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A NOTE ON THE AXIOMATIZATION OF RUBIN'S SYSTEM (S)

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Rubin, in [3], suggests that the axiomatic for system (**S**) may be simplified. It is here shown that

R5 *If* $\vdash \alpha$ *in* (**S**) *then* $\vdash \Box_2 \alpha$ *in* (**S**)

and

R*5 *If* $\vdash \alpha$ *in* (**S**) *then* $\vdash \Box_1 \alpha$ *in* (**S**)

are derivable from the other axioms and rules of (**S**).

This paper presupposes [3] and adopts the same primitive basis and definitions for (**S**). Thus the axioms and rules of (**S**) are

A1 $(\alpha \wedge \beta) \Rrightarrow (\beta \wedge \alpha)$
A2 $(\alpha \wedge \beta) \Rrightarrow \alpha$
A3 $\alpha \Rrightarrow (\alpha \wedge \alpha)$
A4 $((\alpha \wedge \beta) \wedge \gamma) \Rrightarrow (\alpha \wedge (\beta \wedge \gamma))$
A5 $\alpha \Rrightarrow \sim \sim \alpha$
A6 $((\alpha \Rrightarrow \beta) \wedge (\beta \Rrightarrow \gamma)) \Rrightarrow (\alpha \Rrightarrow \gamma)$
A7 $(\alpha \wedge (\alpha \Rrightarrow \beta)) \Rrightarrow \beta$
A8 $\Box_2 \alpha \Rrightarrow \Box_2 \Box_2 \alpha$
A12 $\Box_2 \alpha \Rrightarrow \Box_1 \alpha$
R1 *If* $\vdash \alpha$ *and* $\vdash (\alpha \Rrightarrow \beta)$ *then* $\vdash \beta$.
R2 *If* $\vdash \alpha$ *and* $\vdash \beta$ *then* $\vdash (\alpha \wedge \beta)$.
R3 *If* $\vdash (\alpha \Leftrightarrow \beta)$ *and* $\vdash \gamma$ *and* δ *results from* γ *by replacing* α *for* β (*or* β *for* α) *then* $\vdash \delta$.

together with *A*1-A*8*, **R*1-R*3**. (*T** is the wff obtained from *T* by replacing all the "$\Diamond_2$'s" by "$\Diamond_1$'s".)

The following theorems, *S1-S7*, follow from *A1-A8* and **R1-R3**, and their proofs can be found in [1].[1]

S1	$\alpha \Rrightarrow \alpha$	[12.1]
S2	$((\alpha \supset \beta) \wedge (\beta \supset \gamma)) \Rrightarrow (\alpha \supset \gamma)$	[15.1]
S3	$\Box_2 \alpha \Rrightarrow \alpha$	[18.42]
S4	$(\alpha \Rrightarrow \beta) \Leftrightarrow \Box_2(\alpha \supset \beta)$	[18.7]

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S5 $(\Box_2 \alpha \wedge \Box_2 \beta) \Leftrightarrow \Box_2(\alpha \wedge \beta)$ [19.81]
S6 $(\Box_2 \alpha \wedge \Box_2 \beta) \Rightarrow (\alpha \Leftrightarrow \beta)$ [19.84]
S7 $\Box_2\Box_2 \alpha \Leftrightarrow \Box_2 \alpha$ [C 10]

Some additional theorems of (**S**) follow

S8 $(\alpha \Rightarrow \Box_2 \beta) \Rightarrow (\alpha \Rightarrow \beta)$
S9 $(\alpha \Rightarrow \beta) \Rightarrow (\Box_2 \alpha \Rightarrow \Box_2 \beta)$

S8 was proved by Parry [2] p. 139, and *S9* is given by Rubin [3] p. 308. Any further theorems of (**S**) that are needed will be given without proof—though their proofs are indicated by the bracketed information.

S10 $\Box_2\Box_2 (\alpha \supset \beta) \Rightarrow \Box_2\Box_1 (\alpha \supset \beta)$ [*A12, S9,* **R1**]
S11 $(\alpha \Rightarrow \beta) \Rightarrow \Box_2 (\alpha \rightarrow \beta)$ [*S10, S4, S7,* **R3**, *S*4,* **R*3**]
S12 $(\alpha \Rightarrow \beta) \Rightarrow \Box_2 (\alpha \Rightarrow \beta)$ [*S1, S4, S7,* **R3**]
S13 $\Box_2(\alpha \rightarrow \alpha)$ [*S1, S11,* **R1**]
S14 $\Box_2 ((\alpha \wedge (\alpha \Rightarrow \beta)) \Rightarrow \beta)$ [*A7, S12,* **R1**]
S15 $\Box_2 (\Box_2 \alpha \Rightarrow \alpha)$ [*S3, S12,* **R1**]
S16 $(\Box_2 \alpha \Rightarrow \alpha) \Leftrightarrow ((\alpha \wedge (\alpha \Rightarrow \beta)) \Rightarrow \beta)$ [*S14, S15,* **R2**, *S6,* **R1**]
S17 $(\Box_2 \alpha \Rightarrow \alpha) \Rightarrow ((\alpha \wedge (\alpha \Rightarrow \beta)) \Rightarrow \beta)$ [*S16,* df. '$\Leftrightarrow$' **R3**, *A2,* **R1**]

The following lemma will be needed for the proof of **R5**.

Lemma I *If* $\vdash \Box_2 \alpha$ *and* $\vdash (\alpha \rightarrow \beta)$ *then* $\vdash \Box_2 \beta$

Proof.
(1) $\Box_2 \alpha$ [Hypothesis]
(2) $\alpha \rightarrow \beta$ [Hypothesis]
(3) $\Box_1 \alpha$ [(1), *A12,* **R1**]
(4) $\Box_1 (\alpha \supset \beta)$ [(2), *S*4,* **R*3**]
(5) $\Box_1 \alpha \wedge \Box_1 (\alpha \supset \beta)$ [(3), (4), **R*2**]
(6) $\alpha \longleftrightarrow (\alpha \supset \beta)$ [(5), *S*6,* **R*1**]
(7) $\Box_2 (\alpha \supset \beta)$ [(1), (6), **R*3**]
(8) $\alpha \Rightarrow \beta$ [(7), *S4,* **R3**]
(9) $\Box_2 \alpha \Rightarrow \Box_2 \beta$ [(8), *S9,* **R1**]
(10) $\Box_2 \beta$ [(1), (9), **R1**]

It is now possible to prove

R5 *If* $\vdash \alpha$ *then* $\vdash \Box_2 \alpha$

Proof. **R5** can be established by induction on the length of the proof of α. If α is one of the axioms, *A1-A8, A12,* then $\Box_2\alpha$ follows by *S12,* **R1** and *A1-A8, A12* respectively. If α is one of the axioms *A*1-A*5* then $\Box_2\alpha$ follows by *S11,* **R1**, and *A1-A5* respectively. Thus it remains to show $\Box_2\alpha$ when α is one of the axioms *A*6-A*8*. But the following are theorems of (**S**).

S18 $\Box_2 (((\alpha \supset \beta) \wedge (\beta \supset \gamma)) \rightarrow (\alpha \supset \gamma))$ [*S2, S11,* **R1**]
S19 $\Box_2 (\Box_1((\alpha \supset \beta) \wedge (\beta \supset \gamma)) \rightarrow \Box_1 (\alpha \supset \gamma))$ [*S18, S*9,* Lemma I]
S20 $\Box_2 (((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma))$ [*S19, S*4, S*5,* **R*3**]
S21 $\Box_2 (\Box_1 \alpha \rightarrow \Box_1 \alpha)$ [*S13, S*9,* Lemma I]
S22 $\Box_2 (\Box_1 \alpha \rightarrow \alpha)$ [*S21, S*8,* Lemma I]

S23 $\Box_2((\alpha \wedge (\alpha \rightarrow \beta)) \rightarrow \beta)$ [*S22, S*17,* Lemma I]

S24 $\Box_2(\Box_1 \alpha \rightarrow \Box_1\Box_1 \alpha)$ [*S21, S*7,* **R*3**]

Thus *S20, S23,* and *S24* complete the basis of the induction for the proof of **R5**.

If α follows from previous theorems of (S) by **R1-R3** or **R*1-R*3**, $\Box_2 \alpha$ may be obtained from the induction hypothesis using *S3, S5, S9* (in case α follows by **R1-R3**, or **R*2**); by *S3* and Lemma I (in case α follows by **R*1**); and by *S3* (in case α follows by **R*3**).

Finally, on the basis of **R5**, it is possible to prove

R*5 *If* $\vdash \alpha$ *then* $\vdash \Box_1 \alpha$.

Proof.

(1)	α	[Hypothesis]
(2)	$\Box_2 \alpha$	[(1), **R5**]
(3)	$\Box_1 \alpha$	[(2), *A12,* **R1**]

NOTE

1) The numbers in brackets following each of the theorems, *S1-S7,* refer to the corresponding theorems in Lewis and Langford [1].

BIBLIOGRAPHY

[1] C. I. Lewis and C. H. Langford, *Symbolic Logic,* Dover, New York (1932), 506 pp.

[2] W. T. Parry, Modalities in the "Survey System" of strict implication, *The Journal of Symbolic Logic,* v. 4 (1939), 137-154.

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