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DECISION PROCEDURES FOR $S2^0$ AND T^0

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Decision procedures for the "zero"-modal systems $S1^0$-$S4^0$, T^0 are so far lacking. Procedures for $S2^0$ and T^0 are obtainable by modifying those of Ohnishi [1] for S2, Ohnishi and Matsumoto [2] for T. Bases chosen are in the style of Lemmon [3], viz. from:

(1) all tautologies;
(2) *CLCpqCLpLq* - axiom;
(3) from $L\alpha$ infer α;
(4) from $C\alpha\beta$, α, infer β;
(5) from α infer $L\alpha$ when α is a tautology or axiom;
(6) from α infer $L\alpha$;
(7) from $LC\alpha\beta$ infer $LCL\alpha L\beta$;
(7*) from $C\alpha\beta$ infer $CL\alpha L\beta$;
(8) rule of substitution;

we take for $S2^0$: (1)-(5), (7), (8): for T^0: (1)-(4), (6), (8). As an auxiliary system we use $E2^0$: (1), (2), (4), (7*), (8), *cf.* E2 in [3].

To decide $S2^0$ we take the system S2* of [1] without the rule $(L\rightarrow)$, i.e. Gentzen's **LK** and the rule $(\rightarrow L)$ here called (**LI**):

$$\frac{\Gamma \rightarrow \alpha}{L\Gamma \rightarrow L\alpha} \qquad (\mathbf{LI})$$

with α a single formula, Γ a series of formulas perhaps empty, in which case (**LI**) becomes the Rule of Tautology (**RT**). Restrictions: (**RT**) may not be used previous to (**LI**) or (**RT**) in one and the same string of a proof-figure. For decision of T^0 the restriction is dropped. We call these systems S_0^2, T_0^0. As an auxiliary system we use E_0^2, viz. S_0^2 without (**RT**).

Lemma. The cut-theorem is provable in both systems as in §2 *of* [1] *where* Case 3 *is alone relevant now.*

Theorem 1. *If* α (*is provable*) *in* $S2^0$, *then* $\rightarrow \alpha$ *in* S_0^2.

Proof. If α is a tautology, the theorem holds by **LK**. If α is (2), use **LK**, (**LI**). Since (**RT**) is the only way of producing $\rightarrow L\alpha$, the theorem holds for

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the conclusion of (3) if it holds for the premiss. The lemma settles (4). For (5) use **LK** (**RT**). In respect of (7) use Lemma 3.2 in [1] with E_0^2 in place of E2*. (8) is proved for S_0^2 as usual in such cases.

Theorem 2. *If* α *in* T^0, *then* $\dashv \alpha$ *in* T_0^0.

Proof the same, where relevant, and (6) holds in T_0^0 by unrestricted (**RT**).

Theorem 3. *The converse of* Theorem 1.

The proof is as in 3.3, 3.4 of [1], using $E2^0$, E_0^2, instead of E2, E2*. (And clearly E_0^2 gives a decision procedure for $E2^0$, as E2* does for E2.)

Theorem 4. *The converse of* Theorem 2.

The proof is obvious, since $C\alpha\beta$ yields $CL\alpha L\beta$ in T^0 corresponding to (**LI**), and (6) corresponds to unrestricted (**RT**).

The same process will not work for $S3^0$, since both rules used in [1] for S3* are needed to prove the characteristic axiom of S3.

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