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MODAL LOGIC WITH FUNCTORIAL VARIABLES AND A CONTINGENT CONSTANT

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The World as a Propositional Constant

1. The present section is by Prior; the two which follow it, by Meredith, Meredith's sections were originally produced in 1953 and circulated among colleagues; subsequent references to them in the literature, e.g. in [3], [4], [5], [6] and [7], may be clarified if these two notes are now made more widely available. They were provoked by Łukasiewicz's development of the modal system which he presented in [2]. The importance of this system as a limiting case has been made clear by Smiley in [9]; a number of commentators have noted its intuitive peculiarities. Meredith was interested in it as a first attempt to incorporate functorial variables in a modal system, and sought in the system (C, Γ, 0, δ, p) below to incorporate the same feature in a more normal type of modal logic, namely Lewis's S5. This system is equivalent to S5 supplemented by the qualified law of extensionality $C\Gamma EpqC\delta p\delta q$ (Meredith takes over Łukasiewicz's symbol Γ for necessity and Δ for possibility).

2. The system $(C, \Gamma, 0, n, \delta, p)$ introduces the more original feature of a constant n to represent "the world" in the Wittgensteinian sense of "everything that is the case." Its most distinctive feature is the law $Cp\Gamma Cnp$, "What is true is necessarily implied by the totality of what is the case"—necessarily because this totality is equivalent to a conjunction of which all true propositions are conjuncts, and we have $\Gamma CKpqp$. In a sense, of course, unless all truths are necessary, the totality of what is the case might not have contained (and so implied) the given truth p; but in the symbol n, "the totality of what is the case" is not given by this description of it but given simply as the actual totality of what *is* the case.

Meredith's proof, in his second item, of the independence of this fundamental law $Cp\Gamma Cnp$, is instructive. To distinguish a contingent truth from a necessary one we need two possible worlds, a contingent truth holding in one of them only and a necessary truth in both. Two such worlds generate four truth-values, "truth in both", "true in the actual world but not in the other", "true in the other world but not in the actual one", "true in

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neither". In some of his matrices Meredith represents these four values as 1, n, $\bar{n}$, 0; the symbol $\bar{n}$ is of course the alternative world to the actual one. But to distinguish "the world" from a less comprehensive contingent truth we need three possible worlds; "the world" being true in (in fact being) the actual world only; a less comprehensive contingent truth, true in this world and one other; a necessary truth, true in all three. Three possible worlds generate 8 truth-values, "true in all three", "true in the first two but not in the third", etc. Meredith verifies all of his axioms but $Cp\Gamma Cnp$ by using such an 8-valued matrix, and identifying n with a "value" that is or characterises a contingent truth but not a world.

3. Formally, the system is elegant and ingenious; philosophically, it may well give rise to misgivings.

We may look at the matter from the point of view of propositional identity. It is argued in [8], I still think plausibly, that no proposition can be identical with a logical complication of itself, and this is an assumption which can be put to many uses. For example, it may be used to solve a medieval paradox which appears, e.g. in Ralph Strode (my attention has been drawn to this *sophisma* by P. T. Geach). The paradox is developed in two stages, thus:—

Stage I. The argument in Stage I is sound ($= p$)
Therefore, I am the Pope (or anything at all) ($= q$)

Stage II. :—

(1) If the argument in Stage I is not sound, then possibly (p and not q). ($N\Gamma Cpq = \triangle KpNq$)

(2) If possibly (p and not q), then possibly p, i.e. possibly Stage I is sound. ($C\triangle Kpq\triangle p$)

Therefore

(3) If Stage I is not sound, it possibly *is* sound (syllogistically from (1) and (2)).

Therefore

(4) Stage I is possibly sound (by $CCNp\triangle p\triangle p$, a modal law obtainable syllogistically from $CCN\triangle p\triangle p\triangle p$ and $CN\triangle pNp$).

But (5) Stage I *could* be sound only if it *is* sound ($C\triangle\Gamma Cpq\Gamma Cpq$, from $C\triangle\Gamma p\Gamma p$, S5).

Therefore

(6) Stage I is sound, i.e. p [(4), (5), modus ponens]

Therefore

(7) I am the Pope (from (6), by Stage I).

The first step in all this is simply the identification of p (the premiss in Stage I) with ΓCpq ("p necessarily implies q"). Given $Ip\Gamma Cpq$ (using I for propositional identity, with the usual laws Ipp and $CIpqC\delta p\delta q$), the proof is easily formalized, and valid; but we escape its conclusion if we can argue that no proposition of such a form as $Ip\Gamma Cpq$, identifying a proposition with a logical complication of itself, can ever be true.

Another use of this assumption is the following:—M. J. Cresswell has suggested in [1] that, given the notion of propositional identity, we may develop a whole system of arithmetic without introducing individual variables at all, using sentential variables as the lowest-type variables in our definitions of characteristic arithmetical concepts. In such an arithmetic, we could easily obtain what would function as axioms of infinity from the above assumption, e.g.

1. $NIp\delta Np$

(No proposition is identical with any function of its negation. From this we obtain

2. $NIpNp$ $(1, \delta p/p)$
3. $NIpNNp$ $(1, \delta/N)$
4. $NINpNNp$ $(2, p/Np)$

2 gives us two distinct propositions straight away (p and Np), 2 to 4 together give us three (p, Np, NNp), and continuing the process secures us an infinite number.

4. Given the above assumption, it is clear that there can be no such proposition as Meredith's n. For the conjunction of all truths would have to contain as conjuncts (a) itself, (b) its own double negation, (c) every fact as to what it implies; to name only a few of the impossibilities.

This objection might be met by conceiving n as the *minimum* conjunction by which all truth would be entailed—the conjunction of a possibly infinite set of independent axioms from which whatever is the case would follow. But if there is one there are bound to be also other conjunctions meeting this condition, e.g. the given one with its conjuncts variously regrouped.

There can in fact only be such a unique proposition if propositional identity is reduced to strict equivalence. This reduction, indeed, is assumed even in the n-free portion of Meredith's system, in the extensionality thesis $C\Gamma EpqC\delta p\delta q$. This is clearly inconsistent with our "axioms of infinity" above.

5. We would not, however, require to equate I with ΓE if we enlarged modal logic not by adding functorial variables but by adding propositional quantifiers, and (following a suggestion made to me by Roman Suszko) introduced not a propositional constant n but a function Wp, to be read as something like "p comprehends all truths", and defined by

$$Wp = Kp\Pi qCq\Gamma Cpq$$

We can then obtain Meredith's axioms under a condition. Instead of his axiom n ("The world is the case"), we have $CWpp$; instead of $Cp\Gamma Cnp$, $CWpCq\Gamma Cpq$; and instead of $N\Gamma n$, $CWpN\Gamma p$. Only the last, which is Meredith's form can be dispensed with by defining the standard impossible proposition 0 as Γn, need in our form be laid down separately; the other two, in our form, follow from Df. W, $CKpqp$, $CKpqq$ and the usual rules for Π. Moreover $CWpN\Gamma p$, which when written in full is rather long, can be replaced by $\Sigma pKpN\Gamma p$, i.e. "There is at least one contingent truth". This does not follow from S5 plus the rules for Π, but it is a reasonable addition to those in a modal logic with propositional quantification.

The thesis $CKWpWq\Gamma Epq$, stating that all "world" propositions are strictly equivalent, and securing for "the world" such limited individuality as it does possess, does follow from S5 plus the rules for Π. On the other hand, nothing so far laid down secures that there are any "world" propositions at all. What is easily provable (in S5 plus the rules for Π) is

$$C\,K\,p\,q\,\Sigma\,r\,K\,r\,K\,\Gamma\,C\,r\,p\,\Gamma\,C\,r\,q,$$

i.e. for any two truths there is a truth which strictly implies both of them, (for their conjunction will meet this condition). From this we have "For every pair of truths such that one does not strictly imply the other, there is a truth that strictly implies both"; but this is consistent with "Every truth has some truth which it does not strictly imply", i.e. the contradictory of "There is a truth which strictly implies all truths", ΣpWp. On the other hand, ΣpWp is also consistent with the rest that we have; the consistency of Meredith's n calculus in effect guarantees this.

6. Alternatively, W could be defined by

$$Wp = K\,p\,\Pi\,q\,A\,\Gamma\,C\,p\,q\,\Gamma\,C\,p\,N\,q,$$

Wp if p and, for every q, p either strictly implies q or strictly implies its negation. This definition brings out the fact that a "world" proposition is a *maximum* proposition; if we conjoin with it the least thing that it does not imply we shall have a contradiction, since among the things it does imply will be the negation of the added item. As Meredith puts it, n, though true, is "next to impossibility". (If there were no "worlds", there would be no maximum truths; for any given truths there would be still more comprehensive ones, and between any given item in the series and sheer impossibility one could insert an intermediate item.)

Change the component p in the new Df. W to Mp and we have the definition of a possible world,

$$W(\mathrm{m})p = K\,M\,p\,\Pi\,q\,A\,\Gamma\,C\,p\,q\,\Gamma\,C\,p\,N\,q$$

Given this definition, S5, the rules for Π and ΣpWp, one provable thesis is $C\,M\,p\,\Sigma\,q\,K\,W(\mathrm{m})q\,\Gamma\,C\,p\,q$, i.e. if p is possible there is some possible world in which it is true. (The propositions that are "true in" a possible world are those that are strictly implied by that world proposition).

7. The concept of a set of "possible world" propositions has a tense-logical analogue in that of a set of descriptions of the total state of the world at given instants—a concept required for the logical discussion of Laplacean determinism.

The Two Systems $(C, \Gamma, 0, \delta, p)$ *and* $(C, \Gamma, 0, n, \delta, p)$

1. Both systems I derive as part of the calculus of properties.

$(Cpqx) = C(px)(qx)$, where the 'C' on the right has its usual interpretation.

(Ox) = Falsum x (e.g. $N\varepsilon xx$)

$(\mathrm{I}\,px) = \Pi x(px)$

In the second system we have also

$(nx) = \varepsilon ax$, where a is a certain constant value of x.

This gives

$(\Gamma Cnpx) = pa$

In the second system there are two degrees of validity of α:

(i) universal validity, $\Pi x\ (\alpha x)$
(ii) accidental validity, (αa)

In the first system only universal validity is considered.

2. I have given the axioms of the first system as

1. $\Gamma C \delta CCp0Cqr \delta CCrpCqp$
2. $C\Gamma pC \delta Cpq \delta q$
3. $C \delta 0C \delta C00 \delta \Gamma p$

and added, for the second system

4. n
5. $Cp\Gamma Cnp$
6. $C\Gamma np$

All but (4) and (5) have strong validity.

A slight condensation is possible in the second system by the definition of 0 as Γn. We discard (6) and replace (1) and (3) by

1a. $\Gamma C \delta CCp\Gamma nCqr \delta CCrpCqp$
3a. $C \delta \Gamma nC \delta C\Gamma np \delta \Gamma q$

But in considering decision it is simpler to think of 0 as separately given. (Also vide final section.)

3. A very brief account of the first system is:

(i) If α is a consequence of (1)(2)(3) so is $\Gamma \alpha$
(ii) We have an extensionality law which may be stated in various ways, e.g. $C \delta pC\Gamma Epq \delta q$, $C\Gamma Epq\Gamma C \delta p \delta q$.

Using this law we can eliminate all elementary functors. The primary reduction is to expressions of the form

$$C\Gamma\alpha\, A\beta A\Gamma\, \delta_1 \ldots\ldots \Gamma\, \delta_k$$

where α, β, $\delta_1 \ldots\ldots \delta_k$ are Γ-free and all but α are elementary alternations (α,β or all the δ's may be missing). We can prove this if we can prove any of

$$C\alpha\beta,\ C\alpha\,\delta_1 \ldots\ldots C\alpha\,\delta_k$$

Otherwise an analysis in terms of the basis of the system shows the expression to be invalid. This shows that the axiomatization is complete as far as is possible for this infinite basis. We may complete the system internally by introducing formal rejection and the additional rule

$$\dashv C\alpha\beta,\ \dashv C\alpha\delta_1 \ldots\ldots\ldots \dashv C\alpha\delta_k$$
$$\rightarrow \dashv C\Gamma\alpha A\beta A\Gamma\delta_1 \ldots\ldots\ldots \Gamma\delta_k$$

The smallest matrix satisfying the axioms is

C	1	0	Γ
$*\ 1$	1	0	1
0	1	1	0

showing that $Cp\Gamma p$ requires for its rejection an application of the rejection rule.

I believe, but have not proved, that the addition as an axiom of an expression valid in the 2-valued matrix but rejected in the main system renders the system finite.

4. In the second system the smallest matrix satisfying the axioms is

C	1	n	$\bar{n}$	0	Γ
$*\ 1$	1	n	$\bar{n}$	$\bar{0}$	1
$*\ n$	1	1	n	n	0
$\bar{n}$	1	n	1	n	0
0	1	1	1	1	0

$Cp\Gamma p$ does not require any application of the special rejection rule, since

$$(p/n)\ Cp\Gamma p = Cn0 = \bar{n}$$

We may achieve, without the use of the n-axioms the same primary reduction

$$C\Gamma\alpha A\beta A\Gamma\delta_1 \ldots\ldots \Gamma\delta_k$$

except that here any, or all, of the α, β, $\delta_1 \ldots\ldots \delta_k$ may contain the component n, in which case the rejection rule no longer applies, e.g.

$$\vdash A\Gamma Cnp\Gamma CnCp0,\ \vdash C\Gamma CpnA\Gamma Cp0p$$

It is in fact possible to reduce all those cases further to

$$C\Gamma\alpha_1 C\Gamma CCn0\alpha_2 A\beta A\Gamma\delta_1 \ldots\ldots \delta_k$$

where α_1, α_2, β, $\delta_1 \ldots\ldots \delta_k$ are Γ-free and n-free and all but α_1 are elementary alternatives, but I have not been able to modify the main rejection rule to cope with such expressions.

I have fallen back on a very different kind of n-elimination.

n appears as the unique solution to three theses

(i) ν
(ii) $A\Gamma C\nu p\Gamma C\nu Cp0$
(iii) $C\Gamma\nu 0$

That is to say, from (i) and (ii), with the axioms, we can derive $\Gamma E\nu n$; while (i), (ii), (iii), with n instead of ν, are equivalent to (4), (5), (6).

I abbreviate $A\Gamma Cqp\Gamma CqCp0$ to Bqp. It has the following properties:

$Bq0$
Bqq
$Bq\Gamma p$
$CBqpCBqrBqCpr$

Suppose now $\mathfrak{f}n$ is an expression containing n and the elementary variables $p_1 \ldots\ldots p_k$. The elimination rule is

$$\mathfrak{f}n \sim CqCC\Gamma q0CBqp_1 \ldots\ldots CBqp_k \mathfrak{f}q$$

where q is a new elementary variable and $\mathfrak{f}q$ is the result of replacing n by q in $\mathfrak{f}n$.

First, if the right-hand side can be proved, the substitution q/n and detachments give the left-hand side.

Secondly, if the left-hand side can be proved from the axioms (1) to (6), then the right-hand side can be proved from the axioms (1) to (3).

(a) the axioms (4), (5), (6) are correlated with valid n-free theses.

(b) substitutions on the left-hand side are reducible to the primitives

$$p_i/r,\ p_i/0,\ p_i/Crs,\ p_i/\Gamma r,\ p_i/n.$$

On the right-hand side we have the same substitutions (except p_i/q instead of p_i/n) followed by the use of the above properties of Bqp with the elementary propositional calculus.

(c) $C\mathfrak{f}n\mathfrak{g}n,\ \mathfrak{f}n \rightarrow \mathfrak{g}n$

and cases where n does not appear in $\mathfrak{f}$ or does not appear in g may be similarly paralleled. That the full complement of $p_1 \ldots\ldots p_k$ may not occur in either $\mathfrak{f}n$ or $\mathfrak{g}n$ is of no difficulty; we insert the extra antecedents in $\mathfrak{f}n$ and, by substitution, remove the extra antecedents in $\mathfrak{g}n$.

Thus, if we reject the right-hand expression we must reject the left-hand expression.

Note on the Modal System $(C, \Gamma, 0, n, \delta, p)$

1. '0' is definable as 'Γn', so we may condense the axioms to

1. $\Gamma C\delta CCp\Gamma nCqr\delta CCrpCqp$
2. $C\Gamma pC\delta Cpq\delta q$
3. $C\delta\Gamma nC\delta C\Gamma np\delta\Gamma q$
4. n
5. $Cp\Gamma Cnp$

$\Gamma 1$, $\Gamma 2$, $\Gamma 3$ are theses of the system; $\Gamma 4$, $\Gamma 5$ are not.

2. Independence

For 1.

C	1	n	0	Γ
$* \; 1$	1	n	0	1
$* \; n$	1	1	0	0
0	1	1	1	0

$1(\delta/\Gamma, p/n, q/1, r/0) = 0$

For 2, 3.

C	1	n	$\bar{n}$	0	Γ_2	Γ_3
$* \; 1$	1	n	n	0	1	n
$* \; n$	1	1	$\bar{n}$	$\bar{n}$	0	0
$\bar{n}$	1	n	1	n	0	0
0	1	1	1	1	1	0

$2(p/0, \delta/{}^{\prime}, q/0) =$
$= C\Gamma_2 0CC000 = 0$
$3(\delta/C{}^{\prime}\Gamma_3{}^{\prime}, p/1, q/1)$
$= \bar{n}$

For 4.

C	1	n	Γ
$* \; 1$	1	n	1
n	1	1	n

i.e. $n = 0$, $\Gamma p = p$.

For 5.

C	1	2	3	4	5	6	7	8	Γ
$* \; 1$	1	2	3	4	5	6	7	8	1
$* \; 2$	1	1	3	3	5	5	7	7	8
3	1	2	1	2	5	6	5	6	8
4	1	1	1	1	5	5	5	5	8
5	1	2	3	4	1	2	3	4	8
6	1	1	3	3	1	1	3	3	8
7	1	2	1	2	1	2	1	2	8
8	1	1	1	1	1	1	1	1	8

$n = 2$, $0 = 8$.

$C6\Gamma Cn6 = 3$.

(With this C-matrix, the possible worlds are 4, 6, 7).

3. From 1 and 2 we have Γ-Boole (viz. if $\alpha = \beta$ in an equation in Boolean Algebra then $\Gamma C\delta\alpha\delta\beta$ is a thesis) and $(C, 0, p)$. We have also $C\Gamma p\Gamma C\delta Cpq\delta q$, $C\Gamma CpqC\Gamma Cqp\Gamma C\delta p\delta q$, $C\Gamma pp$. With 3 we can prove $\Gamma 1$, $\Gamma 2$, $\Gamma 3$. Since 2 gives the rule

$$\Gamma C\alpha\beta,\ \Gamma\alpha \rightarrow \Gamma\beta$$

all consequences of 1, 2 and 3 hold Γ.

4. Primary reduction, using only 1, 2, 3 is to expressions of the form

$$C\Gamma\alpha_1 C\Gamma\alpha_2 \ldots\ldots C\Gamma\alpha_r A\beta A\Gamma\gamma_1 \ldots\ldots \Gamma\gamma_s$$

(with the usual lax convention about absence—e.g. there may be no 'α's) where the 'α's, 'γ's and β are elementary Γ-free alternations (except for the occurrences of Γn).
The original rejection rule—rejecting this by alternatives—no longer holds unconditionally.

(i) If any of the α, β or γ elements is of the form $Cn\theta$ the expression may be simplified by

$$Cp\Gamma Cnp,\ CCnpp,\ C\Gamma Cnpp$$

viz. we may replace either $\Gamma Cn\theta$ or $Cn\theta$ by θ.
(ii) If β is of the form $An\theta$ the expression is a thesis.

(iii) If some α is of the form $An\theta$ we use

$$E\Gamma AnpA\Gamma p\Gamma EpCn0$$

to obtain

$$C\Gamma An\theta\beta \sim C\Gamma\theta\beta,\ C\Gamma E\theta Cn0\beta.$$

The second expression is further reducible if there are any other n's (not occurring in Γn) in it, by

$$\text{-}C\Gamma E\theta Cn0fn \sim C\Gamma E\theta Cn0fC\theta 0,$$

(this equivalence being an obvious consequence of $C\Gamma EpqC\delta p\delta q$).

The reduced expressions now have two forms

$$C\Gamma\alpha A\beta A\Gamma\gamma_1 \ldots\ldots \Gamma\gamma_s$$
$$C\Gamma\alpha C\Gamma E\theta Cn0A\beta A\Gamma\gamma_1\ldots\ldots \gamma_s$$

where α is n-free, but not necessarily a simple alternation; β is n-free and a simple alternation; the γ's are simple alternations, possibly containing the alternative 'n' in the first, but not in the second case.

To either of these I think we may apply the rejection rule, leaving for consideration

(a) $C\Gamma\alpha\beta$
(b) $C\Gamma\alpha\Gamma\gamma$
(c) $C\Gamma\alpha\Gamma An\gamma$
(d) $C\Gamma\alpha C\Gamma E\theta Cn0\beta$
(e) $C\Gamma\alpha C\Gamma E\theta Cn0\Gamma\gamma$

'n' appearing only in the indicated places.

These expressions are demonstrable or refutable by substitution, without additional rules.

(a) $C\Gamma\alpha\beta \sim C\alpha\beta$
(b) $C\Gamma\alpha\Gamma\gamma \sim C\alpha\gamma$
(c) $C\Gamma\alpha\Gamma An\gamma \sim C\alpha\gamma$
(d) $C\Gamma\alpha C\Gamma E\theta Cn0\beta \sim C\alpha A\theta\beta$ *or* $C\alpha C\theta 0$
(e) $C\Gamma\alpha C\Gamma E\theta Cn0\Gamma\gamma \sim C\alpha\theta$ *or* $C\alpha C\theta 0$ *or* $C\alpha\gamma$

Explanation is desirable in the last three cases:—

(c) Suppose $C\alpha\gamma$ is rejected: there is a substitution which gives $\alpha' = 1$, $\gamma' = p$, $C\Gamma\alpha'\Gamma An\gamma' = \Gamma Anp$. The substitution p/n completes the refutation. (We may have $\gamma' = 0$ which is simpler.)

(d) $E\Gamma EpCn0KCp0\Gamma Anp$

$$C\Gamma\alpha C\Gamma E\theta Cn0\beta \sim C\Gamma\alpha C\Gamma An\theta A\theta\beta$$
$$\vdash C\Gamma\alpha A\theta\beta \rightarrow \vdash C\Gamma\alpha C\Gamma E\theta Cn0\beta$$
$$\vdash C\Gamma\alpha\Gamma C\theta 0 \rightarrow \vdash C\Gamma\alpha C\Gamma E\theta Cn0\beta$$

Now suppose $\dashv C\alpha A\theta\beta$, $\dashv C\alpha C\theta 0$. There is a substitution such that $\alpha' = 1$, $\theta' = Cp0$, $\beta' = Cp0$ or 0.

$$C\Gamma\alpha' C\Gamma E\theta' Cn0\beta' = C\Gamma EpnCp0 \text{ or } C\Gamma Epn0$$

The substitution p/n completes the refutation.

(e) $C\Gamma\alpha C\Gamma E\theta Cn0\Gamma\gamma \sim C\Gamma\alpha C\Gamma An\theta A\theta\Gamma\gamma$

The procedure for proof is as in (d).

Suppose $\dashv C\alpha\theta$, $\dashv C\alpha C\theta 0$, $\dashv C\alpha\gamma$. There is a substitution such that $\alpha' = 1$, $\theta' = Cp0$, $\gamma' = p$ or $Cp0$ or 0. Now, with p/n,

$$C\Gamma\alpha' C\Gamma E\theta' Cn0\Gamma\gamma' = \Gamma n \text{ or } \Gamma Cn0 \text{ or } \Gamma 0 = 0$$

I am here aiming at a minimal rule of rejection, which is likely to be more easily proved. The matter requires a lot of tidying; but I think the additional complication introduced by n' is interesting.

5. Interesting n-theses are

$$C\Gamma CpnC\delta 0C\delta c\delta p$$
$$C\Gamma Cn0C\delta_1\delta Cnp$$

'n', though true, is next to absolute falsum; '$Cn0$', though false, is next to absolute verum.

$$C\Gamma CpnC\Gamma CqrC\Gamma CrnA\Gamma EpqA\Gamma EprEqr$$

6. Defining Hp by

$$C\delta\Gamma Cnp\delta Hp$$

we will have $H\alpha \sim \alpha$, by $CpHp$, $CHpp$. But we have neither $\Gamma CpHp$ nor $\Gamma CHpp$.

Also $\quad C\delta HCpq\delta CHpHq,\ C\delta H0\delta 0,\ C\delta 0C\delta C00\delta Hp.$

I do not know if there are any philosophical applications of this system. I can only suggest that these philosophers who think that logic *must* be two-valued are confusing Hp and p.

While the use of 'δ' and the constant propositions 'n' and 'o' render the system extremely explicit, it would perhaps appear more natural if axiomatised in terms of C, N, Γ, H, p.

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