can be taken to be a *canonical array* (i.e., the $\omega_{\xi(n)}$ are finite and the cardinality of $\omega_{\xi(n)}$ is a recursive function of n). We exhibit below a complete chart of the relations between these classes (where no arrow or negated arrow appears, one follows, from those which do appear).



Most of these implications and non-implications are very easily deducible from results in [1] and [5]; in particular, Yates proved the result $\alpha \in \mathsf{IRS} \Longleftrightarrow \alpha \in \mathsf{SV}$ explicitly in [5]. The fact that $\alpha \in \mathsf{SV} \not\Longrightarrow \alpha \in \mathsf{D}$ will be proved, in a stronger form, in another paper. The fact that $\alpha \in \mathsf{CD} \not\Longrightarrow \alpha \in \mathsf{PD}$ merely requires a little juggling with non-**SV** sets and number pairs $(2k, 2k+1)$.

We remark, finally, that (at least) two further questions interest us in connection with sections 3 and 4:

(a) What is the extent, within the class of simple sets, of the class of sets belonging to $\overline{\mathsf{PD}}$? (We do not even know if there are any simple sets not in **HHS** which do *not* belong to this class, but conjecture strongly that there are.)

(b) Is the hypothesis of r.e. complement really necessary in Theorem 4? (Presumably, it is.)

Since the completion of this paper, P. R. Young has obtained the following result in connection with question (a): *there exists an element of* **HS-HHS** *whose complement is not point-decomposable.*

REFERENCES

[1] K. I. Appel and T. G. McLaughlin, *On properties of regressive sets*, Trans. Amer. Math. Soc., 115 (1965), 83-93.

[2] J. C. E. Dekker, *A theorem on hypersimple sets*, Proc. Amer. Math. Soc., 5(1954), 791-796.

[3] J. C. E. Dekker, *Infinite series of isols*, Proc. Symposia in Pure Math., Vol. 5 (1962), 77-96.

[4] R. M. Friedberg, *Three theorems on recursive enumeration*, J. Symb. Logic, 23 (1958), 309-316.

[5] C. E. M. Yates, *Recursively enumerable sets and retracing functions*, Zeitschr, f. Math. Logik und Grundl. d. Math., 8 (1962), 331-345.

[6] P. R. Young, *On the structure of recursively enumerable sets*, Doctoral Dissertation, M.I.T., 1963.

[7] P. R. Young, *Notes on the structure of r.e. sets*, Notices Amer. Math. Soc., 63T-368 (Oct. 1963).

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