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FAMILY $\mathcal{K}$ OF THE NON-LEWIS MODAL SYSTEMS

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In [6], p. 109, a regular modal formula α is defined as a modal formula which after deleting the modal functors L and M, if they occur in α, and after replacing the modal functors for more than one argument, if they occur in α, by the corresponding classical functors, throughout α, this formula becomes a thesis of the bi-valued propositional calculus. If a regular modal formula α is such that its addition as a new axiom to S5 reduces such extension of the latter system to the classical propositional calculus, I say that α is non-Lewis modal formula. Correspondingly, the modal systems which are irreducible to the classical propositional calculus and which are obtained by the addition of one or more non-Lewis modal formulas to the proper subsystems of S5 will be called here the non-Lewis modal systems. Such a system, for example, is constracted by McKinsey, *cf.* [2], by adding to S4 the new axiom

K1 $\quad \mathfrak{C}KLMpMLqMKpq$

which, clearly, is non-Lewis modal formula. As I have proved in [7], pp. 77-78, in this system, which is called by McKinsey S4.1, but which I call more conveniently system K1, axiom *K1* can be substituted equivalently by several other formulas, as, e.g., by

K2 $\quad \mathfrak{C}LMpMLp$

or by

K4 $\quad LMLCpLp$

This fact will be used later.

In this paper I shall present some investigations, which are far from being complete, concerning certain family $\mathcal{K}$ of the non-Lewis modal systems. I define this family $\mathcal{K}$ as a class of such and only such modal systems that each of them satisfies the following three conditions:

1) it is a proper normal extension of S4,
2) it is irreducible to the classical propositional calculus,
3) it contains as its axiom or its consequence formula *K2*.

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It is obvious that besides K1 systems K2 and K3 defined in [7], pp. 78-79, obtained by the addition of *K2* to S4.2 and S4.3 respectively, belong to family ***K***.

1 In this section I shall investigate the following three non-Lewis modal formulas

H1 $\mathfrak{C}pLCMpp$

J1 $\mathfrak{C}LCLCpLppp$

and *K2* which are verified by Group II of Lewis-Langford, *cf.* [1], p. 493, i.e. by the matrices $\mathfrak{M}1$ and $\mathfrak{M}2$ given in [4], p. 305. Since $\mathfrak{M}1$ and $\mathfrak{M}2$ falsify S5, the addition of one of the formulas *H1*, *J1* and *K2* to any normal extension of S4 which is verified by $\mathfrak{M}1$ and $\mathfrak{M}2$ gives a system belonging to the family ***K***.

I shall prove here that in the field of S4 *H1* implies *J1* which in its turn gives *K2*. Besides, some additional deductions needed for further discussion will be presented in this section.

1.1 Assume S2 and *H1*. Then:

Z1 $\mathfrak{C}NpLCpLp$ [*H1*,*p*/*Np*;S1°]
Z2 $\mathfrak{C}NLCpLpp$ [*Z1*;S1°]
Z3 $\mathfrak{C}CLCpLppp$ [*Z2*;S1°]
J1 $\mathfrak{C}LCLCpLppp$ [*Z3*;S2]

Thus, in the field of S2 *H1* implies *J1*.

1.2 **A.** Assume S4° and *J1*. Then:

Z1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}prs\mathfrak{C}\mathfrak{C}qr\mathfrak{C}\mathfrak{C}pqs$ [S3°]
Z2 $\mathfrak{C}LCLCpLpLpp$ [*Z1*,*p*/*LCpLp*,*q*/*Lp*,*r*/*p*,*s*/*p*;*J1*;S1°]
Z3 $\mathfrak{C}LLCLCpLpLpLp$ [*Z2*;S2°]
J2 $\mathfrak{C}LCLCpLpLpLp$ [*Z3*;S4°]

B. Now, let us assume S4 and *J2*. Then:

Z1 $\mathfrak{C}\mathfrak{C}LLCpLpLpLp$ [*J2*;S4°]
Z2 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}LpLq$ [S3°]
J1 $\mathfrak{C}LCLCpLppp$ [*Z2*,*p*/*LCpLp*,*q*/*p*;*Z1*;S1]

Thus, {S4;*J1*} ⇄ {S4;*J2*}.

1.3 Assume S4 and *J1*. Then:

Z1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqr\mathfrak{C}LNpr$ [S2°]
Z2 $\mathfrak{C}LNLCpLpp$ [*Z1*,*p*/*LCpLp*,*q*/*p*,*r*/*p*;*J1*]
Z3 $\mathfrak{C}NpMLCpLp$ [*Z2*;S1°]
Z4 $\mathfrak{C}MLCpqMCMpMq$ [S2°]
Z5 $\mathfrak{C}MCpqCLpMq$ [S2°, *cf.* [7] p. 71, lemma 1]
Z6 $\mathfrak{C}MLCpqCLMpMq$ [*Z4*;*Z5*,*p*/*Mp*,*q*/*Mq*;S4°]
Z7 $\mathfrak{C}NpCLMpMLp$ [*Z3*;*Z6*,*q*/*Lp*;S1°]
Z8 $\mathfrak{C}pCLMNpMLNp$ [*Z7*,*p*/*Np*;S1°]

Z9 ℭ*pCLMpM1p* [*Z8*;S1°]
K2 ℭ*LMpMLp* [*Z9*;*Z7*;S1°]

Thus, {S4°;*J1*} implies *K2*.

1.4 Since, by **1.2**, {S4;*J1*} ⇄ {S4;*J2*}, it is clear that in the field of S4 *J1* implies

M1 ℭ*LCLCpLpLpCMLpLp*

and

N1 ℭ*LCLCpLppCMLpp*

i.e. the proper axioms of the systems S4.1.1 and S4.1 defined in [4].

1.5 In the field of S4, *J1* follows from *K2* and *M1* or *N1*. Let us assume S4 and *K2*. Therefore, having *K4* at our disposal, *cf.* [7], pp. 77-78, we can procede as follows:

Z1 ℭ*MLCpLpqq* [*K4*; S1°]
Z2 ℭℭ*pqCMpMq* [S1°]
Z3 ℭℭ*LCpLpqMq* [*Z2,p/LCpLp*;*Z1,q/Mq*; S1°]
Z4 ℭ*LLCLCpLpqLMq* [*Z3*;S2°]
Z5 ℭ*LCLCpLpqLMq* [*Z4*;S4°]
Z6 ℭ*LCLCpLpqMLq* [*Z5*;*K2*;*p/q*;S1°]
Z7 ℭ*CpCqrCCpqCpr* [S1°]

Hence, if we assume *M1*, we have

J2 ℭ*LCLCpLpLpLp* [*Z7,p/LCLCpLpLp,q/MLp,p/Lp*;*M1*;*Z3,q/Lp*;S1°]

and, if we assume *N1*, then

J1 ℭ*LCLCpLppp* [*Z7,p/LCLCpLpp,q/MLp,r/p*;*N1*;*Z6,q/p*;S1°]

Then, since, by **1.2**, {S4;*J1*} ⇄ {S4;*J2*}, in virtue of 1.4 we obtain {S4;*M1*;*K2*} ⇄ {S4;*N1*;*K2*} ⇄ {S4;*J1*}:

1.6 Assume S4.2 and *H1*. Then:

Z1 ℭ*pCLMpLp* [*H1*; S1°]
G1 ℭ*MLpLMp* [S4.2]
R1 ℭ*pCMLpLp* [*G1*;*Z1*;S1°]

Thus, the addition of *H1* to S4.2 implies *R1*, i.e. the proper axiom of the system S4.4 discussed in [4].

1.7 Assume S4.4 and *K2*. Since S4.4 contains S4, we have, *cf*. **1.5**, *K4*. Hence

Z1 ℭ*pCMMLpLp* [*R1*;S4°]
Z2 ℭ*CMpLq*ℭ*pq* [S2°, *cf.* [5]]
Z3 ℭ*pLCMLpp* [*Z1*;*Z2,p/MLp,q/p*;S1°]
Z4 ℭ*pCLMLpLp* [*Z3*;S1°]
Z5 ℭ*CpLpLCpLp* [*Z4,p/CpLp*;*K4*;S1°]

Z6 ℭ*NpLCpLp* [*Z5*;S1

H1 ℭ*pLCMpp* [*Z6*,*p*/*Np*;S1

Thus, {S4.4;*K2*} implies *H1*, and, therefore, points **1.1**, **1.3**, **1.6** and **1.** allow us to establish that {S4.4;*H1*} ⇄ {S4.4;*K1*} ⇄ {S4.4;*J1*} ⇄ {S4.2;*H1*}.

2 Since 𝔐1 and 𝔐2, which falsify S5, verify formulas *H1*, *J1* and *K2* a the systems S4-V1 defined in [4], and since it is proved above that in t field of S4 *H1* implies *J1* and that *K2* follows from *J1*, the addition of a formula *H1*, *J1* and *K2* to one of the systems S4-V1 generates a syste which clearly belongs to family *K*. Thus, we have:

1) K1 = {S4;*K2*}
2) K2 = {S4.2;*K1*}
3) K3 = {S4.3;*K1*}

which was defined previously in [7]. In [3] Prior has proved recently th K1 is a proper subsystem of K2, and that K3 is a proper extension of K Thus, these three systems are distinct.

We define now the other such systems as follows:

4) K1.1 = {S4;*J1*}
5) K1.2 = {S4;*H1*}
6) K2.1 = {S4.2;*J1*}
7) K3.1 = {S4.3;*J1*}
8) K4 = {S4.4;*K2*}
9) K5 = {V1;*K2*}

The inspection of the deductions given in **1** and of the properties of sy tems S4-V1 which are discussed in [4] shows without any difficulty and once that using only the formulas *H1*, *J1* and *K2* and the systems S4-V1 v cannot construct other systems belonging to family *K* than K1-K5. Tl connections existing among the systems under consideration can be d scribed as follows:

𝔞) In virtue of **1.3** and **1.1** we know that K1.1 contains K1 and K1.2 co tains K1.1. I have no proof that K1.1 is a proper extension of K1. On t other hand, matrices 𝔐4 and 𝔐6 given in [4], p. 306, verify K1.1, but falsi *H1* for *p*/*2*: ℭ*2LCM22* = *LC2LC12* = *LC2L2* = *LC26* = *L5* = *5*. Hence K1 is a proper extension of K1.1.

𝔟) K1.2 is a proper subsystem of K4. Matrices 𝔐4 and 𝔐7, *cf*. [4 p. 306, verify K1.2, but falsify *R1* for *p*/*2*: *C2CML2L2* = *LC2CM66* *LC2C26* = *LC25* = *L5* = *5*.

𝔠) In [3] Prior used the same matrices 𝔐4 and 𝔐7 in order to pro that K1 is a proper subsystem of K2. Since 𝔐4 and 𝔐7 verify K1.2, th verify K1.1 too. But, they falsify *G1* for *p*/*2*: *CML2LM2* = *LCM6L2* = *LC* = *L5* = *5*. Hence no system K1, K1.1 and K1.2 contains S4.2 or any extensi of it. Thus, K1.1 is a proper subsystem of K2.1. No proof is known that I is a proper subsystem of K2.1.

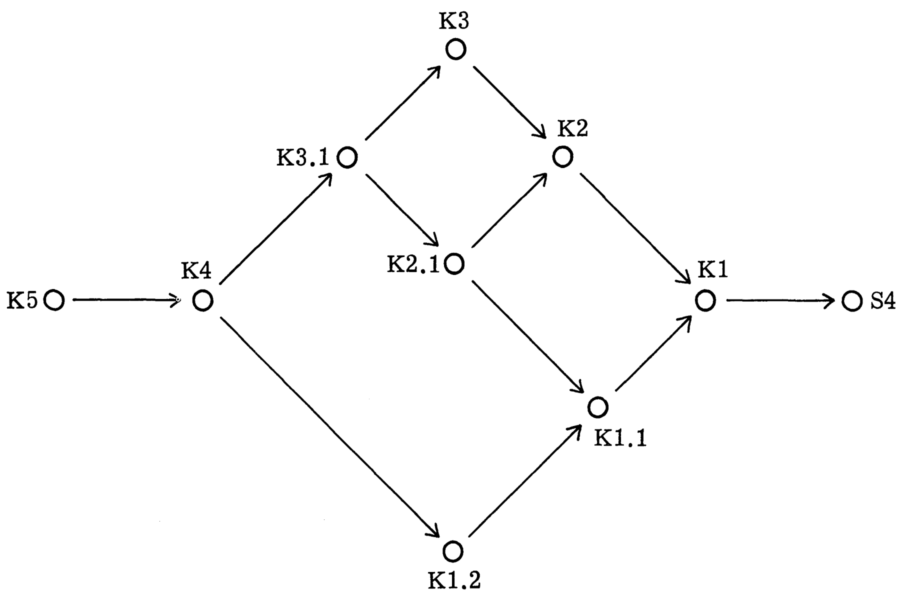
𝔡) Prior's matrix 𝔐8, defined in [3], section 3, and presented e plicitly in [4], p. 310, and which verifies K2, but falsifies K3, verifies al

J1. It shows that K3.1 is a proper extension of K2.1. I have no proof that K3 is a proper subsystem of K3.1.

𝔢) Matrices 𝔐4 and 𝔐6, *cf.* [4], p. 306, 𝔞), verify K3.1, but as we know falsify *H1*. Hence, K3.1 is a proper subsystem of K4, and K1.2 is not contained in it. A problem remains open whether K3.1 is a proper extension of K3.

𝔣) No proof exists yet that K5 is a proper extension of K4.

Thus, the connections existing among the known elements of the family *K* can be presented by the following diagram



supposing that K1.1, K2.1, K3.1 and K5 are proper extensions of K1, K2, K3 and K4 respectively.

NOTE

1. Concerning symbolism, rules of procedure, terminology etc. used in this paper see [4] , p. 311, note 1.

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