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THE ARITHMETIC OF THE TERM-RELATION NUMBER THEORY

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1. *Purpose*. In a previous paper [1], as a result of the formalization of the concept of internal relation, a term-relation number theory was introduced. The former work showed that term-relation numbers are either terms or relations obtained by following determined formation rules and by imposing certain postulates. Addition and multiplication of terms on the one hand, and relations on the other, were defined, and the following properties proved: associativity, commutativity, and distributivity of addition and multiplication; the existence of identity elements for addition $(0,\overline{0})$; the nonexistence of identity elements for multiplication; and the invalidity of the well-ordering principle for a concept of order similar to the one usually defined for natural numbers. The present paper will provide further definitions and examine further properties of term-relation numbers. These will include: definition of negative numbers; study of rings of term-relation numbers as partially ordered sets, leading to the characterization of such rings as modular lattices; definition of prime numbers; and study of divisibility and factorization. The paper will end with a question about the universal relevance of Weierstrass' final theorem of arithmetic.

2. *Terminology and notation*. The set of all terms T^∞, defined in [1], contains as a proper subset the set of all terms without proper components T^* ($T^* = \{0,1,2,\ldots\}$). The set of all relations R^∞, also defined in [1], contains as a proper subset the set of all relations without proper components R^* ($R^* = \{\overline{0},\overline{1},\overline{2},\ldots\}$). By "final component" of a term or relation, we mean a component without proper components, i.e., a component belonging either to T^* or to R^*. Every term or relation can be analyzed and expressed in its final components. Let us now introduce the set $T' = \{0,\pm 1,\pm 2,\ldots\}$, which can be obtained by an extension of T^* into a system of integers. Similarly, we obtain $R' = \{\overline{0},\pm\overline{1},\pm\overline{2},\ldots\}$ as an analogous extension of R^*. By "s.f.r," we mean the phrase "similarly for relations." Obviously, terms from T' (like terms from T^*) do not have proper components. S.f.r.

3. *Negative term-relation numbers*. If we let terms and relations have final components from T' and R', instead of from T^* and R^* only, we have

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extended T^{∞} and R^{∞} into two new sets, $T^{\infty\prime}$ and $R^{\infty\prime}$ respectively. Then, for every term x of $T^{\infty\prime}$ there exists in $T^{\infty\prime}$ a term $-x$ such that $x+(-x) = 0$. The term $-x$ will be called the *negative* of the term x—and clearly, it is unique. S.f.r. The *difference* between x and y is defined $x+(-y)$. S.f.r. In order to avoid confusion in notation, the difference will be indicated only by the expression $x+(-y)$, and not by $x-y$. The sign - will be reserved to represent negative terms or relations.

The triple $(T^{\infty\prime},+,.)$ is a commutative ring without identity (e.g., $3\bar{2}1.1 = 3\bar{0}1$). This ring is not an integral domain (e.g., $0\bar{1}2\cdot 1\bar{0}0 = 0$), and every element in the additive group is of infinite order. As a consequence, equations of the form $a+x=b$ are always solvable, but this is not so for equations of the form $ax = b$. S.f.r.

4. *Rings of term-relation numbers as modular lattices.* Given the two terms x and y, we define $x \geq y$ if and only if $x+(-y)$ belongs to T^{∞}; e.g., $-3\bar{4}(6\bar{9}-2) \geq -5\bar{2}-3$, since $-3\bar{4}(6\bar{9}-2)+5-\bar{2}3 = 2\bar{2}(9\bar{9}1)$. S.f.r. Clearly, the Archimedean property is satisfied for term-relation numbers. The relation $\geq$ introduces a partial ordering in $T^{\infty\prime}$ and $R^{\infty\prime}$. This order is not total (e.g., $2\bar{0}1$ is neither greater than, equal to, nor less than $1\bar{0}2$).

In order to prove that $T^{\infty\prime}$ and $R^{\infty\prime}$ are lattices, we must prove this existence of a l.u.b. and a g.l.b. for every pair of elements in each ring. Let us consider the term x of order m [1] and the term y of order n. Two cases are possible: (a) x and y are comparable (i.e., either $x \geq y$ or $y \geq x$); (b) x and y are not comparable. Case (a) is trivial. Let us assume case (b) and write $x = w\bar{w}z$ and $y = u\bar{u}v$ (w and z terms of order less than m; u and v terms of order less than n; $\bar{w}$ and $\bar{u}$ relations of order greater than or equal to 1). We again have two cases: (b.1) w and u, $\bar{w}$ and $\bar{u}$, and z and v are respectively comparable; (b.2) one, two, or all of the three pairs w,u; $\bar{w},\bar{u}$; z,v are not comparable. Case (b.1) allows four possibilities concerning the sense of the relation $\geq$. Let us assume, e.g., $w \leq u$, $w \geq u$, $z \geq v$ (the senses of the first and third pairs must be opposite in order that x and y not be comparable). In this case, $u\bar{w}z$ is the l.u.b. and $w\bar{u}v$ is the g.l.b. (the other three cases lead to the obviously similar l.u.b. and g.l.b.). Clearly, $u\bar{w}z+(-(w\bar{w}z)) = (u+(-w))\bar{0}0\,\varepsilon\, T^{\infty}$ and $u\bar{w}z+(-(u\bar{u}v)) = 0(\bar{w}+(-\bar{u}))(z+(-v))\,\varepsilon\, T^{\infty}$; i.e., $u\bar{w}z \geq w\bar{w}z$ and $u\bar{w}z \geq u\bar{u}v$. It is also clear that every other term t such that $t \geq x$ and $t \geq y$ will also satisfy $t \geq u\bar{w}z$. Similar reasoning follows for the g.l.b. Case (b.2), in turn, allows several possibilities, but all of them can be treated in like fashion. Let us assume, e.g., $w \geq u$, $\bar{w} \leq \bar{u}$, and z and v not comparable. Let $z = p\bar{p}q$ be of order $h < m$ and $v = r\bar{r}s$ be of order $k < n$. Again, two cases are possible: (b.21) p,r; $\bar{p},\bar{r}$; q,s are respectively comparable, say, e.g., $p \geq r$; $\bar{p} \leq \bar{r}$; $q \leq s$. In this case, reasoning as for (b.1), we find that $w\bar{u}(p\bar{r}s)$ would be the l.u.b. and $u\bar{w}(r\bar{p}q)$ the g.l.b. Case (b.22) assumes that one or more of the pairs p,r; $\bar{p},\bar{r}$; q,s are not comparable. Repeating the analysis of case (b.2), and reiterating it as many times as necessary, we shall always find some components of x and y (eventually their final components) that are comparable for every possible case in which x and y are not comparable. In other words, the l.u.b. and g.l.b. exist and can always be determined by following the indicated analysis. S.f.r. $T^{\infty\prime}$ and $R^{\infty\prime}$ are then lattice rings.

Such lattices are not complete, since they contain neither a zero nor an all element. However, these lattices are modular, since $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$, as the following example shows. Let us write $x = a\bar{a}b$, $y = c\bar{c}d$, $z = e\bar{e}f$, and assume: $a \geq c$, $\bar{a}$ and $\bar{c}$ not comparable, b and d not comparable, $a \leq e$, $\bar{a} \geq \bar{e}$, $b \leq f$, $c \leq e$, $\bar{c} \geq \bar{e}$, $d \leq f$. Let us also write $\bar{a} = \bar{s}s\bar{t}$, $\bar{c} = \bar{u}u\bar{v}$, $b = h\bar{h}k$, $d = p\bar{p}q$, and assume: $\bar{s} \geq \bar{u}$, $s \geq u$, $\bar{t} \leq \bar{v}$, $h \geq p$, $\bar{h} \geq \bar{p}$, $h \leq q$. With these assumptions.

$$x \cap (y \cup z) = a(\bar{s}s\bar{t})b \cap e(\bar{u}u\bar{v})f = a(\bar{u}u\bar{t})b$$

and

$$(x \cap y) \cup (x \cap z) = c(\bar{u}u\bar{t})(p\bar{p}k) \cup a\bar{e}b = a(\bar{u}u\bar{t})(h\bar{h}k).$$

This example can be considered as part of a longer proof by cases in which incomparable pairs occur in different numbers and in different places as components of x, y, and z. The proof is a matter of routine and follows lines that are similar to those of the example just given. S.f.r.

As a consequence of $T^{\infty\prime}$ being a modular lattice, intervals of the form $\mathrm{I}[a \cup b, a]$ and $\mathrm{I}[b, a \cap b]$ (where $\mathrm{I}[a,b] = \{x\colon a \geq x \geq b\}$) are isomorphic. S.f.r. This lattice ring is also a lattice-ordered group ([2], p. 214) in the sense that $T^{\infty\prime}$ is a lattice ring in which the relation $\geq$ is invariant under translations $x \to a+x+b$. (Because of monotony of order with respect to addition, $x \geq y$ implies $a+x+b \geq a+y+b$ for all a,b in $T^{\infty\prime}$.) S.f.r. The absolute value $|x|$ of a term x is defined as follows. Let $x = a\bar{a}b\ldots\bar{m}n$ be the term expressed in its final components (with some adequate arrangement of the corresponding parentheses [1]), then $|x| = |a|\_|\bar{a}|\ |b|\ldots|\bar{m}|\ |n|$ (with the same parenthesis arrangement) where $|i|$ and $|i|$ have the meaning of absolute value in the ordinary sense. S.f.r. It is clear, then, that $|x|$ is positive in the sense that $|x| \geq 0$ (or equivalently $|x| \,\varepsilon\, T^{\infty}$). It is also true that the product of two positive terms is positive. However, nonpositive terms do not necessarily satisfy $0 \geq y$; then, even if the product of the negatives of positive terms is positive, the product of the negatives of nonpositive terms is not necessarily positive (though it may be in some cases). This is a consequence of the fact that $\geq$ is not a total order; for this reason, $T^{\infty\prime}$ is a lattice-ordered group, but not a lattice-ordered ring in Birkhoff's sense ([2], pp. 214, 227). S.f.r.

We can now define a "lattice-ordered ideal" (or 1-ideal) of the ring $T^{\infty\prime}$ as a normal subgroup of the additive lattice-ordered group of $T^{\infty\prime}$ that with every a also contains every x with $|x| \leq |a|$. Then the congruence relations in $T^{\infty\prime}$ become the partitions of $T^{\infty\prime}$ into the cosets of its different 1-ideals. S.f.r.

4. *Multiplies, divisibility, prime elements, factorization.* Let x be an arbitrary term-relation number, namely, a term. By a *multiple* of x, we mean a term z of the form $z = xy$. We shall also say that x and y are *factors* of z. S.f.r. The algorithm of division can be defined as follows: let $x = r\bar{r}s$ and $y = t\bar{t}u$; then $x/y = (r/t)(\bar{r}/\bar{t})(s/u)$ (Definition 1). This is a recursive definition, and x/y will have a definite value if and only if the following three conditions are satisfied: (i) y has no 0 or $\bar{0}$ as component;

(ii) the order of y is greater than or equal to the order of x, (so that postulate P22 [1] does not have to be applied for y); (iii) the parenthesis structures of x and y are such that, by applying **Def. 1**, the orders of r, $\bar{r}$, and s are respectively less than or equal to the orders of t, $\bar{t}$, and u, this being true recursively at every stage of the parallel analysis of x and y in their components, until the final components of x are reached. If these three conditions are not satisfied, division by 0 or $\bar{0}$ would be unavoidable at some stage during the iteration of **Def. 1**. With (i), (ii), and (iii) satisfied as necessary conditions, it is easy to see that they are also sufficient to let **Def. 1** provide an answer to the problem of finding, for every x and y, a pair of terms q and r such that $x = qy + r$ with r in T^{∞}, and such that $r \leq |y|$. The following example shows that the proof of sufficiency consists of an iteration of Euclid's fundamental theorem. Let us consider the quotient

$$x/y = (-1)\bar{7}((2\bar{0}(-4))\bar{1}5)/(3(-\bar{7})(-6))\bar{5}((2\bar{7}6)\bar{6}3).$$

Here x is of order 4, y is of order 5. Applying **Def. 1**,

$$\begin{aligned} x/y = {} & [(-1)\bar{0}(-1)/3(-\bar{7})(-6))][\bar{7}/\bar{5}][((2\bar{0}(-4)\bar{1}5)/((2\bar{7}6)\bar{6}3)) = [(-1+2/3)(\bar{0}+\bar{0}/(-\bar{7})) \\ & (1+5/(-6))] [\bar{1}+(\bar{2}/\bar{5})][((1+0/2)(\bar{0}+\bar{0}/\bar{7})(-1+2/6))(\bar{0}+\bar{1}/\bar{6})(1+2/3)] = \\ & ((-1)\bar{0}1)\bar{1}((1\bar{0}(-1))\bar{0}1)+((2\bar{0}5)\bar{2}((0\bar{0}2)\bar{1}2))/y \end{aligned}$$

The remainder $(2\bar{0}5)\bar{2}(0\bar{0}2)\bar{1}2)$ is less than $|y|$. S.f.r.

In order to determine the g.c.d. of two-term-relation numbers, two conditions must be satisfied. Since the g.c.d. z of two terms x and y divides them both, condition (iii) for divisibility must be satisfied for x/z and y/z. The second condition covers situations such as this: pairs of terms like $2\bar{0}1$ and $1\bar{0}2$ do not have a g.c.d. because every number of the form $1\bar{a}1$, with any arbitrary relation $\bar{a}$, is a common divisor of both terms; however, the set of terms of the form $1\bar{a}1$, for every arbitrary $\bar{a}$, has no upper bound. Therefore, condition (iv), x and y may not have zeros (either 0 or $\bar{0}$) in the same respective places in their given parenthesis structures. With conditions (iii) and (iv) satisfied, the g.c.d. always exist and can be characterized recursively in this way: if $x = r\bar{r}s$ and $y = t\bar{t}u$, then

$$\text{g.c.d.}(x,y) = (\text{g.c.d.}(r,t))(\text{g.c.d.}(\bar{r},\bar{t}))(\text{g.c.d.}(s,u).$$

Because there is no g.c.d. in any other case, (iii) and (iv) are necessary and sufficient conditions for the existence of the g.c.d.

The l.c.m. presents similar problems, except that instead of condition (iv), x and y must satisfy condition (i), since they must divide the l.c.m. Further, condition (v), x and y must have the same order. Then,

$$\text{l.c.m.}(x,y) = (\text{l.c.m.}(r,t))(\text{l.c.m.}(\bar{r},\bar{t}))(\text{l.c.m.}(s,u).$$

(Condition (v) assures that r,t; $\bar{r},\bar{t}$; s,u will not yield zeros at any stage of the division process). Conditions (i), (iii), and (v) are necessary and sufficient for the existence of the l.c.m. S.f.r.

By a *prime* term-relation number, we mean a term or relation such that its final components are exclusively prime terms $(1,2,3,5,7,\ldots)$ and prime relations $(\bar{1},\bar{2},\bar{3},\bar{5},\bar{7},\ldots)$. E.g., $2\bar{1}3((79\bar{1}7)\bar{5}61)$ is a prime term. The factorization theorem holds, then, for term-relation numbers: any term of T^{∞} (0 excluded) admits a unique factorization as a product of prime terms $P_1 P_2 \ldots P_r$ such that $P_1 \leq P_2 \leq \ldots \leq P_r$. S.f.r. E.g., the term $21(\overline{33}45\overline{7})2$ has the factorization: $(1(\overline{1}3\overline{1})1)(3(\overline{3}3\overline{1})1)(7(\overline{11}5\overline{7})2)$, and this is the only one that satisfies $P_1 \leq P_2 \leq \ldots \leq P_r$.

5. *Rational term-relation numbers. Equations.* In the multiplicative semigroups of $T^{\infty\prime}$ and $R^{\infty\prime}$, cancellation laws for multiplication do not hold (unless we exclude from $T^{\infty\prime}$ and $R^{\infty\prime}$ all terms and relations with zero components 0 or $\overline{0}$, limiting, then, the scope of multiplication to pairs of terms and pairs of relations that have the same order and the same parenthesis structure). However, we can construct commutative rings of rational term-relation numbers (which will not be fields) by allowing the final components of every term and relation to be expressions of the form a/b or $\overline{a}/\overline{b}$, where a and b are elements of $T^{\infty\prime}$ and $\overline{a}$ and $\overline{b}$ elements of $R^{\infty\prime}$, with $b \neq 0$ and $\overline{b} \neq 0$. The arithmetic of these rational term-relation numbers can be obtained by generalizing the arithmetic of $T^{\infty\prime}$ and $R^{\infty\prime}$ developed above.

With respect to the solvability of equations with coefficients in $T^{\infty\prime}$ and $R^{\infty\prime}$, when a term or relation has a zero component, it plays a role of indetermination similar to the role that the integer zero plays in the theory of equations with integral coefficients. Therefore, $ax = b$ is uniquely solvable if a does not have a zero component, and if b and a satisfy conditions (ii) and (iii) for divisibility. Also, the linear equation $ax + by = c$ has an infinity of ordered pairs of solutions (x,y) if a and b do not have zero components, and if c and a on the one hand, and c and b on the other, both satisfy conditions (ii) and (iii) for divisibility.

6. *The relevance of the final theorem of arithmetic.* Term-relation number systems are new generalizations of the system of natural numbers T^* that differ greatly from the usual generalizations considered in theory of numbers. A question arises as to how term-relation number systems compare with hypercomplex systems. In this comparison, differences are more striking than similarities, since there is no complex or hyper-complex system isomorphic to the rings $T^{\infty\prime}$ or $R^{\infty\prime}$ or to any extension of them that we can foresee. However, some definitions and concepts from the theory of hypercomplex systems can easily be transferred to term-relation number systems. For example, we can define the modulus of a term-relation number in this form: given the term $x = a\overline{a}b\overline{b}\ldots\overline{m}n$ (with some arrangement of the corresponding parentheses), the modulus of x is the pair $(r,\overline{r})$ where $r = \sqrt{a^2+b^2+\ldots+n^2}$ and $\overline{r} = \sqrt{\overline{a}^2+\overline{b}^2+\ldots+\overline{m}^2}$. S.f.r. It is clear that a necessary condition for equality of terms or relations is that their respective moduli be equal in the sense that $r = r'$ and $\overline{r} = \overline{r}'$.

An important issue develops at this point. It is well known that Weierstrass' final theorem of arithmetic eliminates the possibility of obtaining hypercomplex systems by generalizing the field of complex numbers C to hold all the formal laws of arithmetic. This theorem is "final" in the sense that the usual extensions of number systems reach an end in C, insofar as the validity of all those formal laws is concerned. But Weierstrass' proof does not apply to all possible extensions of rings obtained from T^*. The question here is: Is Weierstrass' theorem "final" also in the sense that it can be properly generalized for all possible field extensions obtainable from rings that are generalizations of T^*, field extensions eventually not isomorphic to the field of rational, real, or complex numbers? The term-relation number systems presented here do not answer the question, but they certainly open it for consideration. (Note: It is not possible to map $T^{\infty\prime}$ onto a difference ring that would be an integral domain, since the least prime ideal of $T^{\infty\prime}$ that contains the set of all terms with 0 or $\bar{0}$ as components—the set of proper zero divisors of $T^{\infty\prime}$—is $T^{\infty\prime}$ itself).

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